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On The Buckling Modes and Static Buckling Of a Column Lying On Nonlinear Elastic Foundations but With One End Simply-Supported While the Other End Is Clamped

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Abstract:*The investigation seeks to determine the buckling modes and the static buckling load of a finite imperfect column lying on a cubic nonlinear elastic foundation but with one end simply-supported while the other end is clamped. Perturbation and asymptotic procedures are employed to obtain the asymptotic results. The formulation contains a small non-dimensional parameter upon which asymptotic expansions are initiated. The results which are strictly asymptotic are valid in the limit as the small non-dimensional parameter becomes increasingly small relative to unity.*

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I. Introduction

Investigations of stability of finite columns lying on nonlinear elastic foundations but with certain prescribed boundary conditions have been a common pre-occupation of civil and structural engineers as well as Applied Mathematicians for a long time now. Such early studies include investigations by Amazigo et al. [1], Amazigo and Frazer [2], Artem and Aydin [3], Ette [4]andJabareen and Izhak [5].Ziolkoloski and Imieowski [6] investigated the buckling and post-buckling of Prismatic aluminium columns submitted to a series of compressive loads, while Huang and Li [7] studied analytic approach for exactly determining critical loads of buckling of non-uniform columns. In the same token, Gross et al. [8] studied buckling of bars, while Gabr et al. [9] investigated the effect of boundary conditions on buckling of friction piles. Equally of note is the investigation by Huang and Luo [10], who investigated a simple method to determine the critical buckling loads for axially inhomogeneous beams with elastic restraint. A similar investigation was undertaken by Magnucki and Mackiewicz [11], who studied elastic buckling of axially compressed cylinder panel with three edges simply supported and one edge free, while Kruzelecki and Ortwein [12] studied optimal design of clamped columns for stability under combined compression and torsion. Our interest here is the investigation by Lee et al. [13], who investigated numerical methods for determining strongest cantilever beam with constant volume whileKripka and Martin [14] studied cold-formed steel channel columns optimization with simulated annealing method. Of equal importance is the investigation by Jatav and Datta [15] while Wang et al. [16] studied refined modelling and free vibration of in-extensional beams on elastic foundations.

In engineering, nonlinear elastic foundations, which may be classified as ‘softening’ or ‘hardening’ provide a simplified model for some complex or complicated nonlinear systems. Such structures include but not limited to columns, shells and plates, among others. If such structures are limited to have finite lengths, then, the usual boundary conditions associated with them at the edges or boundary pointsare, in most cases, simply-supported or clamped boundary conditions. However, the curiosity of investigating the case of two distinct boundary conditions at the ends of a column, in our judgement, deserves some attention. This is the issue we are addressing in this investigation, where simply-supported and clamped boundary conditions are imposed on the opposite ends of the finite column.

In this investigation, we shall apply perturbation and asymptotic techniques as expounded by Bender and Orszag [17]. Such an approach was adopted by Ette and Osuji [18] and Ette and Udo-Akpan [19].

II. Formulation of the Problem

The dimensional differential equation of motion satisfied by the deflection $W(X)$ of a finite column lying on a nonlinear (cubic) elastic foundation trapped by a static load P, but with simply-supported boundary condition at $X = 0$, and clamped boundary condition at $X = \pi$, is

$$EIW_{,XXXX} + 2PW_{,XX} + k_1 W - \alpha k_3 W^3 = -2P\bar{W}_{,XX}, \quad 0 < X < \pi \tag{2.1}$$

$$W(0) = W_{,XX}(0) = 0, \tag{2.2}$$

$$W(\pi) = W_{,X}(\pi) = 0, \tag{2.3}$$

where$X$ is the spatial variable and a comma denotes total derivative. The product EI is the bending stiffness, where E and I are the Young's modulus and moment of inertia respectively. The nonlinear (cubic) elastic foundation exerts a force per unit length of $k_1W - \alpha k_3 W^3$ on the column, where $k_1, k_3 > 0$ are constants and α is the imperfection sensitivity parameter which is such that for $\alpha < 0$, the structure is said to be 'hardening' whereas for $\alpha > 0$, the structure is said to be 'softening'. In this formulation, we have excluded all nonlinearities higher than cubic and also neglected all nonlinear derivatives of the deflection W. Similarly, we have assumed that the imperfection $\overline{W}$ is twice-differentiable and stress-free.
We now introduce the following non-dimensional quantities

