IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 14, Issue 1 Ver. II (Jan. - Feb. 2018), PP 58-63
www.iosrjournals.org

Non-Homogeneousbirth andDeath Processes (Particular Case)

M. EL OUSSATI[1], M.ELMEROUANI[2]

[1](Department of Mathematics, Probability laboratory and statistics, University AbdelmalekEssaâdi, Morocco)
[2](Department of Mathematics, Probability laboratory and statistics, University AbdelmalekEssaâdi, Morocco)

Abstract

- ***Homogeneous*** *and* ***non-HomogeneousBirth and Death Processes (BDPs)*** *have a great importance for many fields of applied probability.For example,****Non-homogeneous****(See* ***[3]****,****[4]****and****[5]****) versions of the* ***Birth and Death Processes (NHBDPs)*** *have an important role in queuing theory.*
- *In this paper we consider* ***Non-Homogeneous Birth and Death Processes (NHBDPs)****(See* ***[3]****)(The rates are taken to be* ***non-homogeneous****, i.e. they can change over time, more particularly we will assume the rates vary with time with constant coeficients,* $(\lambda_i(t) = a_i t\ ;\ \mu_i(t) = b_i t\ ;\ \forall i = 1, \dots, n)$*).*
- *For the same purpose, we are going to complete the resolution of the* ***Chapman Kolmogorov's equation*** *in this case, whose coefficients depend on time t.*

Keywords: *Problems modeling, Birth and death rates, Kolmogorov differential equations, Reduction of the matrix, Recurrent Sequences of Order 2 and Identification of the law.*

Date of Submission: 29-01-2018 Date of acceptance: 19-02-2018

I. Introduction

- **Birth and Death Processes (BDPs)** were introduced by Feller (1939) and have since been used as models for population growth, queue formation, in epidemiology and in many other areas of both theoretical and applied interest.From the standpoint of the theory of stochastic processes they represent an important special case of Markov processes with countable state spaces and continuous parameters.
- We consider in this paper a special case of the **Non-Homogeneous** version of **the Birth and Death Processes (NHBDPs)**, this model describes changes in the size of a population. New population members can appear with a certain rate $(\lambda_i(t) = a_i t)$, called **the Birth rate** or the reproductive power, and members can leave the population with a rate $(\mu_i(t) = b_i t\ )$called **the Death rate**. These rates are taken to be **non-homogeneous**, i.e. they can change over time.
- In the sequel, and in order to complete the resolution of **Chapman Kolmogorov's equation**(See **[1]**and **[2]**) with a special caseof the**Non-Homogeneous Birth and Death Processes (NHBDPs)**(See **[3]**), we present in the first section a reminder on the **(NHBDPs)** model and in the second section the procedure followed in the resolution of this equation and also the solution found for this model.

II. Presentationof The Model

- Our study will be limited on the states between **1** and **n**, so we identify the states of this process with:

  $$P_i(t)\ (i=1,\dots,n)$$

  Knowing that: $\mathbf{P_i(t) = P(X_t = i)}$; (Where $\sum_{i=1}^{n} P(X_t = i) = 1$)

  (Where $X_t$is **adiscrete** and **nonhomogeneousstochasticprocess** $(t \in \mathbb{R}^+)$)
- Let **P(t)**the column matrix of type**(n, 1)**such as:$P(t)^t = (P_1(t), P_2(t), \dots, P_n(t)\ \ ;\ \ (t \in \mathbb{R}^+)$
- Let,$P_{ij}(\Delta t) = P(X_{t+\Delta t} = j / X_t = i)$ (1)**(The transition probability** of the state**I** to the state**j)**
- **The Non-Homogeneous Birth and Death Processes (NHBDPs)** is defined as follows (See [**3**]):

➢ **Definition:**

- Let $\lambda_k(t)(k = 1, \dots, n)$ is **the Birth rate** and$\mu_k(t)(k = 1, \dots, n)$ is **the Death rate,** then we have:

$$①P_{ij}(\Delta t) = \begin{cases} \lambda_i(t)\Delta t + o(\Delta t) & ;\quad \text{if} j = i + 1 \\ \mu_i(t)\Delta t\ + o(\Delta t) & ;\quad \text{if} j = i - 1 \\ 1 - \left(\lambda_i(t) + \mu_i(t)\right)\Delta t + o(\Delta t) & ;\quad \text{if} j = i \\ o(\Delta t) & ;\quad \text{if} |j - i| \geq 2 \end{cases} (2)$$

