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The Convergence of the Approximated Derivative Function by Chebyshev Polynomials

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Abstract: *Let f(x) be a differentiable function on the interval [-1, 1]. Finding an approximation of the derivative of the function through values of the function at points* $\{x_j\}_{j=0}^{N}$ is a very interesting problem. It is also important for solving differential equation. In this paper, we study the error bound, in particular for first and second derivatives by Chebyshev polynomials. Moreover, a generalisation for error bound is found.
Keywords: *Chebyshev polynomials, Chebyshev interpolation, Convergence rate, Error function.*

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I. Introduction

In many problems one of interested in finding the approximating the derivative of the function f debending on the value of the function f at x_j. One of the method is to consider $(p_N\ (f))'$ as an approximation to f'. Let p_N be the Lagrange interpolation polynomial p_N for f which it may not converge to f in the sup-norm. We wish to find conditions such that $p_N' \to f'$.

The Chebyshev approximation method works best when the function is smooth, and particularly when *f(x) can be continued into the complex plane as a function f(z) which is analytic in an open neighborhood of [-1,1]. In this case, the error*

$E_N(x)=\max_{0\le j\le N}\left|f'(x_j)-\ p'\ (x_j)\right|$,

decay at least exponentially fast as $N \to \infty$.

The Chebyshev polynomial of the first kind of degree $\boldsymbol{N}$ is defined as:

$T_N(x) = cos(N\ cos^{-1}\ x) = cos\ N\theta,$ (1.1)

where x = cos θ, $-1 \le x \le 1$, $0 \le \theta \le \pi$, and n is a non negative integer [1].

The Chebyshev polynomials $T_N(x)$ satisfy $|T_N(x)| \le 1$.

This follows from the bound $-1 \le \cos x \le 1$, which leads to

$|T_{N+1}(x) - T_{N-1}(x)| \le 2.$ (1.2)

The Chebyshev polynomial $T_n(x)$ of degree $n \ge 1$ has n zeros on the interval $[-1, 1]$. The zeros x_j are given by: $x_j=\cos\left(\frac{(2j-1)\pi}{2n}\right)$, j=1,…N

Moreover, the extrema, or points $\tilde{x}_j$ such that $T_N(\tilde{x}_j)= (-1)^j$ are given by:

$\tilde{x}_j=\cos\left(\frac{j\pi}{N}\right)$, j=1,…N

The Chebyshev polynomials of the first kind have a generating function of the form

$$\sum_{N=0}^{\infty} \mathrm{T_N}(x).t^N = \frac{1-tx}{1-2xt+t^2};\ |x|<1, |t|<1 \dots .... \qquad (1-3)$$

The Chebyshev polynomials of the second kind $U_N(x)$ is defined as

$$U_N(\cos\theta) = \frac{\sin((N+1)\theta)}{\sin\theta},$$

where $-1 \le x \le 1$, $0 \le \theta \le \pi$, $x = \cos\theta$

and have a generating function of the form [1]

$$\sum_{N=0}^{\infty} \mathrm{U_N}(x)t^N = \frac{1}{1-2xt+t^2} \quad ;\ |x|<1, \quad |t|<1 \dots .... \qquad (1-4)$$

The Chebyshev polynomials have interesting properties that make them a very attractive tool to minimize the maximum error in uniform approximation.

The derivatives of the Chebyshev polynomials satisfy the following:

$\left|\frac{d}{dx}\ \mathrm{T_N}(x)\right| \le N^2.$ (1 − 5)

This comes from the definition of $\mathrm{T_N}(x)$ and $\frac{d}{dx}\ \mathrm{T_N}(x) = \frac{N \sin N\ \cos^{-1} x}{\sqrt{1-x^2}} = \frac{N \sin N\theta}{\sin\theta}$.
We have $|\sin n\theta| \le n\,|\sin\theta|$ and thus $\left|\frac{d}{dx}\ \mathrm{T_N}(x)\right| \le N^2$. For x=± 1, by L'Hopital's rule we get $\lim_{\theta\to 0\ or\ \pi} \frac{N\sin N\theta}{\sin\theta} = N^2$. For second derivative, we have

$T_N''(x) = T_N''(\cos\theta) = \frac{N\ \sin N\,\theta\cos\theta - N^2\cos N\theta \sin\theta}{\sin^3\theta}$.

