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On Performance ofTests forPaired Samples: An Empirical Approach

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Abstract: *This article investigates performance of Sign test, Wilcoxon Signed Rank test and t-tests with and without transformation on paired samples by considering scenarios where assumptions of parametric test may or may not meet. For example, a paired t-testor a specified mean difference requires the assumption of normality of populations the samples come from. Inreal-life, such an assumption may not meet. Given this reality, atransformed t-test or alternately,non-parametric tests (Sign or Signed rank tests)may be employed. To recommend the best test, it is imperative that we compare the performanceof these tests using Type I error probability and power of the test via Monte Carlo simulation at various sample sizes. Unlike the recent study ([1]), this article simulates paired samples from symmetric normal and uniform distributions, and skewed gamma and exponential distributions with varying levels ofcorrelation and skewness in the paired populations.*
Keywords*:Paired t-test, Sign-test, Wilcoxon Signed Rank test, correlation, skewness, Type I error rate, Power of a test.*

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I. Introduction

Very often, we are interested in comparing two means or medians for two populations whose measurements are paired. For example, the two populations of interest may refer to before and after measurements or related measurements of two different treatments. In order to perform such comparison or test for means, one may use paired t-test if the population of paired measurement differences follows a normal distribution. In the violation of the normality assumption, one could have several options:

1) Ignore the violation of the assumption as if there were no violation of the normality assumption and apply the paired t-test.
2) Utilize nonparametric tests such as Sign test and Singed Rank test, which are distribution free methods in that they do not require any distributional assumption.
3) Employ t-test on transformed data by a suitable transformation.

The paired sample Wilcoxonsigned rank test and sign-test are nonparametric methods used in the comparison of the equality of the mediansof two paired populations especially when the normality assumption of the data required for the paired t-test is violated.In other words, when the normality isquestionable, the paired sample Wilcoxon signed rank and sign testare popular alternative tests to use to substitute the pairedsample t-test.

In a recent study ([1]), the consistency and limitation of paired t-test and nonparametric Sign and Wilcoxon Singed Rank testshave been investigated using paired samples from normal and non-normal populations. If t-test is implemented on samples from a normal population, this test is uniformlymost powerful test. However, if t-test is employed on non-normal data, it may end up with a misleading conclusion due to the violation of the normality assumption ([2]). As such, desperate users of t-test may consider re-expressing data by some transformation and then implement$t$-test ([3], [4]).Alternately, one could make use of nonparametric tests, also termed as distribution free methods ([1], [5]), to compare medians of paired populations using paired samples from related populations by means of Wilcoxon Signed rank test or Sign test ([1], [6]). Of course, a comparative study may lead to a better recommendation on the choice of an appropriate test.

While studying consistency and limitation of paired t-test, Sign test and Wilcoxon Sign Rank test, it is important that oneconsider the correlation and skewness of the paired populations into account since the analysis results are likely to be sensitive to the degree of correlation between the paired populations and their skewness. Therefore, unlike [1], this paper considers paired$t$-test, log-transformedpaired$t$-test,Sign test and Signed rank test for varying level of correlation between the paired populations and of their skewness in case the paired populations are skewed.To support the use of transformed paired t-test in the comparison, it is worth pointing out that Sign test and Signed Rank test both replace original data values by Sign (Sign test) and Sign and rank(Wilcoxon Signed Rank test) for conducting the testing procedure. Therefore, indeed, Sign and Signed Rank tests are a particular form of transformed test in the violation of the normality of the data. If Sign or Singed Rank Test iscompared with paired$t$-test, it makes sense to use a log-transformed paired t-test with Students' paired t-test. Of course, if an underlying sample comes from a normal distribution, t-test by itself is expected to perform reasonably well than any transformed t-test or nonparametric test.On the other hand, if normality or one or more requirements of parametric tests are not satisfied, then non-parametric methods can be used which focuses particularly on the fact that the distribution of the sampling statistic is not known ([1], [7]).

Nonparametric tests are often used in conjunction with small samples, because for such samples the central limit theorem cannot be invoked ([1]). Nonparametric tests can be directed towards hypothesis concerning theform, dispersion or location (median) of the population. In the majority of the applications, the hypothesis isconcerned with the value of a median, the difference between medians or the differences among severalmedians ([1]). This contrasts with the parametric procedures, which focus principally on population means. Ifnormal model cannot be assumed for the data then the tests of hypothesis on means are not applicable.Nonparametric tests were created to overcome this difficulty. Nonparametric tests are often (but not always)based on the use of ranks; such as Wilcoxon rank test, Sign test, Wilcoxonrank sum test, Kruskal Wallis test,Kolmogorov test, etc. ([1], [6], [8]).

The objectives of this paper are of three folds:

i. To examine the effect of non-normality on parametric paired$t$-test, log-transformed paired$t$-test, and the nonparametric Sign and Wilcoxon signed rank test.
ii. To examine the effect of varying sample size on the four underlying testsin terms of the Type I error rate andpowerof the test.
iii. Examine the sensitivity of these tests in terms of the Type I error rate and powerdue to the presence of varyingskewnessand correlation in the paired populations.

II. Materials and Methods

Like [1], the materials used for the analysis are simulated paired samples from symmetric normal and uniform distributions and skewed exponential and gamma distributions. However, unlike [1], various specified correlation and skewness between the paired distributionsare considered in the simulation of the paired samples. Gamma distribution is considered so that effects of varying skewness on underlying four tests can be examined. Since it is very difficult to get data that follows these distribution patterns, even if there is, it isvery difficult to get the required number of replicates for the sample sizes of interest ([1]). Given these facts, twoparametric tests (untransformed paired$t$-test and log-transformed paired$t$-test) and two nonparametric tests (Sign test and Wilcoxon signed rank test) are applied on simulated paired sample from related populations towards assessing their performance in terms of Type I error rate (level of significance) and power of the test.

2.1 Paired-sample Simulation Procedures and Analysis

Note that the paired t-test is applied to test for a specified mean difference using a paired sample, where two samples forming the pair refer to before-after or related measurements of two different treatments. In other words, a paired t-test applies to correlated paired samples due to the existence of correlation between the two underlying populations the samples come from.

A well-known open source software R has been utilized for all simulation and computation. To control the degree of dependence on the paired samples, paired samples are generated from paired population distributions with varying levels of correlation using the desired and specified variance-covariance matrix of the variables. For more detail on the implementation of the method, one could look at the random number generator using mvrnorm function available via package MASS in R. Sample programs to generate a paired sample with specified correlation are provided in the appendix for exponential, gamma and uniform distribution.

In this paper, paired samples were simulated from Normal, Uniform, Gamma and Exponential distributions, respectively, for sample sizes of 10, 15, 20, 25, 30 and 35,to allow finite small, moderate and large sample performance of underlying four tests. A specified variance covariance matrix was utilized to control correlation between two paired distributions. The paired population correlations are arbitrarily set to 0.25, 0.50 and 0.75 to assess any sensitiveness of underlying four tests on estimated level of significance (Type I error rate) and power of the test. The performance of four tests are evaluated based on estimated Type I error rate and power of underlying tests from a Monte Carlo simulation of 5000 repeated paired samples for each of the specified sample sizes, with a specified correlation(0.25, 0.50 and 0.75) and skewness (for gamma distribution). As such, an estimated power is the proportion of rejection of the null hypothesis of no mean difference in the paired populations over 5000 repetitions, when two means in the paired populations are actually different. The estimated level of significance or probability of Type I erroris the proportion of rejection of the null hypothesis of no mean difference in the paired populations over 5000 repetitions, when two means in the paired populations are actually the same. The mean differences in the paired populations are arbitrarily considered to be 0.15, 0.25, 0.35 and 0.45 so that a meaningful comparisons of the underlying test can be made in terms of the Type I error rate and power of the test. The skewness of the paired populations are considered arbitrarily to be 0.25, 0.50 and 0.75 so that a meaningful comparisons of the underlying test can be made in terms of the Type I error rate and power of the test.