$x = \left(\frac{k_1}{EI}\right)^{\frac{1}{4}} X, w = \left(\frac{k_3}{k_1}\right)^{\frac{1}{2}} W, \epsilon\overline{w} = \left(\frac{k_3}{k_1}\right)^{\frac{1}{2}} \overline{W},\ 0 < \epsilon \ll 1, \lambda = \frac{P}{2(EIk_1)^{\frac{1}{2}}},\ 0 < \lambda < 1$

On introducing these non-dimensional quantities into (2.1) – (2.3) and simplifying, we get

$$w_{,xxxx} + 2\lambda w_{,xx} + w - \alpha w^3 = -2\lambda\epsilon\overline{w}_{,xx}, \quad 0 < x < \pi \tag{2.4}$$

$$w(0) = w_{,xx}(0) = 0, \tag{2.5}$$

$$w(\pi) = w_{,x}(\pi) = 0, \tag{2.6}$$

1. PERTURBATION AND ASYMPTOTIC SCHEME FOR THE SOLUTION OF THE PROBLEM

We now solve the problem (2.4) – (2.6) by first assuming the asymptotic series

$w(x) = \sum_{i=0}^{\infty} w^{(i)}\epsilon^i$(3.1)

On substituting (3.1) into (2.4) – (2.6) and equating the coefficients of orders of ϵ, we get

$$O(\epsilon): \quad Lw^{(1)} \equiv w^{(1)}_{,xxxx} + 2\lambda w^{(1)}_{,xx} + w^{(1)} = -2\lambda\overline{w}_{,xx} \tag{3.2}$$

$$O(\epsilon^2): \quad Lw^{(2)} = 0 \tag{3.3}$$

$$O(\epsilon^3): \quad Lw^{(3)} = \alpha\left(w^{(1)}\right)^3 \tag{3.4}$$

etc.

We let

$$\overline{w} = \bar{a}_m \sin mx, \quad 0 < x < \pi \tag{3.5}$$

A convenient form of the deflection $w^{(i)}(x)$ is

$$w^{(i)}(x) = \sum_{n=1}^{\infty} 4w_n^{(i)} \sin^3 nx = \sum_{n=1}^{\infty} [3\sin nx - \sin 3nx] w_n^{(i)} \tag{3.6}$$

On substituting (3.5) into (3.2) and simplifying, we get, for $n = m$

$$w^{(1)}(x) = (3\sin mx - \sin 3mx) w_m^{(1)} \tag{3.7a}$$

where

$$w_m^{(1)} = \frac{2m^2\lambda\bar{a}_m}{3(m^4 - 2m^2\lambda + 1)} \tag{3.7b}$$

Next, (3.6) is further substituted into (3.3) and simplified to get

$$w^{(2)}(x) = 0 \tag{3.8}$$

We now substitute (3.7a,b) into (3.4) and simplify to get

$$Lw^{(3)} = \alpha\left(w_m^{(1)}\right)^3 \left[\frac{27}{4}(3\sin mx - \sin 3mx) - \frac{27}{4}(2\sin 3mx - \sin mx - \sin 5mx) + \frac{9}{4}(2\sin mx + \sin 5mx - \sin 7mx) - \frac{1}{4}(3\sin 3mx - \sin 9mx)\right] \tag{3.9}$$

We remark that to get (3.9), we have assumed

$$w^{(3)}(x) = \sum_{n=1}^{\infty} (3\sin nx - \sin 3nx)\, w_n^{(3)} \tag{3.10}$$