- Where all $\mathbf{o}(\Delta\mathbf{t})$ are uniform with respect to $\mathbf{i}$.
- We also suppose that for almost all $\mathbf{t} \geq \mathbf{0}$: $\quad \mathbf{Sup_i}\left(\boldsymbol{\lambda}_\mathbf{i}(\mathbf{t}) + \boldsymbol{\mu}_\mathbf{i}(\mathbf{t})\right) < \infty$

♦ We have the following proposition(See [1]):

➢ **Proposition:**

- Let $\mathbf{P_i(t)}$ **a Birth and Death Process** $(\mathbf{i = 1, \dots, n})$.
  With $\boldsymbol{\lambda}_\mathbf{k}(\mathbf{t})(\mathbf{k = 1, \dots, n})$ is **the Birth rate** and $\boldsymbol{\mu}_\mathbf{k}(\mathbf{t})(\mathbf{k = 1, \dots, n})$ is **the Death rate.**
- Thus we have the following **system of linear differential equations** (See [7]):

$$\begin{cases} \mathbf{P}_1'(\mathbf{t}) = -(\boldsymbol{\lambda}_1(\mathbf{t}) + \boldsymbol{\mu}_1(\mathbf{t}))\mathbf{P}_1(\mathbf{t}) + \boldsymbol{\mu}_2(\mathbf{t})\mathbf{P}_2(\mathbf{t}) \\ \dots \\ \mathbf{P}_\mathbf{j}'(\mathbf{t}) = \boldsymbol{\lambda}_{\mathbf{j}-1}(\mathbf{t})\mathbf{P}_{\mathbf{j}-1}(\mathbf{t}) - (\boldsymbol{\lambda}_\mathbf{j}(\mathbf{t}) + \boldsymbol{\mu}_\mathbf{j}(\mathbf{t}))\mathbf{P}_\mathbf{j}(\mathbf{t}) + \boldsymbol{\mu}_{\mathbf{j}+1}(\mathbf{t})\mathbf{P}_{\mathbf{j}+1}(\mathbf{t}) \quad ; \quad \mathbf{j = 2, \dots, n-1} \\ \dots \\ \mathbf{P}_\mathbf{n}'(\mathbf{t}) = \boldsymbol{\lambda}_{\mathbf{n}-1}(\mathbf{t})\mathbf{P}_{\mathbf{n}-1}(\mathbf{t}) - (\boldsymbol{\lambda}_\mathbf{n}(\mathbf{t}) + \boldsymbol{\mu}_\mathbf{n}(\mathbf{t}))\mathbf{P}_\mathbf{n}(\mathbf{t}) \end{cases} (3)$$

➢ **Proof:**

♦ According to **the Bays formula**, we have: $\mathbf{P}(\mathbf{X}_{\mathbf{t}+\Delta\mathbf{t}} = \mathbf{j}) = \sum_{\mathbf{i=1}}^{\mathbf{n}} \mathbf{P}(\mathbf{X}_{\mathbf{t}+\Delta\mathbf{t}} = \mathbf{j},\ \mathbf{X_t} = \mathbf{i})$
So, $\mathbf{P}(\mathbf{X}_{\mathbf{t}+\Delta\mathbf{t}} = \mathbf{j}) = \sum_{\mathbf{i=1}}^{\mathbf{n}} \mathbf{P}(\mathbf{X}_{\mathbf{t}+\Delta\mathbf{t}} = \mathbf{j}\ /\mathbf{X_t} = \mathbf{i}) \,.\, \mathbf{P}(\mathbf{X_t} = \mathbf{i})$
Thus we obtain the relation: ② $\mathbf{P_j}(\mathbf{t} + \Delta\mathbf{t}) = \sum_{\mathbf{i=1}}^{\mathbf{n}} \mathbf{P_i}(\mathbf{t}) \,.\, \mathbf{P_{ij}}(\Delta\mathbf{t})$
Replacing ① in ②, we will have:

$$\mathbf{P_j}(\mathbf{t} + \Delta\mathbf{t}) = \mathbf{P}_{\mathbf{j}-1}(\mathbf{t}).\mathbf{P}_{\mathbf{j}-1\mathbf{j}}(\Delta\mathbf{t}) + \mathbf{P_j}(\mathbf{t}).\mathbf{P_{jj}}(\Delta\mathbf{t}) + \mathbf{P}_{\mathbf{j}+1}(\mathbf{t}).\mathbf{P}_{\mathbf{j}+1\mathbf{j}}(\Delta\mathbf{t}) + \mathbf{o}(\Delta\mathbf{t})$$