2.2Student's Paired$t$-test

Suppose we are interested to compare two population means where the two populations are paired. For example, let μ_1 and μ_2 be the means of two populations that represent before and after measurements or related measurements of two different treatments. By letting $\mu_d = \mu_1 - \mu_2$, we wish to test $H_0: \mu_d = 0$ (or, $H_0: \mu_d = \mu_0$) against $H_a: \mu_d \neq 0$ (or, $H_0: \mu_d \neq \mu_0$), where $\mu_d = E(X - Y)$ for the paired population (X, Y). In order to carry out this test, a sample of n paired measurements $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ is considered. By letting $d_i = x_i - y_i$; $i = 1, 2, \dots, n$; $\bar{d} = \bar{x} - \bar{y}$ and $s_d^2 = \sum_{i=1}^{n} \left(d_i - \bar{d}\right)^2 / (n-1)$, the test statistic for $H_0: \mu_d = 0$ (or, $H_0: \mu_d = \mu_0$) against $H_a: \mu_d \neq 0$ (or, $H_0: \mu_d \neq \mu_0$) is given by

$$T = \frac{\bar{d} - \mu_d}{s_d / \sqrt{n}} \quad (1)$$

The test using the test statistic T in (1) is called a paired t-test, which follows a Student's t-distribution with $(n-1)$ degrees of freedom. Details of paired t-test can be found in any standard texts (e.g., [9]). This test requires that the sample of paired differences follow a normal distribution. In the violation of this assumption, one could reach an invalid or a misleading conclusion. To address this limitation, Sign and Wilcoxon tests along with a log-transformed t-testhave been employed following ([1], [6])with the contention that if the p-valueproduced by the t-test in any distribution is close to the p-value produced by the sign test, Wilcoxon signed rank test (Wsr) or the log-transformed t-test (Ltt), thenthe t-test could be trusted ([1]). Unlike [1], this article takes into account the varying values of correlation and skewness of the paired population distributions to study the effect of correlation and skewness, along with sample sizes, in estimated significance level and power of the underlying tests. Note that Sign test and Wilcoxon signed rank tests may be considered a particular form of transformed tests in that they replace the original sample data values by the sign of (x-y) for Sign test, and by the sign of (x-y) and rank of |x-y| for the Wilcoxon test. Given these facts, it is inspiring to compare these tests (Sign test and Wilcoxon test) to another simple transformation based test, called a log-transformed paired$t$-test.

2.3 Log-transformed Paired t-test (Ltt)

Log-transformation is widely used for transforming substantially skewed data to reduce skewness. This transformation is particularly recommend when effects are multiplicative, e.g., time series data, substances in blood, etc. The substances in blood (e.g., cholesterol) are subject to a continuous metabolic reaction, the rate of which depends on many factors multiplied together, which results in skewed distribution. The log-transformation of $x$is defined as follows:

(i) $x' = log_{10}(x)$, where$x$ is substantially positive skewed.

(ii) $x' = log_{10}(x + c)$, where$x$ is substantially positive skewed with existence of zero values; c is a constant added to each data value so that log is defined.

(iii) $x' = log_{10}(K - x)$, where x is negatively substantially skewed; $K = \max(x) + 1$

Since exponential and gamma distribution are skewed, it is expected that the log-transformed paired t-test (Ltt) would perform relatively better for skewed distribution. In this article, we employ paired t-test to transformed pair (x',y') and call it a log-transformed paired$t$-test (Ltt).

2.4 Sign Test

The sign test has a null hypothesis that thepaired samples have the same (identical) distribution and thus the same median. If two distributions are identical, they indeed will have identical means, medians or any other location parameters. Thus, sign test can be compared with t-test ([1], [6]).

The test statistic for the sign test is the number of pairs for which x-valuesare different from y-values.Under the null hypothesis, the test statistic has the binomial distribution with the number of trials being the totalnumber of non-tied pairs.To test $H_0: \mu_d = 0$against $H_0: \mu_d \neq 0$given a paired sample $(x_1, y_1), (x_2, y_2) \dots, (x_n, y_n)$ of n ordered pairs, the Sign test uses sign of $(x_i - y_i)$, $i = 1,2, \dots, n$. Let T^+ and T^- denote the total number of +'s and –'s in the sample, $minT = \min\{T^+, T^-\}$ and$n^* = T^+ + T^- \leq n$ such that any pair with $x_i - y_i = 0$is being ignored, and as such the number of trials is reduced for each tied pair ([10]). Then, it follows that $T^+$and T^- follow binomial distributions, and one can compute the p-value of the test$H_0: \mu_d = 0$, using binomial table or any software. For detail about sign test one could consult with any standard texts (such as [11], [12], [13]). In this article, SIGN.testfunction available from R package BSDA has been utilized to implement Sign test.

2.5 The Wilcoxon Singed Rank (Wsr) Test

The null hypothesis of the Wilcoxon signed rank ([11], [12], [13]) is the identity of the medians.While the Sign test takes into account the signs of $(x_i - y_i)$'s, the Wilcoxon Signed Rank test takes into account the signs of $(x_i - y_i)$'s and their magnitudes using their ranks for testing for no median difference $H_0: \mu_d = 0$ against $H_0: \mu_d \neq 0$ for paired sample, and hence the test is called the Signed rank test.

Under the null hypothesis,$(x_i - y_i)$'s will be distributed symmetrically about 0. In other words, these differences are equally likely to be positive and negative.

Let$r_i = rank\ of |x_i - y_i|andu_i = 1$ if $x_i - y_i > 0$, 0 elsewhere, for paired sample $(x_1, y_1), (x_2, y_2) .., (x_n, y_n)$. Then, it follows that $\sum_{i=1}^{n} r_i = \frac{n(n+1)}{2}$and $\sum_{i=1}^{n} {r_i}^2 = \frac{n(n+1)(2n+1)}{6}$.

The two test statistics

$$T^+ = \sum_{i=1}^{n} u_i \times r_i \tag{2}$$

and

$$T^- = \sum_{i=1}^{n} (1 - u_i) \times r_i \tag{3}$$

refer to sum of positive ranks and sum of negative ranks, respectively. One can reject $H_0: \mu_d = 0$ against $H_0: \mu_d \neq 0$ if $T^+ \geq \frac{n(n+1)}{2}$ or p-value= Pr$(T \leq T^+) \leq \alpha$, the level of significance.Note that under the null hypothesis, u_i's are i.i.d. Bernoulli random variables with $p = 1/2$. That is, $E[u_i] = \frac{1}{2}$and $V[u_i] = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$. Therefore, $E[T^+] = E[T^-] = \frac{n(n+1)}{4}$and $V[T^+] = V[T^-] = \frac{n(n+1)(2n+1)}{24}$. Since there is no explicit form of the exact probability distribution of $T^+$and T^-, a transformed test statistic given by

$$Z = \frac{T^+ - \frac{n(n+1)}{4} - 0.5}{\sqrt{\frac{n(n+1)(2n+1)}{24}}} \tag{4}$$

or

$$Z = \frac{T^- - \frac{n(n+1)}{4} - 0.5}{\sqrt{\frac{n(n+1)(2n+1)}{24}}} \tag{5}$$

can be used. The statistic Z follows N(0,1) distribution, approximately. The half-unit correction is used for the continuity of the transformed signed rank statistic.