Again by L'Hopital's rule, we get $\frac{N^3-N}{3}\lim_{\theta\to 0\ or\ \pi}\frac{\sin n\theta}{\sin\theta\cos\theta} = \frac{N(N^3-N)}{3}\lim_{\theta\to 0\ or\ \pi}\frac{\cos N\theta}{\cos^2\theta-\sin^2\theta}$.
Therefore

$$|T_N''(x)| \le \frac{N^2(N-1)(N+1)}{3}. \qquad (1-6)$$

The values of $\mathrm{T_N}(x)$ *and their derivatives at some points are of interest:*

$$|T_{N+1}'(x) - T_{N-1}'(x)| \le 4\mathrm{N}, \qquad |T_{N+1}''(x) - T_{N-1}''(x)| \le \frac{4}{3} N\,(2N^2+1). \qquad (1-7)$$

In general

$$T_N^{(r)}(x) \le T_N^{(r)}(1) = \frac{N^2\,(N^2-1)\ldots\ldots(N^2-(r-1)^2)}{(2r-1)!}. \qquad (1-8)$$

II. Convergence Rate

The convergence of Chebyshev series is determined by a property of the function $f(x)$. If the function f is smooth, then its Chebyshev expansion coefficients decrease rapidly. Two notions of smoothness were considered: an r^{th} derivative with bounded variation, or analyticity in a neighborhood of [-1,1].

Theorem2.1 *[2,p.66] The truncation error when approximating a function* $f(x)$ *in terms of Chebyshev polynomials satisfies*

$$|f(x) - f_n(x)| \le \sum_{k=n+1}^{\infty} |a_k|$$

If all a_k are rapidly decreasing, then the error is dominated by the leading term $a_{k+1}T_{k+1}$.
The coefficients a_k for $k > n+1$ are negligibly small, where the rest of the terms will be neglected if $a_{n+1} \neq 0$.

Theorem 2.2 *[2, p.51] If* $f, f', \ldots, f^{(r-1)}$ *are absolutely continuous for* $r \ge 0$ *on* $[-1,1]$, *where the* r^{th} *derivative* $f^{(r)}$ *has bounded variation* $V=\| f^{(r)}\|$, *then the coefficients of the Chebyshev series satisfy the fallowing inequality*

$$|a_k| \le \frac{2V}{\pi nk\,(k-1)\ldots(k-r)}, \quad k \ge r+1 \qquad (2.1)$$

for each $k \ge r+1$.

Theorem 2.3 *[2, p.51] Let a function f be analytic on* $[-1, 1]$ *and analytically continuable to the ellipse* $E_\rho := \{z \in C : z = \rho\,(e^{i\theta} + e^{-i\theta})/2,\ \theta \in [0, 2\pi]\}$ *in which* $|f(z)| \le M$ *for some M . For all* $k \ge 0$ *the Chebyshev coefficients* a_k *of f exponentially decay as* $k \to \infty$ *and satisfying*

$$|a_k| \le 2M\rho^{-k}, \quad \rho > 1. \qquad (2.2)$$

Theorem 2.4 *[2, p.53] If f is absolutely continuous for* $r \ge 0$ *on* $[-1, 1]$, *where the* r^{th} *derivative* $f^{(r)}$ *has bounded variation* $V=\| f^{(r)}\|$, *then the Chebyshev truncation satisfies*

$$\|f - f_N\|_\infty \le \frac{2V}{\pi r\,(N-r)^r} \qquad (2.3)$$

Theorem 2.5 *[2, p.58] Let a function f be analytic on* $[-1, 1]$ *and analytically continuable to the open ellipse* E_ρ, *in which* $|f| \le M$ *for some M . Then the Chebyshev truncation error satisfies*

$$\|f - f_N\|_\infty \le \frac{2M\rho^{-N}}{\rho-1} \qquad (2.4)$$

III. Chebyshev Interpolation

Given a function f that is interpolated at $n+1$ points in term of Chebyshev polynomials and that satisfies the interpolation condition $p_n(x_j)= f(x_j)$, we have the following theorem:

Theorem 3.1 *[2] Let f (x) be a Lipschitz continuous function on* $[-1, 1]$, *where*

$$f(x) = \sum_{k=0}^{\infty} \mathrm{a_k}\, \mathrm{T_k}\,(\mathrm{x}), \qquad a_k = \frac{2}{\pi} \int_{-1}^{1} \frac{f(x)T_k(x)}{\sqrt{1-x^2}}\, dx, \quad k \geq 1 \qquad (3.1)$$