$$\mathbf{P_j}(\mathbf{t} + \Delta\mathbf{t}) = \boldsymbol{\lambda}_{\mathbf{j}-1}(\mathbf{t})\Delta\mathbf{t}.\mathbf{P}_{\mathbf{j}-1}(\mathbf{t}) + \left(\mathbf{1} - \left(\boldsymbol{\lambda}_\mathbf{j}(\mathbf{t}) + \boldsymbol{\mu}_\mathbf{j}(\mathbf{t})\right)\Delta\mathbf{t}\right)\mathbf{P_j}(\mathbf{t}) + \boldsymbol{\mu}_{\mathbf{j}+1}(\mathbf{t})\Delta\mathbf{t}\mathbf{P}_{\mathbf{j}+1}(\mathbf{t}) + \mathbf{o}(\Delta\mathbf{t})$$

$$\mathbf{P_j}(\mathbf{t} + \Delta\mathbf{t}) - \mathbf{P_j}(\mathbf{t}) = (\boldsymbol{\lambda}_{\mathbf{j}-1}(\mathbf{t})\mathbf{P}_{\mathbf{j}-1}(\mathbf{t}) - \left(\boldsymbol{\lambda}_\mathbf{j}(\mathbf{t}) + \boldsymbol{\mu}_\mathbf{j}(\mathbf{t})\right)\mathbf{P_j}(\mathbf{t}) + \boldsymbol{\mu}_{\mathbf{j}+1}(\mathbf{t})\mathbf{P}_{\mathbf{j}+1}(\mathbf{t}))\Delta\mathbf{t} + \mathbf{o}(\Delta\mathbf{t})$$

$$\frac{\mathbf{P_j}(\mathbf{t} + \Delta\mathbf{t}) - \mathbf{P_j}(\mathbf{t})}{\Delta\mathbf{t}} = \boldsymbol{\lambda}_{\mathbf{j}-1}(\mathbf{t})\mathbf{P}_{\mathbf{j}-1}(\mathbf{t}) - \left(\boldsymbol{\lambda}_\mathbf{j}(\mathbf{t}) + \boldsymbol{\mu}_\mathbf{j}(\mathbf{t})\right)\mathbf{P_j}(\mathbf{t}) + \boldsymbol{\mu}_{\mathbf{j}+1}(\mathbf{t})\mathbf{P}_{\mathbf{j}+1}(\mathbf{t}) + \boldsymbol{\varepsilon}(\Delta\mathbf{t})$$

With $\boldsymbol{\varepsilon}(\Delta\mathbf{t}) = \frac{\mathbf{o}(\Delta\mathbf{t})}{\Delta\mathbf{t}}$ (Such as: $\mathbf{lim}_{\Delta\mathbf{t}\to\mathbf{0}}\ \boldsymbol{\varepsilon}(\Delta\mathbf{t}) = \mathbf{0}$)

$$(\Delta\mathbf{t} \to \mathbf{0}) \Longrightarrow \mathbf{P}_\mathbf{j}'(\mathbf{t}) = \boldsymbol{\lambda}_{\mathbf{j}-1}(\mathbf{t})\mathbf{P}_{\mathbf{j}-1}(\mathbf{t}) - \left(\boldsymbol{\lambda}_\mathbf{j}(\mathbf{t}) + \boldsymbol{\mu}_\mathbf{j}(\mathbf{t})\right)\mathbf{P_j}(\mathbf{t}) + \boldsymbol{\mu}_{\mathbf{j}+1}(\mathbf{t})\mathbf{P}_{\mathbf{j}+1}(\mathbf{t})$$

Therefore, we obtain: $\mathbf{P}'(\mathbf{t}) = \mathbf{A}(\mathbf{t}).\mathbf{P}(\mathbf{t})$ (4)
With $\mathbf{A}(\mathbf{t}) \in \mathbf{M_n}(\mathbb{R})$ and $\mathbf{P}(\mathbf{t})$ a column matrix of type $(\mathbf{n},\ \mathbf{1})$:

$$\mathbf{A}(\mathbf{t}) = \begin{pmatrix} -(\boldsymbol{\lambda}_1(\mathbf{t}) + \boldsymbol{\mu}_1(\mathbf{t})) & \boldsymbol{\mu}_2(\mathbf{t}) & \mathbf{0} \\ \boldsymbol{\lambda}_1(\mathbf{t}) & \ddots & \boldsymbol{\mu}_\mathbf{n}(\mathbf{t}) \\ \mathbf{0} & \boldsymbol{\lambda}_{\mathbf{n}-1}(\mathbf{t}) & -(\boldsymbol{\lambda}_\mathbf{n}(\mathbf{t}) + \boldsymbol{\mu}_\mathbf{n}(\mathbf{t})) \end{pmatrix}$$

This represents the famous **Chapman Kolmogorov's equation** (See **[6]** and **[11]**).