By noting the fact that

$$S = \frac{T^+ - T^-}{2} = \sum_{i=1}^{n} u_i \times r_i - \frac{n(n+1)}{4} \tag{6}$$

, the test statistic based on the statistic S can also be used as signed rank test (SAS system outputs the value of this statistic). It appears that $E[S] = 0$and $V[S] = \sum_{i=1}^{n} V[u_i] \times {r_i}^2 = \frac{n(n+1)(2n+1)}{24}$, and so one can use the test statistic $Z = \frac{S - 0.5}{\sqrt{V[S]}} = \frac{S - 0.5}{\sqrt{\frac{n(n+1)(2n+1)}{24}}}$which is distributed as a standard normal distribution, approximately.

In this article, wilcox.test available from the software R has been utilized to implement Wilcoxon Signed rank test.

2.6 Assessment Criteria forCompetitive Tests

Two assessment criteria have been used to decide which test is best among the underlying four competitive tests:

(i) Type I error rate, also called the level of significance, which is the estimated probability of rejection of the null hypothesis when it is actually true.

(ii) The power of the test, which is the estimated probability of rejection of the null hypothesis which is indeed false.

For all paired samples generated under the null hypothesis or under the alternative hypothesis, the decision regarding the rejection of the null hypothesis of no mean (or median) difference are counted over all 5000 repetitions of the paired samples, as explained in detail in section 3, to determine Type I error rate and the power of the test ([14], [15]).

III. SimulationsAlgorithm and Analysis Results

The simulation algorithm of the paired samples towards assessing performance of the four underlying tests in terms of the Type I error rate and power of the test is as follows:

(a) Generate paired samples of size $n = 10$, 15, 20, 25, 30 and 35 under null and alternative hypotheses from paired populations which are distributed as Normal, Uniform, Exponential or Gamma.Under the null model mean difference is 0 and the alternative model mean difference of 0.15, 0.25, 0.35 and 0.45 are considered arbitrarily so as to make a meaningful comparisons of the four underlying tests.
(b) The correlation between the paired populations in the simulation are considered arbitrarily to be 0.25, 0.50 and 0.75.
(c) For gamma distribution $G(\theta_1, \theta_2)$,the skewness of the paired population is arbitrarily set to 1, 2, 4, 8 by choosing the values of the shape parameter $\theta_1$to be 4, 1, 0.25 and 0.0625, respectively, since the skewness of the gamma distribution is $2/\sqrt{\theta_1}$.The mean of gamma distribution is arbitrarily fixed at 1in all simulation by choosing the values of (θ_1, θ_2) to be (4, 0.25), (1, 1), (0.25, 4) and (0.0625, 16).

θ_1	θ_2	$\mu = \theta_1\theta_2$	Skewness $(2/\sqrt{\theta_1})$
4	0.25	1	1
1	1	1	2
0.25	4	1	4
0.0625	16	1	8

(d) Monte Carlo size of 5000 is considered for the repetition of each paired samplesof a given size defined in (a)-(c).
(e) The simulated Type I error rate, also called size of the test or level of significance, is the proportion of the rejection of the null hypothesis over all 5000 repetitions when the null hypothesis is indeed true. The estimated power of any underlying test is the proportion of the rejection of the null hypothesis over all 5000 repetitions when the null hypothesis is indeed false.
(f) In all simulation, the true level of significance is considered 5%, that is,$\alpha = 0.05$.

The results of simulation for the assessment of the performance of the four underlying tests are presented in Tables 1.1-1.4 for Type I error rates and in Tables 2.1-2.4 for power of the test.

Table 1.1: Estimated Type I error rate at true significance level $\alpha = 0.05$ for $X, Y \sim U(0,2)$with specified paired population correlation$r_{x,y}$

n	$r_{x,y} = 0.25$				$r_{x,y} = 0.50$				$r_{x,y} = 0.75$			
	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt
10	0.044	0.017	0.048	0.038	0.053	0.023	0.053	0.042	0.052	0.023	0.051	0.048
15	0.047	0.036	0.046	0.035	0.043	0.033	0.041	0.038	0.042	0.028	0.041	0.036
20	0.045	0.048	0.044	0.056	0.047	0.031	0.046	0.045	0.051	0.042	0.045	0.045
25	0.046	0.040	0.045	0.047	0.033	0.045	0.033	0.037	0.038	0.041	0.038	0.038
30	0.056	0.053	0.055	0.052	0.037	0.036	0.036	0.042	0.042	0.028	0.046	0.049
35	0.042	0.036	0.043	0.054	0.056	0.043	0.055	0.049	0.060	0.032	0.053	0.054

Table 1.2: Estimated Type I error rate at true significance level $\alpha = 0.05$ for $X, Y \sim N(5,1)$with specified paired population correlation$r_{x,y}$

n	$r_{x,y} = 0.25$				$r_{x,y} = 0.50$				$r_{x,y} = 0.75$			
	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt
10	0.057	0.018	0.056	0.056	0.044	0.018	0.043	0.039	0.060	0.026	0.056	0.056
15	0.055	0.041	0.055	0.062	0.044	0.033	0.047	0.045	0.057	0.037	0.052	0.056
20	0.057	0.053	0.059	0.053	0.047	0.040	0.047	0.046	0.052	0.036	0.042	0.049
25	0.060	0.046	0.054	0.062	0.046	0.042	0.045	0.052	0.055	0.047	0.049	0.051
30	0.038	0.045	0.042	0.041	0.050	0.047	0.056	0.047	0.060	0.048	0.059	0.064
35	0.042	0.031	0.043	0.037	0.055	0.044	0.052	0.052	0.054	0.054	0.056	0.057

Table 1.3: Estimated Type I error rate at true significance level $\alpha = 0.05$ for $X, Y \sim Exp(1)$with specified paired population correlation $r_{x,y}$

n	$r_{x,y} = 0.25$				$r_{x,y} = 0.50$				$r_{x,y} = 0.75$			
	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt
10	0.047	0.025	0.061	0.064	0.046	0.029	0.050	0.054	0.028	0.025	0.046	0.041
15	0.042	0.031	0.045	0.044	0.044	0.044	0.052	0.062	0.036	0.042	0.046	0.053
20	0.052	0.040	0.054	0.046	0.056	0.044	0.061	0.052	0.051	0.047	0.051	0.044
25	0.045	0.035	0.041	0.045	0.045	0.038	0.035	0.050	0.030	0.039	0.031	0.036
30	0.055	0.046	0.063	0.049	0.042	0.051	0.054	0.048	0.038	0.042	0.044	0.040
35	0.033	0.040	0.037	0.048	0.051	0.036	0.044	0.048	0.053	0.039	0.057	0.050