Then the function f (x) can be presented by interpolation in Chebyshev points as

$$p_N = \sum_{k=0}^{\infty''} \mathrm{b_k}\, \mathrm{T_k}\,(\mathrm{x}), \quad \mathrm{b_k} = \frac{2}{\mathrm{N}} \sum_{\mathrm{j}=0}^{\mathrm{N}} \mathrm{f(x_j)T_k\,(x_j)}\,, \ \tilde{x}_j = \cos\left(\frac{j\pi}{N}\right) \qquad (3.2)$$

and

$$p_N = \sum_{k=0}^{\infty''} \mathrm{c_k}\, \mathrm{T_k}\,(\mathrm{x}), \quad \mathrm{c_k} = \frac{2}{\mathrm{N+1}} \sum_{\mathrm{j}=0}^{\mathrm{N}} \mathrm{f(x_j)T_k\,(x_j)}, \ x_j = \cos\left(\frac{(2j-1)}{2N}\right)\pi \qquad (3.3)$$

Here $\mathrm{a_k}$ are the exact coefficients, and $\mathrm{b_k}$ and $\mathrm{c_k}$ are coefficients of $\mathrm{p_n}$.

Theorem 3.2 *[3] Assume that* $\{x_j\}_{j=0}^{N}$ *are distinct points in [a,b] and that f(x) is a function in* C^{N+1} *[a,b] and* $|f^{N+1}| \leq M$. *Let* p_N *be a sequence of polynomial interpolating f. Then for each* $x \in [a, b]$, *there is* $\zeta \epsilon$ *(a, b) such that*

$$|f(x) - p_N(x)| \leq |\prod_{k=0}^{N}(x - x_k)| \left|\frac{f^{(N+1)}}{(N+1)!}\right| \qquad (3.4)$$

Theorem 3.3 *Let f (x) be a continuous function,* $p_N(x)$ *its polynomials interpolation at n+1 points and* $(p_N(f))'$ *an approximation to* f'. *Then*

$$\|f - p_N\|_\infty \leq \left|\frac{d}{dx} \prod_{k=0}^{N}(x - x_k)\right| \left|\frac{f^{(N+1)}}{(N+1)!}\right| \qquad (3.5)$$

IV. Main Results

The choice of Chebyshev points minimizes the terms $\prod_{k=0}^{N}(x - x_k)$ on [-1,1]. *This choice ensures* uniform convergence for a Lipschitz continuous function f. This condition is more important than the condition of continuity of the function f.

Theorem 4.1 *Let f (x) be a continuous function on [a, b] and let* $p_n(x)$ *be interpolant polynomials of* f at Chebyshev zeros. Then the error is given by

$$\|f - p_n\|_\infty \leq \left\|\frac{2(b-a)^{n+1}}{4^{n+1}(n+1)!}\right\|_\infty \|f^{n+1}(\zeta)\|_\infty \qquad (4.1)$$

Similarly, the error at Chebyshev extrema is given by:

$$\|f - p_n\|_\infty \leq \left\|\frac{1}{2^{n-1}(n+1)!}\right\|_\infty \|f^{n+1}(\zeta)\|_\infty \qquad (4, 2)$$

Now, we will investigate the interpolation convergence bound at zeros and extrema of Chebyshev polynomials:

Theorem 4.2 *If f is absolutely continuous and* $\|f^{(r)}\| = V < \infty$. *Then for every* $N \geq r+1$,

$$\|f' - p_N'\|_\infty \leq 4V \left[\frac{N^2(r-1) - 2r\,(N+1)}{(r-1)(r-2)(N-r)^r}\right], \qquad r \geq 2 \qquad (4, 3)$$

and

$$\|f'' - p_N''\|_\infty \leq \frac{2V}{3}\left[\frac{1}{(\mathrm{r}-4)(\mathrm{N}-\mathrm{r})^{\mathrm{r}-4}} + \frac{4\mathrm{r}}{(\mathrm{r}-3)(\mathrm{N}-\mathrm{r})^{\mathrm{r}-3}} + \frac{6\mathrm{r}^2-1}{(\mathrm{r}-2)(\mathrm{N}-\mathrm{r})^{\mathrm{r}-2}} + \frac{4\mathrm{r}^2-2\mathrm{r}}{(\mathrm{r}-1)(\mathrm{N}-\mathrm{r})^{\mathrm{r}-1}} - \frac{\mathrm{r}^4-\mathrm{r}^2}{\mathrm{r}(\mathrm{N}-\mathrm{r})^{\mathrm{r}}}\right], \ \mathrm{r} \geq 4 \qquad (4, 4)$$

Proof.