Thus, for $n = m$ in (3.9), we simplify to get the buckling mode as

$$(3\sin mx - \sin 3mx) w_m^{(3)} \tag{3.11a}$$

where

$$w_m^{(3)} = \frac{51\alpha\left(w_m^{(1)}\right)^3}{4(m^4 - 2m^2\lambda + 1)} \tag{3.11b}$$

For $n = 3m$ in (3.9), we get the buckling mode as

$$(3\sin 3mx - \sin 9mx) w_{3m}^{(3)} \tag{3.12a}$$

where

$$w_{3m}^{(3)} = \frac{7\alpha\left(w_m^{(1)}\right)^3}{(81m^4 - 18m^2\lambda + 1)} \tag{3.12b}$$

The term $-\sin 3nx$ gives a buckling mode when $n = m$ and this is given as

$$(3\sin 3mx - \sin 9mx) w_{3m_1}^{(3)} \tag{3.13a}$$

where

$$w_{3m_1}^{(3)} = \frac{-7\alpha\left(w_m^{(1)}\right)^3}{(m^4-2m^2\lambda+1)} \tag{3.13b}$$

For $n = 5m$ in (3.9), we get the buckling mode as

$$(3\sin 5mx - \sin 15mx)w_{5m}^{(3)} \tag{3.14a}$$

where

$$w_{5m}^{(3)} = \frac{3\alpha\left(w_m^{(1)}\right)^3}{(625m^4-50m^2\lambda+1)} \tag{3.12b}$$

For $n = 7m$ in (3.9), we get the buckling mode as

$$(3\sin 7mx - \sin 21mx)w_{7m}^{(3)} \tag{3.15a}$$

where

$$w_{7m}^{(3)} = \frac{-3\alpha\left(w_m^{(1)}\right)^3}{4(2401m^4-98m^2\lambda+1)} \tag{3.15b}$$

For $n = 9m$ in (3.9), we get the buckling mode as

$$(3\sin 9mx - \sin 27mx)w_{9m}^{(3)} \tag{3.16a}$$

where

$$w_{9m}^{(3)} = \frac{-\alpha\left(w_m^{(1)}\right)^3}{12(6561m^4-162m^2\lambda+1)} \tag{3.16b}$$

From the term $-\sin 3nx$in (3.6), when $n = 3m$, we get buckling modeas

$$(3\sin 9mx - \sin 27mx)w_{9m_1}^{(3)} \tag{3.17a}$$

where

$$w_{9m_1}^{(3)} = \frac{\alpha\left(w_m^{(1)}\right)^3}{12(81m^4-18m^2\lambda+1)} \tag{3.17b}$$

As a summary, we can write the deflection so far as

$$w(x) = \epsilon(3\sin mx - \sin 3mx)w_m^{(1)} + \epsilon^3\big[(3\sin mx - \sin 3mx)w_m^{(3)}$$
$$+(3\sin 3mx - \sin 9mx)\left(w_m^{(3)} + w_{m_1}^{(3)}\right) + (3\sin 5mx - \sin 15mx)w_{5m}^{(3)}$$
$$+(3\sin 7mx - \sin 21mx)w_{7m}^{(3)} + (3\sin 9mx - \sin 27mx)\left(w_{9m}^{(3)} + w_{9m_1}^{(3)}\right)\big] + \cdots \tag{3.18}$$

So far, we have obtained the deflection as in (3.18). We shall now evaluate the static buckling loadλ_S which is defined as the largest load parameter for the deflection to remain bounded. As in Amazigo and Ette [20], this is obtained from maximization

$$\frac{d\lambda}{dw} = 0 \tag{3.19}$$

However, we shall obtained λ_S in two separate levels of approximation, first by taking the deflection (3.18) in its simplest mode form, i.e. taking $w(x)$ as

$$w(x) = \epsilon(3\sin mx - \sin 3mx)w_m^{(1)} + \epsilon^3(3\sin mx - \sin 3mx)w_m^{(3)} + \cdots \tag{3.20}$$

Next, we shall take (3.18) in its entirety.