Table 1.4: Estimated Type I error rate at true significance level $\alpha = 0.05$ for $X, Y \sim G(\theta_1, \theta_2)$so as to have mean=1 and skewness =1, 2, 4 and 8 with specified paired population correlation $r_{x,y}$

n	Skewness=1											
	$r_{x,y} = 0.25$				$r_{x,y} = 0.50$				$r_{x,y} = 0.75$			
	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt
10	0.048	0.023	0.052	0.047	0.046	0.023	0.044	0.049	0.042	0.024	0.050	0.048
15	0.053	0.039	0.052	0.052	0.049	0.031	0.041	0.050	0.049	0.036	0.045	0.049
20	0.055	0.049	0.059	0.059	0.060	0.049	0.055	0.046	0.059	0.033	0.060	0.057
25	0.049	0.039	0.049	0.043	0.041	0.046	0.050	0.056	0.045	0.041	0.047	0.045
30	0.059	0.031	0.049	0.050	0.056	0.049	0.058	0.051	0.053	0.038	0.052	0.048
35	0.060	0.035	0.054	0.054	0.053	0.047	0.061	0.057	0.051	0.045	0.049	0.046
n	Skewness=2											
	$r_{x,y} = 0.25$				$r_{x,y} = 0.50$				$r_{x,y} = 0.75$			
	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt
10	0.035	0.021	0.047	0.044	0.048	0.020	0.056	0.048	0.042	0.022	0.051	0.057
15	0.034	0.035	0.037	0.042	0.055	0.040	0.061	0.053	0.047	0.047	0.057	0.048
20	0.046	0.037	0.056	0.044	0.041	0.047	0.053	0.046	0.049	0.043	0.045	0.040
25	0.046	0.036	0.043	0.055	0.050	0.041	0.047	0.046	0.048	0.045	0.047	0.053
30	0.054	0.049	0.054	0.055	0.045	0.045	0.056	0.051	0.041	0.045	0.049	0.040
35	0.054	0.043	0.059	0.054	0.048	0.035	0.058	0.051	0.046	0.042	0.052	0.055
n	Skewness=4											
	$r_{x,y} = 0.25$				$r_{x,y} = 0.50$				$r_{x,y} = 0.75$			
	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt
10	0.024	0.029	0.056	0.038	0.021	0.028	0.062	0.045	0.023	0.025	0.054	0.042
15	0.037	0.032	0.046	0.054	0.034	0.045	0.056	0.054	0.032	0.039	0.054	0.049
20	0.029	0.039	0.045	0.040	0.040	0.043	0.057	0.056	0.031	0.032	0.042	0.040
25	0.038	0.046	0.044	0.041	0.026	0.047	0.042	0.055	0.036	0.046	0.048	0.043
30	0.038	0.037	0.053	0.050	0.033	0.053	0.053	0.054	0.033	0.045	0.043	0.045
35	0.036	0.036	0.042	0.049	0.031	0.031	0.038	0.041	0.038	0.043	0.051	0.056
n	Skewness=8											
	$r_{x,y} = 0.25$				$r_{x,y} = 0.50$				$r_{x,y} = 0.75$			
	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt
10	0.004	0.027	0.056	0.041	0.002	0.016	0.050	0.037	0.001	0.027	0.046	0.033
15	0.011	0.036	0.049	0.048	0.010	0.028	0.052	0.034	0.005	0.037	0.043	0.045
20	0.013	0.050	0.055	0.057	0.012	0.048	0.054	0.051	0.009	0.042	0.043	0.038
25	0.013	0.035	0.045	0.034	0.014	0.031	0.045	0.039	0.019	0.037	0.047	0.043
30	0.020	0.037	0.042	0.043	0.015	0.042	0.057	0.047	0.016	0.045	0.048	0.046
35	0.021	0.041	0.048	0.039	0.017	0.044	0.045	0.045	0.018	0.044	0.043	0.041

Ta**ble 2.1:** Estimated power of the test for paired samples generated from $X \sim U(0,2) + \mu_d, Y \sim U(0,2)$with specified paired population correlation $r_{x,y}$ and $\mu_d \neq 0$

n	$\mu_d = 0.15$											
	$r_{x,y} = 0.25$				$r_{x,y} = 0.50$				$r_{x,y} = 0.75$			
	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt
10	0.086	0.040	0.088	0.066	0.103	0.050	0.112	0.089	0.185	0.102	0.191	0.154
15	0.089	0.059	0.076	0.097	0.154	0.118	0.153	0.173	0.261	0.207	0.267	0.266
20	0.162	0.112	0.151	0.164	0.187	0.157	0.185	0.213	0.336	0.290	0.330	0.354
25	0.165	0.126	0.158	0.177	0.239	0.202	0.223	0.262	0.432	0.375	0.436	0.493
30	0.203	0.173	0.203	0.246	0.278	0.216	0.278	0.331	0.504	0.442	0.507	0.567
35	0.253	0.173	0.239	0.308	0.301	0.241	0.302	0.399	0.550	0.489	0.554	0.640
n	$\mu_d = 0.25$											
	$r_{x,y} = 0.25$				$r_{x,y} = 0.50$				$r_{x,y} = 0.75$			
	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt
10	0.145	0.077	0.145	0.139	0.224	0.112	0.220	0.187	0.398	0.252	0.389	0.327
15	0.243	0.142	0.231	0.246	0.332	0.255	0.335	0.339	0.557	0.434	0.556	0.521
20	0.308	0.199	0.293	0.321	0.443	0.361	0.447	0.482	0.706	0.632	0.717	0.702
25	0.398	0.294	0.373	0.449	0.516	0.421	0.512	0.569	0.816	0.711	0.817	0.829
30	0.451	0.329	0.430	0.534	0.605	0.449	0.586	0.691	0.871	0.778	0.870	0.886
35	0.529	0.347	0.485	0.604	0.665	0.551	0.663	0.769	0.920	0.857	0.915	0.948
n	$\mu_d = 0.35$											
	$r_{x,y} = 0.25$				$r_{x,y} = 0.50$				$r_{x,y} = 0.75$			
	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt
10	0.263	0.146	0.259	0.226	0.387	0.212	0.382	0.323	0.655	0.432	0.652	0.532
15	0.429	0.287	0.401	0.415	0.570	0.389	0.549	0.570	0.844	0.701	0.830	0.805

20	0.552	0.389	0.519	0.562	0.714	0.551	0.684	0.723	0.947	0.851	0.940	0.931
25	0.657	0.460	0.610	0.692	0.810	0.664	0.782	0.831	0.981	0.922	0.975	0.979
30	0.765	0.576	0.724	0.808	0.892	0.752	0.879	0.921	0.990	0.956	0.990	0.987
35	0.816	0.615	0.791	0.868	0.926	0.814	0.913	0.942	0.996	0.981	0.997	0.998
n	$\mu_d = 0.45$											
	$r_{x,y} = 0.25$				$r_{x,y} = 0.50$				$r_{x,y} = 0.75$			
	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt
10	0.416	0.215	0.408	0.370	0.591	0.348	0.575	0.495	0.818	0.568	0.812	0.695
15	0.600	0.415	0.566	0.592	0.795	0.605	0.773	0.763	0.969	0.874	0.962	0.910
20	0.773	0.558	0.742	0.781	0.899	0.755	0.893	0.889	0.992	0.943	0.991	0.991
25	0.861	0.700	0.827	0.872	0.959	0.850	0.946	0.968	0.996	0.985	0.996	0.995
30	0.927	0.755	0.903	0.945	0.973	0.900	0.972	0.978	1.000	0.995	1.000	1.000
35	0.946	0.816	0.932	0.968	0.995	0.938	0.992	0.994	1.000	0.998	1.000	1.000