We have

$$\|\mathrm{f}' - \mathrm{p}_\mathrm{N}'\| \leq \sum_{k=0}^{N-1}|a_k - b_k|\, \|T'_k\|_\infty + \left|a_N - \frac{b_N}{2}\right| \|T'_N\|_\infty + \sum_{k=N+1}^{\infty}|a_k|\, \|T'_k\|_\infty$$

$$\leq 2 + \sum_{k=N+1}^{\infty}|a_k|\,\mathrm{k}^2 \quad \leq \quad + \sum_{k=N+1}^{\infty} \frac{4V}{\pi r\,(k-r)^{r+1}}\ \mathrm{k}^2$$

Where, a_k, b_k and c_k are defined in (3, 1), (3, 2) and (3, 3).
From the above we have that $\|T'_k\|_\infty = k^2$

$$\sum_{k=N+1}^{\infty} \frac{k^2}{(k-r)^{r+1}} \leq \int_N^\infty \frac{x^2 dx}{(x-r)^{r+1}}$$

$$= \int_{N-r}^{\infty} \frac{(u+r)^2 du}{u^{r+1}} = \frac{N^2(r-1)-2r\,(N+1)}{(r-1)(r-2)(N-r)^r}.$$

Therefore, for the second derivative

$$\|f'' - p''_N\|_\infty \leq \sum_{k=0}^{N-1} |a_k - b_k|\,\|T''_k\|_\infty + \left|a_N - \frac{b_N}{2}\right| \|T''_N\|_\infty + \sum_{k=N+1}^{\infty} |a_k|\,\|T''_k\|_\infty .$$

We have from () that $\|T''_k\|_\infty = \frac{k^2\,(k^2-1)}{3}$ and so

$$\|f'' - p''_N\|_\infty \leq \sum_{k=0}^{N-1} |a_k - b_k| \frac{k^2\,(k-1)(k+1)}{3} + \left|a_N - \frac{b_N}{2}\right| \frac{N^2\,(N-1)(N+1)}{3} + \sum_{k=N+1}^{\infty} |a_k| \frac{k^2\,(k-1)(k+1)}{3}.$$

$$\leq \sum_{k=N+1}^{\infty} \frac{2V}{\pi(k-r)^{r+1}} \frac{k^2\,(k-1)(k+1)}{3}$$

Similarly to the above we have

$$\sum_{k=N+1}^{\infty} \frac{k^2\,(k^2-1)}{(k-r)^{r+1}} \leq \int_N^\infty \frac{x^2(x^2-1)dx}{(x-r)^{r+1}} = \int_{N-r}^{\infty} \frac{(u+r)^2((u+r)^2-1)du}{u^{r+1}}$$

$$\leq \frac{1}{(r-4)(N-r)^{r-4}} + \frac{4r}{(r-3)(N-r)^{r-3}} + \frac{6r^2-1}{(r-2)(N-r)^{r-2}} + \frac{4r^2-2r}{(r-1)(N-r)^{r-1}} - \frac{r^4-r^2}{r(N-r)^r}.$$

Therefore

$$\|f'' - p''_N\|_\infty \leq \frac{2V}{3}\left[\frac{1}{(r-4)(N-r)^{r-4}} + \frac{4r}{(r-3)(N-r)^{r-3}} + \frac{6r^2-1}{(r-2)(N-r)^{r-2}} + \frac{4r^2-2r}{(r-1)(N-r)^{r-1}} - \frac{r^4-r^2}{r(N-r)^r}\right],\ r \geq 4$$