Case 1:

We can write (3.20) simply as

$$w = c_1\epsilon + c_3\epsilon^3 + \cdots \tag{3.21}$$

where

$$c_1 = (3\sin mx - \sin 3mx)w_m^{(1)} \tag{3.22a}$$

$$c_3 = (3\sin mx - \sin 3mx)w_m^{(3)} \tag{3.22b}$$

As in Ette and Udo-Akpan [19], the maximization (3.19) is however preceded by a reversal of the series (3.21) to get

$$\epsilon = d_1w + d_3w^3 + \cdots \tag{3.23a}$$

By substitution for w from (3.21) and equating the coefficients of powers of ϵ, we get

$$d_1 = \frac{1}{c_1} d_3 = -\frac{c_3}{{c_1}^3} \tag{3.23b}$$

wherec_i, $i = 1,3, \ldots$ are functions of the load parameter through $w_m^{(1)}$ and $w_m^{(3)}$.

The maximization (3.19) is now easily accomplished through (3.23a) to get

$$d_1 + 3d_3{w_S}^2 = 0 \tag{3.24}$$

wherew_S is the value of w at buckling. Thus, we have

$$w_S = \left(\frac{-d_1}{3d_3}\right)^{\frac{1}{2}} = \frac{1}{3}\left(\frac{{c_1}^2}{c_3}\right) \tag{3.25}$$

where we have taken the positive square root sign.

To determine the static buckling load λ_S we have to evaluate (3.23a) at buckling and get

$$\epsilon = w_S(d_1 + d_3 w_S{}^2) + \cdots \ = \frac{2}{3}\left(\frac{c_1}{3c_3}\right)^{\frac{1}{2}} \tag{3.26}$$

On simplifying (3.26), we get

$$(m^4 - 2m^2\lambda + 1)^{\frac{3}{2}} = \frac{3}{2}\sqrt{17}\alpha^{\frac{1}{2}} m^2 \bar{a}_m \epsilon \lambda_S \tag{3.27a}$$

The case $m = 1$ yields

$$(1 - \lambda_S)^{\frac{3}{2}} = \frac{3}{4}\sqrt{\frac{17}{2}}\alpha^{\frac{1}{2}}\bar{a}_1 \epsilon \lambda_S \tag{3.27b}$$

We can call (3.27b) the dominant result obtained for the case $m = 1$.

Case 2:

In this case we take the full expression in (3.18). Lacking the simplicity of manipulation that characterized case 1, we shall here determine the deflection w, in this case, by evaluating it at a convenient point, namely, the point $x = x_a$, where the deflection w has a maximum. The condition for the maximum, w_a of w is

$$\frac{dw}{dx} = 0 \tag{3.28}$$

Let

$$x_a = x_0 + \epsilon^2 x_2 + \epsilon^3 x_3 + \cdots \tag{3.29}$$

If we differentiate (3.18) by x and evaluate it at x_a by finding a Taylor series expansion of the function of x_a about x_0 and thereafter, equate the resulting equation to zero, in line with (3.28), we get, for terms of order ϵ

$$\cos mx_0 - \cos 3mx_0 = 0 \tag{3.30}$$

By maintaining just the first three terms in the Taylor expansion of the two terms in (3.30) and solving the resultant equation, we get

$$mx_0 = \sqrt{\frac{6}{5}} \tag{3.31}$$

where we have taken only the positive square root sign. The maximum w_a of w in (3.18) is afterward evaluated as

$$w_a = \epsilon e_1 + \epsilon^3 e_3 + \cdots \tag{3.32a}$$

where

$$e_1 = \left\{3\sin\left(\sqrt{\frac{6}{5}}\right) - \sin\left(3\sqrt{\frac{6}{5}}\right)\right\} w_m^{(1)}$$

$$e_3 = \alpha w_m^{(1)^3}\left\{3\sin\left(\sqrt{\frac{6}{5}}\right) - \sin\left(3\sqrt{\frac{6}{5}}\right)\right\}\left[1 + \left(\frac{1}{3\sin\left(\sqrt{\frac{6}{5}}\right) - \sin\left(3\sqrt{\frac{6}{5}}\right)}\right)\left\{\frac{3\sin\left(3\sqrt{\frac{6}{5}}\right) - \sin\left(9\sqrt{\frac{6}{5}}\right)}{625m^4 - 50m^2\lambda + 1}\right.\right.$$

