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An Analytical Solution To Riccati Equation Through The Use of Variational Iteration Method.

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Abstract: *In this paper, we derive the variational iteration formula for the analytical solution to Riccati equations.The method provides the solutions in terms of rapidly convergent series and does not required small parameters in equation as the perturbation technique do. Few examples were solved analytically to show the effectiveness and efficiency of the method. It was observed that by carefully chosen a very good Lagrange multiplier, a solution in a closed form is obtained.*

Keywords: *variational iteration method, initial value problems, Riccati equation*

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I. Introduction

This paper considers an analytical solution to the first order initial value problems of the form

$$\frac{dy}{dt} = Q(t)y + R(t)y^2 + P(t) \qquad (1)$$

where Q(t), R(t), P(t) and G(t) are scalar functions. This is a general form of Riccati equation.

A survey of the various existing methods (analytical and otherwise) for solving the ODE (1) is given in many reference textbooks. We shall focus on the use of Variational Iteration Method in this paper. Variational iteration is a relatively new method for solving differential equations that is theoretically accurate with simple concepts and fast convergence. The Variational Iteration Method (VIM) was first developed by Chinese mathematician Ji-Huan He, professor at Donghua University.

The VIM was initially proposed toward the end of the most recent century and completely grew in 2006 and 2007.The method VIM is used to solve effectively, easily, and accurately a large class of non- linear problems with approximations, which converge rapidly to the accurate solutions. VIM was derived from the general Lagrange multiplier method used for solving nonlinear equations in quantum mechanics, which was modified by He [1-4] and Biazar et al. [5] into an iteration method named the Variational Iteration method (VIM).

The variational iteration method (VIM) is a simple and yet powerful method for solving a wide class of nonlinear problems, first envisioned by He [1] . The VIM has successfully been applied to many situations. For instance, Abbasbandy [6] solved one example of the quadratic Riccati differential equation (with constant coefficient) by He's Variational Iteration method by using Adomian's polynomials.

The VIM has also recently been applied for solving nonlinear coagulation problem with mass loss by Abulwafa et al. [7]. Bildik et al. [8] used VIM for solving different types of nonlinear partial differential equations.He [9] coupled the iteration method with the perturbation method to solve the well-known Blasius equation. Soliman [10] applied the VIM to solve the KdV-Burger's and Lax's seventh-order KdV equations. Dehghan and Tatari [11] employed VIM to solve a Fokker-Planck equation. Junfeng [12] introduced Variational Iteration Method to solve two-point boundary value problems.

In this paper, in section two, we discussed the derivation of iteration formula for solving non linear initial value problem of ordinary differential equations and also show the iterative formula for Riccati equations. In section three, we confirm the applicability of the method most especially on the Riccati equations. Section four is about discussion and conclusion.

II. Methodology

2.1 Derivation of Variational Iteration Formula

Let us consider the first order equation equation ordinary differential equation of the form (2). This method, which is a modified general Lagrange's multiplier method [10], has been shown to solve effectively, easily and accurately a large class of nonlinear problems [1-5]. The main feature of the method is that the solution of a mathematical problem with linearization assumption is used as initial approximation or trial function. Then a more highly precise approximation at some special point can be obtained. This approximation converges rapidly to an accurate solution. To illustrate the basic concepts of the VIM, we consider the following nonlinear differential equation:

$$\left(u' + f(u,u')\right) = 0,\ u(0) = a \quad (2)$$

Proof: The VIM employs the correction function

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(s)\left[(u_n')_s + \bar{f}_n(u,u',)\right]ds \quad n \geq 0 \quad (3)$$