Theorem 4.3 *Let f be an analytic function such that |f (z)| ≤ M in the region bounded by an ellipse with foci ±1 and major and minor semi-axes summing to ρ >1. Then for each n≥0*

$$\|f' - p'_N\|_\infty \leq \frac{4M}{\rho^{N+1}\,(\rho-1)^3}\,[N^2\rho + (1-2N-2N^2)\rho^2 + (1+2N+2N^2)\rho^3] \qquad r \geq 2 \quad (4, 5)$$

and

$$\|f'' - p''_N\|_\infty \leq \frac{4M}{\rho^N\,(1-\rho)^5}$$
$$[N^4\,(\rho-1)^4 + 4N^3\,1\,(\rho-1)^3\rho + 12\rho^2\,(1+\rho) + + N^2(\rho-1)^2\,(5\rho^2 + 8\rho - 1 + 2N\rho(\rho 3 + 9\rho 2 - 9\rho - 1) \qquad (4, 6)$$

Proof.
As above, we arrive at

$$\|f' - p'_N\| \leq 2 \sum_{k=N+1}^{\infty} |a_k|\,\|T'_k\|_\infty \leq \sum_{k=N+1}^{\infty} \frac{4Mk^2}{\rho^k}$$

By the table value of the last sum $\sum_{k=N+1}^{\infty} \frac{k^2}{\rho^k}$, which can also verified in computer algebra system '' Mathematica'', we get the above result.
For the second derivative

$$\|f'' - p''_N\| \leq 2 \sum_{k=N+1}^{\infty} |a_k|\,\|T''_k\|_\infty \leq \sum_{k=N+1}^{\infty} \frac{4Mk^2\,(k^2-1)}{\rho^k}$$

Again by the table value of the last sum $\sum_{k=N+1}^{\infty} \frac{k^2\,(k^2-1)}{\rho^k}$, which can also verified in computer algebra system '' Mathematica'', we get the above result.
We now consider the case when the function $f(x)$ extends to function $f(z)$ of the complex plane which is analytic in a simple closed contour C the interval $[a, b]$. The complex equivalent to (4, 1) and (4, 2) is given by a contour integral [1, p150]:

Theorem 4.4 [5, p.83] *Assume that f is that extends to an analytic function in a domain Ωthat contains the interval [-1, 1]. Let* $C \subset \Omega$ *be a simple closed contour in the complex plane and let* $x_j \subset C$*, where f is an analytic function on and inside C. Then*

$f(x) - p_N(x) = \frac{1}{2\pi i}\int_C \frac{\emptyset_N(x) f(z)}{\emptyset_N(z)(z-x)}\, dz, \qquad x\epsilon[-1,1],$ (4, 7)

where

$p_N(x) = \frac{1}{2\pi i}\int_C \frac{f(z)(\emptyset_N(z) - \emptyset_N(x))}{\emptyset_N(z)(z-x)}\, dz, \ \emptyset_N(x) = \prod_{k=0}^{N}(x - x_k)$ (4, 8)

Remark. In the case of Interpolation at Chebyshev zeros, we have
$\emptyset_N(x) = \prod_{k=0}^{N}(x - x_k) = T_N(x)$, whereas in the case of interpolation at Chebyshev extrema,
$\emptyset_N(x) = \prod_{k=0}^{N}(x - x_k) = \mathbf{T_{N+1}(x)} - \mathbf{T_{N-1}}(x)$.

Theorem 4.5 *If f is a bounded analytic function such that |f (z)| ≤ M in the region bounded by an ellipse* E_ρ *with foci ±1 and major semi-axis a=*$\frac{\rho+\rho^{-1}}{2}$ *and minor semi-axis b=* $\frac{\rho-\rho^{-1}}{2}$ *summing to ρ >1. Then*

$$\|f' - p_N'\|_\infty \le \left[\frac{N^2}{\left(\frac{1}{2}(\rho+\rho^{-1})-1\right)} + \frac{1}{\left(\left(\frac{1}{2}(\rho+\rho^{-1})-1\right)\right)^2}\right]\frac{M\sqrt{\rho^2+\rho^{-2}}}{(\rho^N - \rho^{-N})} \qquad (4, 9)$$

And, for second derivative

$$\|f'' - p_N''\|_\infty \le \left[\frac{N^2(N^2-1)}{\left(\frac{1}{2}(\rho+\rho^{-1})-1\right)} + \frac{2N^2}{\left(\left(\frac{1}{2}(\rho+\rho^{-1})-1\right)\right)^2} + \frac{2}{\left(\left(\frac{1}{2}(\rho+\rho^{-1})-1\right)\right)^3}\right]\frac{M\sqrt{\rho^2+\rho^{-2}}}{(\rho^N - \rho^{-N})} \qquad (4, 10)$$

Where p_N is the polynomial interpolant of degree $\le N$ at Chebyshev zeros.