$$+7\left(3\sin\left(3\sqrt{\frac{6}{5}}\right) - \sin\left(9\sqrt{\frac{6}{5}}\right)\right)\left(\frac{1}{81m^4 - 18m^2\lambda + 1} - \frac{1}{m^4 - 2m^2\lambda + 1}\right)$$

$$\left.\left.-3\left(\frac{3\sin\left(7\sqrt{\frac{6}{5}}\right) - \sin\left(21\sqrt{\frac{6}{5}}\right)}{2401m^4 - 98m^2\lambda + 1}\right)\right\}\right] \tag{3.32c}$$

As in (3.19) the condition for buckling in this case is

$$\frac{d\lambda}{dw_a} = 0 \tag{3.33}$$

By first reversing the series (3.32a – c) and invoking the condition (3.33), we determine the static buckling λ_S by the equation

$$\epsilon = \frac{2}{3}\left(\frac{e_1}{3e_3}\right)^{\frac{1}{2}} \tag{3.34}$$

On simplifying (3.34), we get

$$(m^4 - 2m^2\lambda_S + 1)^{\frac{3}{2}} = \frac{3}{2}\sqrt{17}\alpha^{\frac{1}{2}} m^2 \bar{a}_m \epsilon \lambda_S\left[1 + \left(\frac{1}{3\sin\left(\sqrt{\frac{6}{5}}\right) - \sin\left(3\sqrt{\frac{6}{5}}\right)}\right)\left\{\frac{3\sin\left(3\sqrt{\frac{6}{5}}\right) - \sin\left(9\sqrt{\frac{6}{5}}\right)}{625m^4 - 50m^2\lambda_S + 1}\right.\right.$$

$$+7\left(3\sin\left(3\sqrt{\frac{6}{5}}\right) - \sin\left(9\sqrt{\frac{6}{5}}\right)\right)\left(\frac{1}{81m^4 - 18m^2\lambda_S + 1} - \frac{1}{m^4 - 2m^2\lambda_S + 1}\right)$$

$$-3\left(\frac{3\sin\left(7\sqrt{\frac{6}{5}}\right)-\sin\left(21\sqrt{\frac{6}{5}}\right)}{2401m^4-98m^2\lambda_S+1}\right)\Bigg\}\Bigg]^{\frac{1}{2}} \qquad (3.36)$$

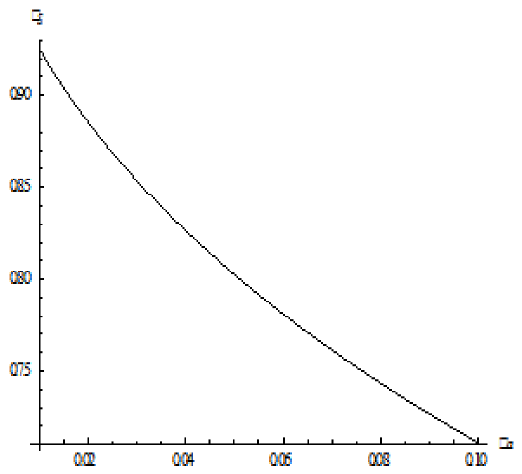
In particular, if we restrict equation (3.35) to the case $m = 1$, we get

$$(1-\lambda_S)^{\frac{3}{2}} = \frac{3}{4}\sqrt{17}\alpha^{\frac{1}{2}}\bar{a}_1\epsilon\lambda_S\left[1+\left(\frac{1}{3\sin\left(\sqrt{\frac{6}{5}}\right)-\sin\left(3\sqrt{\frac{6}{5}}\right)}\right)\left\{\frac{3\sin\left(3\sqrt{\frac{6}{5}}\right)-\sin\left(9\sqrt{\frac{6}{5}}\right)}{2(313-2\lambda_S)}\right.\right.$$