T**able 2.2:**Estimated power of the test for paired samples generatedfrom $X{\sim}N(5,1) + \mu_d, Y{\sim}N(5,1)$with specified paired population correlation $r_{x,y}$ and $\mu_d \neq 0$

n	$\mu_d = 0.15$											
	$r_{x,y} = 0.25$				$r_{x,y} = 0.50$				$r_{x,y} = 0.75$			
	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt
10	0.059	0.025	0.054	0.057	0.053	0.025	0.052	0.051	0.093	0.039	0.085	0.083
15	0.061	0.036	0.058	0.059	0.080	0.048	0.078	0.077	0.120	0.088	0.117	0.116
20	0.105	0.058	0.097	0.098	0.078	0.051	0.082	0.081	0.142	0.096	0.135	0.144
25	0.078	0.058	0.072	0.075	0.103	0.063	0.097	0.099	0.155	0.113	0.146	0.156
30	0.100	0.077	0.094	0.105	0.121	0.080	0.123	0.123	0.198	0.128	0.192	0.195
35	0.111	0.085	0.115	0.105	0.136	0.098	0.134	0.129	0.224	0.147	0.209	0.228
n	$\mu_d = 0.25$											
	$r_{x,y} = 0.25$				$r_{x,y} = 0.50$				$r_{x,y} = 0.75$			
	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt
10	0.092	0.035	0.088	0.092	0.114	0.047	0.101	0.103	0.163	0.059	0.154	0.162
15	0.117	0.081	0.108	0.126	0.151	0.088	0.138	0.141	0.253	0.138	0.232	0.232
20	0.122	0.101	0.116	0.125	0.178	0.128	0.169	0.172	0.300	0.193	0.290	0.298
25	0.160	0.104	0.150	0.158	0.226	0.161	0.218	0.218	0.399	0.267	0.374	0.389
30	0.181	0.133	0.176	0.177	0.271	0.194	0.256	0.266	0.495	0.344	0.473	0.475
35	0.212	0.127	0.208	0.194	0.292	0.191	0.272	0.291	0.548	0.352	0.521	0.540
n	$\mu_d = 0.35$											
	$r_{x,y} = 0.25$				$r_{x,y} = 0.50$				$r_{x,y} = 0.75$			
	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt
10	0.147	0.061	0.140	0.134	0.162	0.064	0.164	0.170	0.303	0.137	0.287	0.291
15	0.186	0.112	0.182	0.181	0.229	0.141	0.216	0.221	0.404	0.261	0.385	0.383
20	0.221	0.160	0.217	0.223	0.366	0.232	0.341	0.362	0.542	0.378	0.527	0.538
25	0.285	0.182	0.273	0.273	0.416	0.259	0.397	0.396	0.649	0.448	0.632	0.632
30	0.340	0.217	0.335	0.335	0.466	0.299	0.451	0.441	0.757	0.562	0.733	0.733
35	0.344	0.226	0.336	0.334	0.530	0.343	0.505	0.519	0.811	0.592	0.781	0.789
n	$\mu_d = 0.45$											
	$r_{x,y} = 0.25$				$r_{x,y} = 0.50$				$r_{x,y} = 0.75$			
	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt
10	0.179	0.069	0.167	0.169	0.248	0.117	0.236	0.247	0.445	0.230	0.422	0.436
15	0.242	0.142	0.234	0.234	0.398	0.231	0.371	0.376	0.617	0.440	0.598	0.597
20	0.364	0.233	0.349	0.356	0.452	0.310	0.441	0.449	0.767	0.562	0.746	0.746
25	0.442	0.292	0.406	0.433	0.556	0.393	0.545	0.560	0.868	0.684	0.845	0.859
30	0.503	0.338	0.481	0.497	0.673	0.467	0.644	0.651	0.922	0.769	0.909	0.915
35	0.562	0.366	0.543	0.549	0.744	0.540	0.724	0.728	0.951	0.812	0.939	0.945

Table 2.3: Estimated power of the test for paired samples generated from $X{\sim}Exp(1) + \mu_d, Y{\sim}Exp(1)$with specified paired population correlation $r_{x,y}$ and $\mu_d \neq 0$

n	$\mu_d = 0.15$											
	$r_{x,y} = 0.25$				$r_{x,y} = 0.50$				$r_{x,y} = 0.75$			
	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt
10	0.059	0.046	0.074	0.093	0.071	0.042	0.079	0.114	0.109	0.075	0.132	0.194
15	0.074	0.067	0.083	0.160	0.092	0.104	0.114	0.192	0.131	0.175	0.165	0.327
20	0.081	0.103	0.108	0.194	0.092	0.131	0.127	0.265	0.174	0.231	0.215	0.428
25	0.083	0.117	0.104	0.221	0.137	0.177	0.168	0.368	0.176	0.282	0.242	0.536
30	0.103	0.133	0.127	0.308	0.133	0.181	0.164	0.428	0.214	0.342	0.292	0.623
35	0.107	0.165	0.155	0.368	0.137	0.208	0.194	0.480	0.231	0.373	0.346	0.722
n	$\mu_d = 0.25$											
	$r_{x,y} = 0.25$				$r_{x,y} = 0.50$				$r_{x,y} = 0.75$			
	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt
10	0.097	0.069	0.113	0.184	0.134	0.108	0.164	0.229	0.184	0.164	0.217	0.348

15	0.142	0.143	0.157	0.290	0.170	0.198	0.207	0.367	0.287	0.347	0.353	0.584
20	0.129	0.192	0.176	0.380	0.189	0.251	0.238	0.499	0.350	0.482	0.456	0.758
25	0.186	0.235	0.232	0.485	0.232	0.341	0.303	0.631	0.398	0.587	0.509	0.851
30	0.210	0.285	0.271	0.587	0.280	0.395	0.369	0.736	0.421	0.622	0.575	0.905
35	0.234	0.338	0.312	0.678	0.289	0.435	0.377	0.791	0.514	0.709	0.675	0.964
n	$\mu_d = 0.35$											
	$r_{x,y} = 0.25$				$r_{x,y} = 0.50$				$r_{x,y} = 0.75$			
	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt
10	0.150	0.106	0.165	0.260	0.212	0.177	0.235	0.329	0.326	0.265	0.352	0.531
15	0.206	0.211	0.228	0.435	0.279	0.314	0.310	0.573	0.447	0.521	0.525	0.766
20	0.263	0.313	0.327	0.573	0.363	0.447	0.456	0.744	0.538	0.656	0.645	0.896
25	0.324	0.419	0.420	0.713	0.413	0.535	0.507	0.854	0.642	0.774	0.754	0.965
30	0.333	0.450	0.438	0.795	0.454	0.604	0.570	0.910	0.677	0.846	0.824	0.990
35	0.408	0.518	0.528	0.877	0.526	0.705	0.670	0.943	0.785	0.909	0.900	0.998
n	$\mu_d = 0.45$											
	$r_{x,y} = 0.25$				$r_{x,y} = 0.50$				$r_{x,y} = 0.75$			
	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt
10	0.188	0.133	0.219	0.343	0.279	0.225	0.300	0.467	0.481	0.383	0.504	0.676
15	0.293	0.319	0.348	0.579	0.408	0.446	0.494	0.736	0.627	0.639	0.691	0.888
20	0.373	0.437	0.455	0.733	0.504	0.605	0.606	0.870	0.717	0.802	0.801	0.970
25	0.446	0.541	0.541	0.855	0.599	0.731	0.707	0.961	0.801	0.894	0.895	0.992
30	0.503	0.625	0.605	0.934	0.647	0.774	0.777	0.975	0.867	0.950	0.949	1.000
35	0.600	0.690	0.704	0.962	0.728	0.837	0.836	0.984	0.928	0.977	0.980	1.000