Proof.
By differentiating (4, 7) we obtain

$$f'(x) - p_N'(x) = \frac{1}{2\pi i}\int_{E_\rho}\left[\frac{\emptyset_N'(x) f(z)}{\emptyset_N(z)(z-x)} + \frac{\emptyset_N(x) f(z)}{\emptyset_N(z)(z-x)^2}\right] dz$$
$$= \frac{1}{2\pi i}\int_{E_\rho}\left[\frac{\emptyset_N'(x)}{(z-x)} + \frac{\emptyset_N(x)}{(z-x)^2}\right]\frac{f(z)}{\emptyset_N(z)} dz$$

From (1, 2), (1, 5) , we have $|\emptyset_N(x)| \le 1$, $|\emptyset_N'(x)| \le N^2$ and
$|z - x| \ge a - 1 = \frac{1}{2}(\rho + \rho^{-1}) - 1$, so

$$\|f' - p_N'\|_\infty \le \left[\frac{N^2}{\left(\frac{1}{2}(\rho+\rho^{-1})-1\right)} + \frac{1}{\left(\left(\frac{1}{2}(\rho+\rho^{-1})-1\right)\right)^2}\right]\frac{M\sqrt{\rho^2+\rho^{-2}}}{(\rho^N - \rho^{-N})}$$

For the second part, we differentiate (4, 7) twice to get

$$f'' - p_N'' = \frac{1}{2\pi i}\int_{E_\rho}\left[\frac{\emptyset_N''(x)}{(z-x)} + \frac{2\emptyset_N'(x)}{(z-x)^2} + \frac{2\emptyset_N(x)}{(z-x)^3}\right]\frac{f(z)}{\emptyset_N(z)} dz$$

From the above, we have $|\emptyset_N''(x)| \le \frac{N^2(N^2-1)}{3}$, thus

$$\|f'' - p_N''\|_\infty \le \left[\frac{N^2(N^2-1)}{\left(\frac{1}{2}(\rho+\rho^{-1})-1\right)} + \frac{2N^2}{\left(\left(\frac{1}{2}(\rho+\rho^{-1})-1\right)\right)^2} + \frac{2}{\left(\left(\frac{1}{2}(\rho+\rho^{-1})-1\right)\right)^3}\right]\frac{M\sqrt{\rho^2+\rho^{-2}}}{(\rho^N - \rho^{-N})}$$

Theorem 4.6 *If f is a bounded analytic function such that |f (z)| ≤ M in the region bounded by an ellipse* E_ρ *with foci ±1 and major semi-axis a=*$\frac{\rho+\rho^{-1}}{2}$ *and minor semi-axis b=* $\frac{\rho-\rho^{-1}}{2}$ *summing to ρ >1. Then*

$$\|f' - p_N'\|_\infty \le \left[\frac{N^2}{\left(\frac{1}{2}(\rho+\rho^{-1})-1\right)} + \frac{1}{\left(\left(\frac{1}{2}(\rho+\rho^{-1})-1\right)\right)^2}\right]\frac{M\sqrt{\rho^2+\rho^{-2}}}{(\rho+\rho^{-1})(\rho^N - \rho^{-N})} \qquad (4, 11)$$

And, for second derivative

$$\|f'' - p_N''\|_\infty \le \left[\frac{N(2N^2-1)}{\left(\frac{1}{2}(\rho+\rho^{-1})-1\right)} + \frac{8N^2}{\left(\left(\frac{1}{2}(\rho+\rho^{-1})-1\right)\right)^2} + \frac{2}{\left(\left(\frac{1}{2}(\rho+\rho^{-1})-1\right)\right)^3}\right]\frac{M\sqrt{\rho^2+\rho^{-2}}}{(\rho+\rho^{-1})(\rho^N - \rho^{-N})} \qquad (4, 12)$$

Where p_N is the polynomial interpolant of degree $\leq N$ at Chebyshev extrema.

Proof.