$$\left.\left.+\frac{7}{2}\left(3\sin\left(3\sqrt{\frac{6}{5}}\right)-\sin\left(9\sqrt{\frac{6}{5}}\right)\right)\left(\frac{1}{42-\lambda_S}-\frac{1}{1-\lambda_S}\right)-\frac{3}{2}\left(\frac{3\sin\left(7\sqrt{\frac{6}{5}}\right)-\sin\left(21\sqrt{\frac{6}{5}}\right)}{1201-\lambda_S}\right)\right\}\right]^{\frac{1}{2}} (3.36)$$

III. Analysis of results

We observe that the result (3.27a) is a particular case of (3.35) obtained if we restrict the buckling modes to only the basic simple mode, i.e. without modes with higher waves numbers. The same observation applies to the results (3.27b) and (3.36). These observations are depicted in the graphs below obtained from the results as indicated below each graph. Mathematica® programs are used in generating and plotting of the graphs.



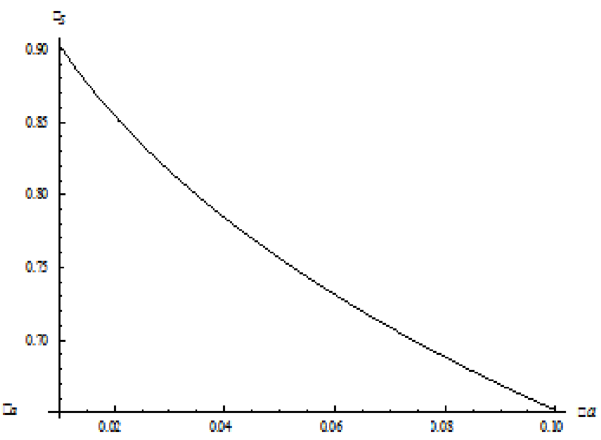


Fig. 1: Graph of λ_S vs $\bar{a}\epsilon$ with m=1 from Eqn(3.27b)Fig. 2: Graph of λ_S vs $\bar{a}\epsilon$ with m=1 from Eqn(3.36)

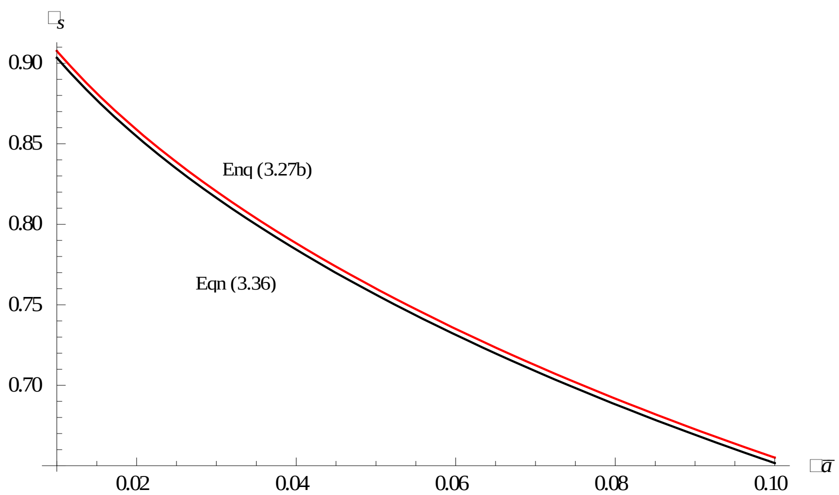


Fig. 3: Plotting of Eqns (3.27b) and (3.36) on the same graph.