Table 2.4.1: Estimated power of the test for paired samples generated from$X \sim G(,) + \mu_d, Y \sim G(,)$, where $G(,)$ is a gamma distribution with mean 1 and skewness=1 with specified paired population correlation $r_{x,y}$ and $\mu_d \neq 0$

n	Skewness=1, $\mu_d = 0.15$											
	$r_{x,y} = 0.25$				$r_{x,y} = 0.50$				$r_{x,y} = 0.75$			
	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt
10	0.089	0.049	0.091	0.096	0.140	0.079	0.154	0.167	0.253	0.131	0.258	0.281
15	0.152	0.100	0.151	0.173	0.200	0.148	0.191	0.234	0.347	0.269	0.354	0.437
20	0.186	0.130	0.182	0.232	0.267	0.187	0.262	0.327	0.451	0.383	0.471	0.570
25	0.206	0.178	0.208	0.269	0.310	0.250	0.315	0.407	0.532	0.450	0.550	0.681
30	0.258	0.191	0.267	0.340	0.336	0.270	0.362	0.466	0.626	0.517	0.644	0.754
35	0.283	0.224	0.290	0.382	0.402	0.298	0.423	0.528	0.634	0.557	0.689	0.801
n	Skewness=1, $\mu_d = 0.25$											
	$r_{x,y} = 0.25$				$r_{x,y} = 0.50$				$r_{x,y} = 0.75$			
	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt
10	0.237	0.113	0.237	0.276	0.315	0.164	0.319	0.380	0.539	0.334	0.529	0.594
15	0.325	0.223	0.321	0.397	0.433	0.327	0.434	0.535	0.709	0.586	0.700	0.810
20	0.450	0.350	0.454	0.555	0.550	0.456	0.556	0.666	0.827	0.735	0.836	0.924
25	0.474	0.412	0.491	0.597	0.672	0.554	0.677	0.794	0.905	0.823	0.911	0.969
30	0.597	0.474	0.605	0.732	0.718	0.626	0.738	0.845	0.956	0.906	0.964	0.991
35	0.654	0.520	0.661	0.780	0.802	0.720	0.821	0.921	0.972	0.939	0.981	0.992
n	Skewness=1, $\mu_d = 0.35$											
	$r_{x,y} = 0.25$				$r_{x,y} = 0.50$				$r_{x,y} = 0.75$			
	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt
10	0.403	0.214	0.390	0.451	0.517	0.294	0.511	0.596	0.797	0.558	0.776	0.856
15	0.551	0.411	0.554	0.647	0.734	0.560	0.733	0.839	0.934	0.832	0.933	0.979
20	0.665	0.552	0.675	0.787	0.825	0.695	0.829	0.904	0.973	0.944	0.977	0.995
25	0.787	0.656	0.812	0.897	0.903	0.843	0.915	0.968	0.994	0.975	0.997	1.000
30	0.854	0.744	0.858	0.937	0.954	0.881	0.963	0.988	1.000	0.988	0.999	1.000
35	0.901	0.796	0.908	0.964	0.975	0.930	0.981	0.997	1.000	0.998	1.000	1.000
n	Skewness=1, $\mu_d = 0.45$											
	$r_{x,y} = 0.25$				$r_{x,y} = 0.50$				$r_{x,y} = 0.75$			
	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt
10	0.575	0.356	0.573	0.647	0.736	0.518	0.730	0.786	0.943	0.778	0.932	0.976
15	0.764	0.618	0.764	0.838	0.896	0.772	0.890	0.952	0.985	0.962	0.991	0.999
20	0.883	0.756	0.887	0.947	0.964	0.901	0.971	0.989	0.999	0.992	0.999	1.000
25	0.939	0.871	0.939	0.983	0.981	0.954	0.985	0.995	1.000	0.998	1.000	1.000
30	0.981	0.915	0.980	0.996	0.993	0.975	0.997	0.999	1.000	0.999	1.000	1.000
35	0.986	0.962	0.990	0.997	0.999	0.992	1.000	1.000	1.000	1.000	1.000	1.000

Table 2.4.2: Estimated power of the test for paired samples generated from $X \sim G(,) + \mu_d, Y \sim G(,)$, where $G(,)$ is a gamma distribution with mean 1 and skewness=2 with specified paired population correlation $r_{x,y}$ and $\mu_d \neq 0$

n	Skewness=2, $\mu_d = 0.15$											
	$r_{x,y} = 0.25$				$r_{x,y} = 0.50$				$r_{x,y} = 0.75$			
	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt
10	0.064	0.040	0.082	0.089	0.070	0.047	0.089	0.125	0.117	0.097	0.149	0.224
15	0.086	0.065	0.093	0.138	0.086	0.108	0.108	0.223	0.127	0.172	0.178	0.295
20	0.091	0.097	0.107	0.191	0.094	0.118	0.114	0.246	0.156	0.224	0.214	0.464
25	0.094	0.105	0.103	0.250	0.102	0.156	0.132	0.307	0.180	0.306	0.251	0.552
30	0.097	0.131	0.130	0.307	0.129	0.194	0.172	0.426	0.230	0.347	0.316	0.663
35	0.118	0.137	0.140	0.362	0.159	0.215	0.195	0.480	0.235	0.381	0.347	0.719
n	Skewness=2, $\mu_d = 0.25$											
	$r_{x,y} = 0.25$				$r_{x,y} = 0.50$				$r_{x,y} = 0.75$			
	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt
10	0.089	0.063	0.102	0.162	0.102	0.092	0.138	0.231	0.206	0.178	0.249	0.361
15	0.122	0.151	0.165	0.309	0.176	0.220	0.237	0.389	0.281	0.325	0.338	0.576
20	0.136	0.188	0.179	0.356	0.226	0.283	0.279	0.516	0.336	0.487	0.423	0.758
25	0.177	0.237	0.232	0.480	0.237	0.365	0.309	0.649	0.399	0.542	0.520	0.848
30	0.195	0.286	0.261	0.581	0.281	0.418	0.378	0.725	0.469	0.666	0.620	0.928
35	0.228	0.308	0.285	0.634	0.321	0.468	0.426	0.804	0.528	0.708	0.682	0.951
n	Skewness=2, $\mu_d = 0.35$											
	$r_{x,y} = 0.25$				$r_{x,y} = 0.50$				$r_{x,y} = 0.75$			
	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt
10	0.157	0.118	0.195	0.276	0.204	0.178	0.244	0.339	0.351	0.302	0.372	0.552
15	0.190	0.219	0.230	0.437	0.270	0.328	0.314	0.576	0.424	0.501	0.503	0.747
20	0.238	0.310	0.313	0.567	0.341	0.448	0.424	0.736	0.521	0.634	0.636	0.903
25	0.306	0.405	0.378	0.688	0.390	0.552	0.513	0.849	0.655	0.781	0.774	0.968
30	0.340	0.463	0.441	0.809	0.460	0.642	0.591	0.914	0.703	0.849	0.840	0.994
35	0.402	0.535	0.539	0.851	0.519	0.662	0.656	0.949	0.780	0.896	0.884	0.996
n	Skewness=2, $\mu_d = 0.45$											
	$r_{x,y} = 0.25$				$r_{x,y} = 0.50$				$r_{x,y} = 0.75$			
	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt
10	0.230	0.160	0.254	0.363	0.291	0.231	0.324	0.462	0.492	0.367	0.504	0.660
15	0.306	0.339	0.366	0.606	0.393	0.414	0.445	0.717	0.645	0.669	0.708	0.892
20	0.395	0.467	0.470	0.749	0.514	0.644	0.613	0.890	0.718	0.810	0.809	0.967
25	0.448	0.552	0.543	0.838	0.580	0.682	0.691	0.944	0.804	0.890	0.887	0.998
30	0.514	0.628	0.632	0.917	0.668	0.798	0.785	0.975	0.876	0.946	0.949	1.000
35	0.570	0.692	0.703	0.954	0.702	0.830	0.828	0.987	0.919	0.969	0.971	1.000