By differentiating ( ) we obtain

$$f'(x) - p_N'(x) = \frac{1}{2\pi i}\int_{E_\rho}\left[\frac{\emptyset_N'(x)}{(z-x)} + \frac{\emptyset_N(x)}{(z-x)^2}\right]\frac{f(z)}{\emptyset_N(z)}\,dz$$

From $|\emptyset_N(x)| \leq 2$, $|\emptyset_N'(x)| \leq 4N$, *then*

$$\|f' - p_N'\|_\infty \leq \left[\frac{N^2}{\left(\frac{1}{2}(\rho+\rho^{-1})-1\right)} + \frac{1}{\left(\left(\frac{1}{2}(\rho+\rho^{-1})-1\right)\right)^2}\right]\frac{M\sqrt{\rho^2+\rho^{-2}}}{(\rho+\rho^{-1})(\rho^N-\rho^{-N})}$$

For the second part

$$f'' - p_N'' = \frac{1}{2\pi i}\int_{E_\rho}\left[\frac{\emptyset_N''(x)}{(z-x)} + \frac{2\emptyset_N'(x)}{(z-x)^2} + \frac{\emptyset_N(x)}{(z-x)^3}\right]\frac{f(z)}{\emptyset_N(z)}\,dz$$

From above, we have $|\emptyset_N''(x)| \leq \frac{4N(2N^2+1)}{3}$, we have

$$\|f'' - p_N''\|_\infty \leq \left[\frac{N(2N^2-1)}{\left(\frac{1}{2}(\rho+\rho^{-1})-1\right)} + \frac{8N^2}{\left(\left(\frac{1}{2}(\rho+\rho^{-1})-1\right)\right)^2} + \frac{2}{\left(\left(\frac{1}{2}(\rho+\rho^{-1})-1\right)\right)^3}\right]\frac{M\sqrt{\rho^2+\rho^{-2}}}{(\rho+\rho^{-1})(\rho^N-\rho^{-N})}$$

Lemma For Chebyshev polynomial, the estimation of r^{th} derivative satisfy the bound

$$\left\|\frac{d^r}{dx^r}\left(\boldsymbol{T_{N+1}(x) - T_{N-1}}\right)\right\|_\infty \leq \frac{(N+r-2)!}{((2r-1)!!)(N-r+11)!}[4rN^2 + r^2]. \quad (4, 13)$$

Proof.

We have [1]

$$\left\|T_N^{(r)}(x)\right\|_\infty \leq \prod_{k=0}^{r-1}\frac{N^2-k^2}{2k+1} \quad (4, 14)$$

From the Stirling formula, the term $(2r-1)!!$ can be written as $\frac{(2r)!}{2^r r!}$ and

$$N^2(N^2-1^2)(N^2-2^2)\ldots((N^2-(r-1)^2) = \frac{N(N+r)!}{N+r(N-r)!} \quad (4, 15)$$

We use induction on r. If $r=1$, then we have N^2. If this hold for $N \geq 2$, and $r = 1, \ldots N-2$, then it also hold for r+1:

$$\frac{N(N+(r+1)!)}{N+(r+1)(N-(r+1)!} = \frac{N+r}{N+(r+1)}(N+(r+1)(N-r)\frac{N(N+r)!}{N+r(N-r)!}$$
$$= (N^2-r^2)(N^2(N^2-1^2)(N^2-2^2)\ldots N^2(r-1)^2.$$

Then by using (4, 14) and (4, 15) to estimate $\left|\frac{d^r}{dx^r}\left(\boldsymbol{T_{N+1}(x) - T_{N-1}}\right)\right|$, we have

$$\frac{d^r}{dx^r}\left(\boldsymbol{T_{n+1}(x) - T_{n-1}}\right) = \frac{1}{(2r-1)!!}\left[\frac{(N+1)(N+r+1)!}{(N+r+1)(N-r+1)!} - \frac{(N-1)(N+r-1)!}{(N+r-1)(N-r-1)!}\right]$$

$$= \frac{(N+r-2)!}{((2r-1)!!)(N-r+11)!}[4rN^2 + r^2].$$

We may generalize the previous result as follows:

Theorem 4.7 *If f is a bounded analytic function such that $|f(z)| \leq M$ in the region bounded by an ellipse E_ρ with foci ±1 and major semi-axis a=$\frac{\rho+\rho^{-1}}{2}$ and minor semi-axis b= $\frac{\rho-\rho^{-1}}{2}$ summing to $\rho > 1$. Then*

$$\left\|f^{(r)} - p_N^{(r)}\right\|_\infty \leq \sum_{k=0}^{(r)}\frac{r!}{k!}\times\frac{(N+r-2)!}{((2r-1)!!)(N-r+11)!}[4rN^2 + r^2]\times\frac{M}{(\rho^N-\rho^{-N})}\times\sum_{k=0}^{r}\left(\frac{2\rho}{(\rho-1)^2}\right)^{r-k+1} \quad (4, 16)$$

Where p_N is the polynomial interpolant of degree $\leq N$ at Chebyshev extrema points.