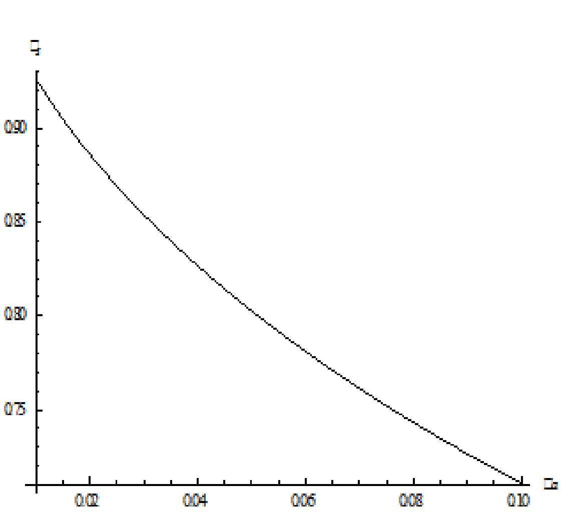


Fig. 4a: Graph of (3.27a) with m=1; same as (3.27b).

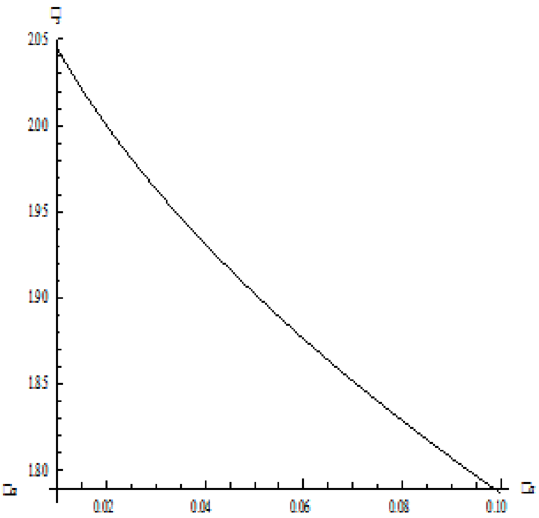


Fig. 4b: Graph of (3.27a) with m=2.

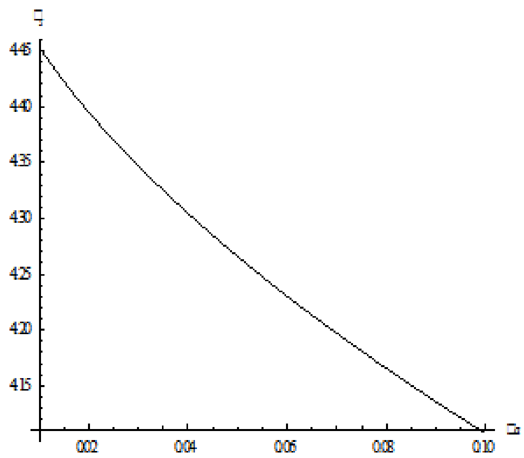


Fig. 4c: Graph of (3.27a) with m=3.

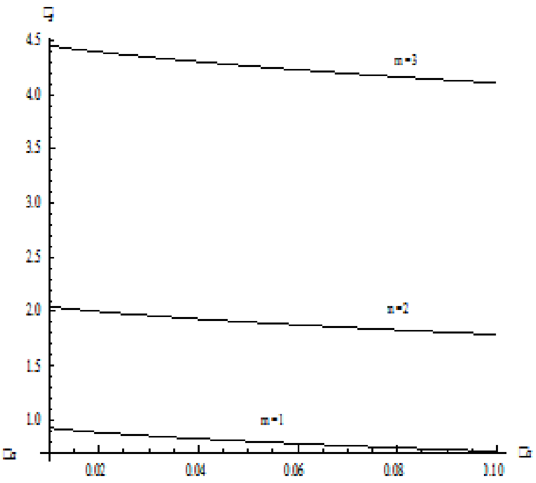


Fig. 4d: Plotting of Eqns (3.27a) with m=1, m=2, m=3
On the same graph.