Where $\bar{f}_n$ is a restricted variation, $d\bar{f}_n = 0$

To find the value of λ(s),we start by taking the variation with respect to $u_n(x)$, which yields

$$\frac{\partial u_{n+1}}{\partial u_n} = 1 + \frac{\partial}{\partial u_n}\left(\int_0^x \lambda(s)\left[(u_n')_s + \bar{f}_n(u,u',)\right]ds\right), \quad (4)$$

or equivaently,

$$\partial u_{n+1} = \partial u_n + \partial\left(\int_0^x \lambda(s)\left[(u_n')_s + \bar{f}_n(u,u',)\right]ds\right) \quad (5)$$

By appying variation to eq(5) gives

$$\partial u_{n+1} = \partial u_n + \partial\left(\int_0^x \lambda(s)(u_n')_s\, ds\right) \quad (6)$$

Integrating the integral in eq(6) by parts we have

$$\int_0^x \lambda(s)(u_n')_s\, ds = \left[\lambda(x)(u_n)(x) - \lambda(0)(u_n)(0)\right] - \int_0^x \lambda'(s)(u_n)_s\, ds = 0 \quad (7)$$

Repacing the integral in Eq(7) by its value in Eq(6) we obtain

$$\partial u_{n+1} = \partial u_n + \partial\left[\lambda(x)(u_n)(x)\right] - \partial\left(\int_0^x \lambda'(s)(u_n)(s)ds\right) = 0 \quad (8)$$

The last equation is satisfied if the following ` stationary conditions' are satisfied

$$\lambda'(s) = 0$$
$$1 + \lambda(s)\ \Big]_{s=x} = 0 \quad (9)$$

By solving (9) for λ(s) we have λ(s)=-1.
Substituting this value of λ(s) into Eq (3) gives the corresponding iterative scheme

$$u_{n+1}(x) = u_n(x) - \int_0^x \left[(u_n')_s + f_n(u,u')\right]ds, \quad (10)$$

Thus, in general, the differential equation of the form $\left(u'(x) + f(u,u')\right) = 0$ has this iteration formula

$$u_{n+1}(x) = u_n(x) - \int_0^x \left[(u_n')_s + f_n(u_n,u_n')\right]ds, \quad (11)$$

In particular however, the iterative formula for general Riccati Equation can be derived using the following procedure

$$\frac{dy}{dt} = Q(t)y + R(t)y^2 + P(t),\ y(0) = G(t) \quad (12)$$

where Q(t), R(t), P(t) and G(t) are scalar functions. To solve equation (12) by means of He's variational method, we construct a correction functional,

$$y_{n+1}(t) = y_n(t) + \int_0^t \lambda(s)\left[\frac{dy_n(s)}{ds} - Q(s)\bar{y}_n(s) + R(s)\bar{y}_n(s) - P(s)\right]ds$$

$$\partial_{n+1}(t) = \partial y_n(t) + \partial \int_0^t \lambda(s)\left[\frac{dy_n(s)}{ds} - Q(s)\bar{y}_n(s) + R(s)\bar{y}_n(s) - P(s)\right]ds$$

$$\partial_{n+1}(t) = \partial y_n(t) + \partial \int_0^t \lambda(s)\left[\frac{dy_n(s)}{ds}\right]ds$$

$$\partial_{n+1}(t) = (1+\lambda)\partial y_n(t) - \int_0^t \partial y_n(s)\lambda(s)ds$$

Where $\widehat{y}_n$ is considered as restricted variations, which mean $\partial \widehat{y}_n = 0$ .its stationary conditions can be obtained as follows

$$1 + \lambda(t) = 0,\ \lambda'(s)\ \Big|_{s=t} = 0 \quad (13)$$

The Lagrange multiplier, therefore, can be identified as λ(s)=-1 and the following variational iterational formula is obtained

$$y_{n+1}(t) = y_n(t) - \int_0^t \left[\frac{dy_n(s)}{ds} - Q(s)\widehat{y}_n(s) - R(s)\widehat{y}_n(s) - P(s)\right]ds \quad (14)$$