Proof.

By considering the error formula (), we have

$$f^{(r)} - p_N^{(r)} = \frac{1}{2\pi i}\int_{E_\rho}\frac{f(z)}{\emptyset_N(z)}\left(\frac{\emptyset_N(x)}{(z-x)}\right)^{(r)} dz\ .$$

By Leibniz's rule we have
$f^{(r)}(x) = \sum_{k=0}^{r} \binom{r}{k} u^{(k)} \cdot v^{(r-k)}$, where f(x) = u(x) · v(x).
Thus
$$f^{(r)}(x) - p_N^{(r)}(x) = \frac{1}{2\pi i}\int_{E_\rho} \frac{f(z)}{\emptyset_N(z)} \sum_{k=0}^{(r)} \frac{r!}{k!} \binom{r}{k} (r-k)! \left(\emptyset_N(x)\right)^{(k)} (z-x)^{k-r-1} dz.$$

$$= \sum_{k=0}^{(r)} \frac{r!}{k!} \frac{1}{2\pi i}\int_{E_\rho} \frac{\left(\emptyset_N(x)\right)^{(k)} f(z)}{\emptyset_N(z)(z-x)^{r-k+1}}$$

$$= \sum_{k=0}^{(r)} \frac{r!}{k!} \frac{1}{2\pi i}\int_{E_\rho} \frac{\left(\emptyset_N(x)\right)^{(k)} f(z)}{w(w^N - w^{-N})(z-x)^{r-k+1}} dw.$$

To estimate $\left|\frac{1}{z-x}\right|$, $let\ z = \frac{w+w^{-1}}{2}$, where w=$\rho e^{i\theta}$ and $0 \le \theta \le 2\pi$. Then
$$\left|\frac{1}{z-x}\right| = \left|\frac{1}{\frac{w+w^{-1}}{2} - x}\right| = \left|\frac{2}{w(1-2xw^{-1}+w^{-2})}\right|.$$
By the definition of the generating function of the second kind (1, 4) of the Chebyshev polynomials $U_n(x)$, we have
$$\left|\frac{2}{w(1-2xw^{-1}+w^{-2})}\right| = \frac{2}{\rho}\left|\sum_{k=0}^{\infty} U_n(x)\, w^{-k}\right| \le \frac{2}{\rho}\sum_{k=0}^{\infty}\frac{k+1}{\rho^k} = \frac{2\rho}{(\rho-1)^2}.$$

From (4, 13) we have
$$\left(\emptyset_N(x)\right)^{(k)} \le \frac{(N+r-2)!}{((2r-1)!!)(N-r+11)!}[4rN^2 + r^2].$$
Therefore
$$\left\|f^{(r)}(x) - p_N^{(r)}(x)\right\|_\infty = \left\|\sum_{k=0}^{(r)} \frac{r!}{k!} \frac{1}{2\pi i}\int_{E_\rho} \frac{\left(\emptyset_N(x)\right)^{(k)} f(z)}{\emptyset_N(z)(z-x)^{r-k+1}}\right\|_\infty$$
$$\le \sum_{k=0}^{(r)} \frac{r!}{k!} \left\|\left(\emptyset_N(x)\right)^{(k)}\right\|_\infty \frac{1}{2\pi}\int_{E_\rho} \frac{|f(z)|}{\rho(\rho^N - \rho^{-N})(z-x)^{r-k+1}} |dw|$$
$$\le \sum_{k=0}^{(r)} \frac{r!}{k!} \times \frac{(N+r-2)!}{((2r-1)!!)(N-r+11)!}[4rN^2 + r^2] \times \frac{M}{(\rho^N - \rho^{-N})} \times \sum_{k=0}^{r}\left(\frac{2\rho}{(\rho-1)^2}\right)^{r-k+1}$$

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