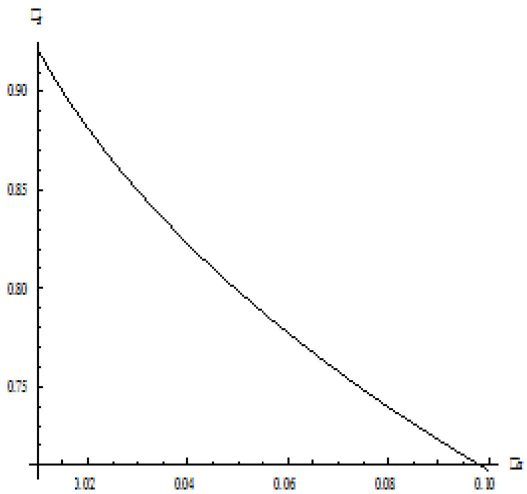


Fig. 5a: Graph of (3.35) with m=1; same as (3.36).

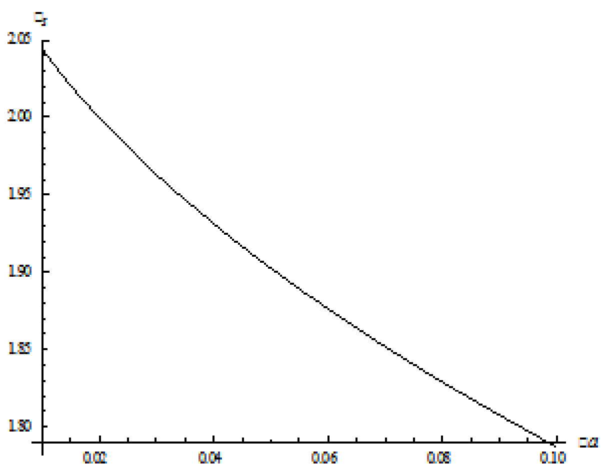


Fig. 5b: Graph of (3.35) with m=2.

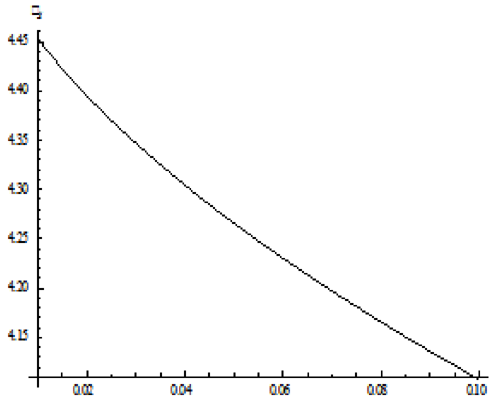


Fig. 5c: Graph of (3.35) with m=3

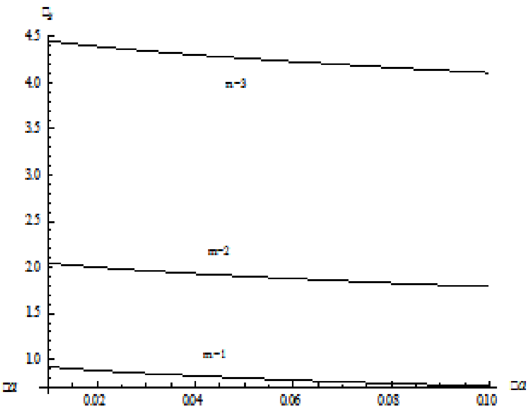


Fig. 5d: Plotting of Eqns (3.35) with m=1, m=2 m=3 Togetheron the same graph.

IV. Conclusion

We have been able to determine the static buckling of a finite imperfect column lying on a nonlinear (cubic) elastic foundation, but with the boundary conditions at both ends different. While one end of the column is simply-supported the other end is clamped. We have been able to obtain a asymptotic results which are valid as the small parameter ϵ becomes increasingly small relative to unity but, of course, nonzero. The results are given in two separate sets of approximations. The earlier results, i.e. (3.27a – b) are seen to be the restricted forms of later results (3.35) and (3.36) respectively.

However, observation shows that, while the end conditions are actually simply-supported at $x = 0$ and clamped at$x = \pi$. The use of (3.36) makes the end conditions to be simply-supported also at $x = \pi$ and not only at$x = 0$. It would be worth the while in our judgement, if we could, develop a scheme where the respective end conditions are restricted to their separate ends without any of them being satisfied at the opposite end.

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