III. Implementation Of The Method

The variational iteration method handles nonlinear problems and linear problems in a parallel manner. In this section, we will apply the VIM for a well known nonlinear equation called Riccati equation

$$u' = u^2 - 2xu + x^2 + 1,\ u(0) = \frac{1}{2} \quad (15)$$

Setting $\lambda = -1$ and use $u(0) = \frac{1}{2}$. The iteration formula becomes

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n(t) - u_n{}^2 + 2tu_n(t) - t^2 - 1\right)dt,\ n \geq 0 \quad (16)$$

$$u_0 = \frac{1}{2}$$

$$u_1(x) = u_0(x) - \int_0^x (u_0(t) - u_0^2 + 2tu_0(t) - t^2 - 1)dt,$$

$$= \frac{1}{2} + \frac{5}{4}x - \frac{1}{2}x^2 + \frac{1}{3}x^3,$$

$$u_2(x) = u_1(x) - \int_0^x (u_1(t) - u_1^2 + 2tu_1(t) - t^2 - 1)dt,$$

$$= \frac{1}{2} + \frac{5}{4}x - \frac{1}{8}x^2 - \frac{7}{48}x^3 + \frac{1}{28}x^4 + \frac{1}{12}x^5 + \ldots,$$

$$u_3(x) = u_2(x) - \int_0^x (u_2(t) - u_2^2 + 2tu_2(t) - t^2 - 1)dt,$$

$$= \frac{1}{2} + \frac{5}{4}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{1}{48}x^4 + \frac{7}{760}x^5 + \ldots,$$

$$u_4(x) = u_3(x) - \int_0^x (u_3(t) - u_3^2 + 2tu_3(t) - t^2 - 1)dt,$$

$$= \frac{1}{2} + \frac{5}{4}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \frac{1}{32}x^4 + \frac{1}{64}x^5 + \ldots,$$

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$$u_n(x) = x + \frac{1}{2} + \frac{5}{4}x + \frac{1}{8}x^2 + \frac{1}{16}x^3 + \frac{1}{32}x^4 + \frac{1}{64}x^5$$

this converges to exact solution

$$u(x) = x + \frac{1}{2-x} \quad |x| < 0 \quad (17)$$

Example 2

Consider the another example

$$\frac{dy}{dt} = -y^2(t) + 1, \; y(0) = 0 \qquad (18)$$

Here Q(t)=0, R(t)=-1, P(t)=1 and G(t)=0. The exact solution was found to be

$$y(t) = \frac{e^{2t} - 1}{e^{2t} + 1} \quad (19)$$

To solve equation (18) by means of He's Variational iteration method, we construct a correction functional using

$$y_{n+1}(t) = y_n(t) - f_0^t \left[\frac{dy_n}{ds} + y_n{}^2 - 1 \right] ds \quad (20)$$

We can take the linearized solution y(t)=t+C as the initial approximation y_0, the condition y(0)=0 gives us C=0.Then we obtained

$$y_1(t) = t - \frac{1}{3}t^3$$

$$y_2(t) = t - \frac{1}{3}t^3 + \frac{2}{15}t^5 - \frac{1}{63}t^7,$$

$$y_3(t) = t - \frac{1}{3}t^3 + \frac{2}{15}t^5 - \frac{1}{63}t^7 + \frac{38}{2835}t^9 - \frac{134}{51975}t^{11} + \frac{4}{12285}t^{13} - \frac{1}{59535}t^{15}$$

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$$y_n(t) = \frac{e^{2t} - 1}{e^{2t} + 1}$$

IV. Discussion And Conclusion

In this paper, we have derived the analytical solution of a class of Riccati equations using Variational Iteration Method which can later be used to characterise the solution of a fairly large class of non-linear first order ODEs.One important advantage of VIM method is that it is capable of greatly reducing the size of calculation while still maintaining high accuracy of numerical solution. Moreover, the power of the method gives it a wider applicability in handling a huge number of analytical and numerical applications in real life problems. The results obtained in this paper will provide a better insight on efficacy and accuracy of Variational Iteration Methods in solving higher order nonlinear problems.

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