Table 2.4.3: Estimated power of the test for paired samples generated from $X \sim G(,) + \mu_d, Y \sim G(,)$, where $G(,)$ is a gamma distribution with mean 1 and skewness=4 with specified paired population correlation $r_{x,y}$ and $\mu_d \neq 0$

n	Skewness=4, $\mu_d = 0.15$											
	$r_{x,y} = 0.25$				$r_{x,y} = 0.50$				$r_{x,y} = 0.75$			
	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt
10	0.035	0.108	0.100	0.255	0.051	0.110	0.116	0.268	0.071	0.207	0.181	0.376
15	0.054	0.217	0.122	0.471	0.051	0.252	0.145	0.539	0.064	0.366	0.199	0.667
20	0.048	0.276	0.148	0.686	0.054	0.384	0.207	0.738	0.075	0.522	0.253	0.854
25	0.045	0.354	0.164	0.771	0.047	0.485	0.210	0.880	0.090	0.619	0.323	0.928
30	0.054	0.402	0.185	0.861	0.076	0.541	0.258	0.929	0.099	0.700	0.390	0.976
35	0.061	0.447	0.191	0.930	0.083	0.601	0.278	0.966	0.095	0.745	0.396	0.997
n	Skewness=4, $\mu_d = 0.25$											
	$r_{x,y} = 0.25$				$r_{x,y} = 0.50$				$r_{x,y} = 0.75$			
	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt
10	0.050	0.119	0.130	0.325	0.061	0.201	0.164	0.433	0.117	0.260	0.235	0.477
15	0.067	0.271	0.164	0.623	0.083	0.359	0.225	0.691	0.132	0.486	0.318	0.795
20	0.074	0.414	0.232	0.811	0.095	0.481	0.290	0.866	0.131	0.663	0.410	0.939
25	0.088	0.511	0.255	0.920	0.106	0.617	0.345	0.943	0.169	0.788	0.486	0.986
30	0.101	0.578	0.319	0.954	0.114	0.712	0.383	0.979	0.186	0.829	0.569	0.991
35	0.101	0.630	0.319	0.976	0.120	0.743	0.436	0.995	0.185	0.884	0.611	0.999
n	Skewness=4, $\mu_d = 0.35$											
	$r_{x,y} = 0.25$				$r_{x,y} = 0.50$				$r_{x,y} = 0.75$			
	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt
10	0.089	0.183	0.184	0.453	0.119	0.247	0.223	0.498	0.184	0.335	0.283	0.581
15	0.101	0.391	0.277	0.724	0.122	0.453	0.297	0.781	0.185	0.580	0.396	0.865
20	0.117	0.507	0.300	0.891	0.145	0.605	0.386	0.921	0.262	0.769	0.560	0.965
25	0.152	0.619	0.382	0.959	0.159	0.719	0.456	0.982	0.276	0.851	0.624	0.994

30	0.131	0.674	0.413	0.982	0.203	0.794	0.546	0.991	0.291	0.913	0.730	0.996
35	0.161	0.765	0.480	0.996	0.188	0.863	0.574	0.998	0.293	0.951	0.772	1.000
n	Skewness=4, $\mu_d = 0.45$											
	$r_{x,y} = 0.25$				$r_{x,y} = 0.50$				$r_{x,y} = 0.75$			
	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt
10	0.118	0.236	0.209	0.502	0.170	0.316	0.272	0.573	0.237	0.399	0.337	0.645
15	0.172	0.460	0.330	0.793	0.188	0.553	0.380	0.860	0.323	0.681	0.510	0.906
20	0.163	0.613	0.399	0.938	0.228	0.713	0.517	0.958	0.337	0.813	0.624	0.983
25	0.190	0.721	0.484	0.985	0.237	0.802	0.576	0.992	0.375	0.910	0.706	0.998
30	0.222	0.788	0.559	0.992	0.253	0.873	0.654	0.995	0.415	0.952	0.808	1.000
35	0.256	0.869	0.627	0.999	0.292	0.904	0.701	1.000	0.452	0.971	0.848	1.000

Table 2.4.4: Estimated power of the test for paired samples generated from $X \sim G(,) + \mu_d, Y \sim G(,)$, where $G(,)$ is a gamma distribution with mean 1 and skewness=8 with specified paired population correlation $r_{x,y}$ and $\mu_d \neq 0$

n	Skewness=8, $\mu_d = 0.15$											
	$r_{x,y} = 0.25$				$r_{x,y} = 0.50$				$r_{x,y} = 0.75$			
	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt
10	0.068	0.426	0.256	0.737	0.086	0.465	0.279	0.761	0.130	0.594	0.345	0.781
15	0.046	0.713	0.347	0.950	0.074	0.777	0.441	0.952	0.073	0.835	0.481	0.956
20	0.033	0.859	0.479	0.992	0.038	0.888	0.504	0.997	0.064	0.925	0.607	0.996
25	0.047	0.927	0.587	1.000	0.046	0.956	0.644	1.000	0.049	0.975	0.700	1.000
30	0.031	0.958	0.641	1.000	0.042	0.975	0.725	1.000	0.047	0.990	0.802	1.000
35	0.039	0.978	0.690	1.000	0.043	0.993	0.774	1.000	0.059	0.998	0.858	1.000
n	Skewness=8, $\mu_d = 0.25$											
	$r_{x,y} = 0.25$				$r_{x,y} = 0.50$				$r_{x,y} = 0.75$			
	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt
10	0.087	0.480	0.304	0.784	0.128	0.516	0.312	0.784	0.184	0.625	0.393	0.804
15	0.068	0.765	0.440	0.970	0.088	0.823	0.470	0.969	0.145	0.866	0.536	0.976
20	0.074	0.917	0.539	0.995	0.087	0.930	0.606	0.998	0.138	0.957	0.694	0.999
25	0.051	0.961	0.638	1.000	0.082	0.977	0.737	1.000	0.118	0.988	0.781	1.000
30	0.063	0.967	0.708	1.000	0.064	0.985	0.780	1.000	0.098	0.996	0.858	1.000
35	0.072	0.990	0.788	1.000	0.082	0.993	0.857	1.000	0.088	0.999	0.915	1.000
n	Skewness=8, $\mu_d = 0.35$											
	$r_{x,y} = 0.25$				$r_{x,y} = 0.50$				$r_{x,y} = 0.75$			
	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt
10	0.113	0.496	0.290	0.791	0.178	0.574	0.374	0.797	0.229	0.639	0.395	0.838
15	0.105	0.781	0.473	0.968	0.150	0.859	0.514	0.983	0.212	0.898	0.616	0.985
20	0.105	0.918	0.621	0.998	0.134	0.947	0.670	0.998	0.192	0.974	0.735	0.998
25	0.110	0.971	0.725	0.999	0.130	0.983	0.772	1.000	0.176	0.993	0.845	1.000
30	0.096	0.986	0.807	1.000	0.119	0.993	0.826	1.000	0.190	0.997	0.913	1.000
35	0.104	0.999	0.872	1.000	0.107	0.999	0.896	1.000	0.166	1.000	0.948	1.000
n	Skewness=8, $\mu_d = 0.45$											
	$r_{x,y} = 0.25$				$r_{x,y} = 0.50$				$r_{x,y} = 0.75$			
	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt	t	$Sign$	Wsr	Ltt
10	0.186	0.578	0.383	0.824	0.221	0.617	0.383	0.812	0.274	0.683	0.437	0.840
15	0.171	0.840	0.519	0.976	0.191	0.865	0.539	0.984	0.241	0.897	0.617	0.979
20	0.129	0.937	0.671	0.998	0.174	0.959	0.706	0.999	0.249	0.980	0.802	0.999
25	0.142	0.975	0.759	1.000	0.159	0.987	0.824	1.000	0.245	0.991	0.891	1.000
30	0.122	0.990	0.845	1.000	0.150	0.997	0.871	1.000	0.248	0.999	0.932	1.000
35	0.129	0.997	0.885	1.000	0.173	1.000	0.934	1.000	0.264	1.000	0.964	1.000

IV. Results Discussions

For the simulated Type I error rates of Tables 1.1-1.4, the correlation between the paired populations are considered to be 0.25, 0.50 and 0.75. In addition, for gamma distribution in Table 1.4, the skewness in the paired populations are considered arbitrarily to be 1, 2, 4 and 8.

From the results in Tables 1.1-1.4, it follows that Sign test underestimates Type I error rates when sample size n =10 with estimated Type I error rates varying between 0.017 to 0.023 for uniform distribution (Table 1.1), between 0.018 to 0.026 for normal distribution (Table 1.2) and between 0.025 to 0.029 (Table 1.3) at 5% level of significance. For increasing sample sizes from n =15 to 35, all tests provide reasonable control over Type I error rates (Tables 1.1-1.3) except for the gamma distribution (Table 1.4), where paired t-test provides decreasing control over Type I error rates with increasing skewness. Thus, skewness seems to affect the Type I error rate of paired t-test, whereas the problem of paired t-test is taken care of by the log-transformed paired t-test (Ltt). This may be due to the fact that log transformation reduces the skewness of the distribution, and hence improves the control over the Type I error rates.

Overall (Tables 1.1-1.4), estimated Type I error rates for paired t-test (t) varies from 0.001 to 0.06, for Sign test ($Sign$) varies from 0.016 to 0.053, for Wilcoxon Signed Rank test (Wsr) varies from 0.031 to 0.063 and for log-transformed paired t-test (Ltt) varies from 0.033 to 0.064. As far as Type I error rate is concerned, the level of correlations do not seem to have any definite trend on the four underlying tests. Given the facts of the study, Wilcoxon Signed Rank test and log-transformed paired t-test provide better Type I error rates than the paired t-test or Sign test at 5% level of significance for skewed distributions (e.g., the gamma distribution considered). For paired samples from uniform, normal and exponential distributions, all four tests seem to have comparable control over the Type I error rates for $n \geq 15$ in most of the cases.

For the simulated power of Tables 2.1-2.3 and Tables 2.4.1-2.4.4, the mean difference between the paired populations are considered to 0.15, 0.25, 0.35 and 0.45, arbitrarily, so that a meaningful comparison of power of the four underlying tests can be made. In addition, for each choice of the mean difference, the correlation between the paired populations are considered to be 0.25, 0.50 and 0.75; for gamma distributions (Tables 2.4.1-2.4.4) the skewness in the paired populations are considered arbitrarily to be 1, 2, 4 and 8.

From the results in Tables 2.1-2.4.4, it follows that the power of all four tests increases towards the nominal level of 0.95 for (i) increasing sample sizes (with a very few exceptions in t-test), (ii) increasing mean differences, (iii) increasing correlation values for a given level of skewness value for gamma distributions (Tables 2.4.1-2.4.4). However, as skewness increases between 1 and 2, the power of all four tests decreases, and as skewness increases between 4 and 8, power of all tests increases except the paired t-test. Indeed, the power of paired t-test always decreases as skewness increases between 1 and 8. It is worth mentioning that the rate of decease or increase in power differ by four tests. In this study, it turns out that for gamma distributions with skewness between 1 and 8, the log-transformed paired t-test (Ltt) performs the best, with the Wilcoxon signed rank test (Wsr) performing the second best.

V. Concluding remarks

The power of the four underlying tests, namely, paired t-test, Sign test, Wilcoxon Signed rank test, and the log-transformed paired t-test, are sensitive to levels of correlation, skewness, mean difference and sample size. In all the skewed cases studied using gamma distribution with specified skewness between 1 and 8, the log-transformed paired t-test outperforms other tests in estimating power of the test. Thepower of the Wilcoxon signed rank test and Sign test is mostly comparable with skewness between 1 and 8. The similar conclusion is applicable for underlying tests in the estimation of the power of testfor paired samples from paired exponential distributions. The power of the test increases as the level of correlation increases for a given mean difference and skewness (for gamma distribution), with log-transformed test performing the best followed by Wilcoxon signed rank test or Sign test for gamma distribution with varying skewness, and exponential distribution. Overall, correlation and skewness affect the power of all tests significantly, with log-transformed test performing the best in power consideration. Regarding the estimated level of significance or Type I error rate, correlation does not seem to have any definite effect on the four underlying tests except for the Sign test, which provides significant underestimation for $n = 10$ for normal, uniform and exponential distributions. For skewed distributionsstudied using the gamma distributions with varying level of skewness between 1 and 8, t-test exhibits severe problem in controlling Type I error rate for increasing skewness. Given the facts of the study, the log-transformed paired t-test could be a better choice than the Sign or the Wilcoxon Signed rank test for tests of hypothesis regarding paired populations.

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Appendix

R Code for generating paired sample with specified variance covariance matrix (**varcovmat**).

The program below generates a paired sample of size $n = 20$from paired population with correlation between two populations to be 0.50 from exponential, gamma and uniform distributions using the bivariate normal distribution.

```
require(MASS);
mu<- rep(0,2);
varcovmat<- matrix(.50, nrow=2, ncol=2) + diag(2)*.50;
rawvars<- mvrnorm(n=20, mu=mu, Sigma= varcovmat);
pvars<- pnorm(rawvars)

# using inverse transformation to get correlated exponential paired samples from Exp(1);
expvars<- qexp(pvars, 1)
cor(expvars[,1], expvars[, 2])

# using inverse transformation to get correlated gamma paired samples from G(2,1);
gamvars<-qgamma(pvars, 2,1)
cor(gamvars[,1], gamvars[,2])

# using inverse transformation to get correlated uniform paired samples from U(0,2);
univars<-qunif(pvars, min = 0, max = 2)
cor(univars[,1], univars[,2])
```

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