♦ As we saw in the two previous articles (See **[1]** and **[2]**), we have presented solutions of the **Chapman Kolmogorov's equation** in four cases, so that we will present the solution of this equation in this particular case.

➢ **Note:**

♦ We'll restrict our study to **Non-Homogeneous Birth and Death Processes** whose rates have the following form:

$$\boldsymbol{\lambda}_\mathbf{i}(\mathbf{t}) = \mathbf{a_i t} \quad ; \quad \boldsymbol{\mu}_\mathbf{i}(\mathbf{t}) = \mathbf{b_i t} \qquad (\text{With } \mathbf{a_i}, \mathbf{b_i} > 0 \quad \forall i = 1, \dots, n)$$

Thus, the matrix **A(t)** becomes in the following form:

$$\mathbf{A}(\mathbf{t}) = \begin{pmatrix} -(\mathbf{a}_1 + \mathbf{b}_1)\mathbf{t} & \mathbf{b}_2\mathbf{t} & \mathbf{0} \\ \mathbf{a}_1\mathbf{t} & \ddots & \mathbf{b_n t} \\ \mathbf{0} & \mathbf{a}_{\mathbf{n}-1}\mathbf{t} & -(\mathbf{a_n} + \mathbf{b_n})\mathbf{t} \end{pmatrix} = \mathbf{t}\begin{pmatrix} -(\mathbf{a}_1 + \mathbf{b}_1) & \mathbf{b}_2 & \mathbf{0} \\ \mathbf{a}_1 & \ddots & \mathbf{b_n} \\ \mathbf{0} & \mathbf{a}_{\mathbf{n}-1} & -(\mathbf{a_n} + \mathbf{b_n}) \end{pmatrix}$$

We consider the matrix $\mathbf{B} \in \mathbf{M_n}(\mathbb{R})$, such that:

$$\mathbf{B} = \begin{pmatrix} -(\mathbf{a}_1 + \mathbf{b}_1) & \mathbf{b}_2 & \mathbf{0} \\ \mathbf{a}_1 & \ddots & \mathbf{b_n} \\ \mathbf{0} & \mathbf{a}_{\mathbf{n}-1} & -(\mathbf{a_n} + \mathbf{b_n}) \end{pmatrix}$$

So, the equation $\mathbf{P}'(\mathbf{t}) = \mathbf{A}(\mathbf{t}).\mathbf{P}(\mathbf{t})$ (See **[3]**) becomes: $\mathbf{P}'(\mathbf{t}) = \mathbf{t}.\mathbf{B}.\mathbf{P}(\mathbf{t})$ (5)

1. Solving the Chapman Kolmogorov's equation in this last case:

1.1 Procedure followed in solving this equation:

- Our objective is solving this equation: $\mathbf{P}'(\mathbf{t}) = \mathbf{t}.\mathbf{B}.\mathbf{P}(\mathbf{t})$(5)

So we have to solve the following **system of linear differential equations** (See [1]):

$$\begin{cases} \mathbf{P}_1'(\mathbf{t}) = -(\mathbf{a}_1 + \mathbf{b}_1)\mathbf{t}.\mathbf{P}_1(\mathbf{t}) + \mathbf{b}_2\mathbf{t}.\mathbf{P}_2(\mathbf{t}) \\ \dots \\ \mathbf{P}_j'(\mathbf{t}) = \mathbf{a}_{j-1}\mathbf{t}.\mathbf{P}_{j-1}(\mathbf{t}) - (\mathbf{a}_j + \mathbf{b}_j)\mathbf{t}.\mathbf{P}_j(\mathbf{t}) + \mathbf{b}_{j+1}\mathbf{t}.\mathbf{P}_{j+1}(\mathbf{t}) \ ; \quad \mathbf{j} = \mathbf{2}, \dots, \mathbf{n} - \mathbf{1} \\ \dots \\ \mathbf{P}_n'(\mathbf{t}) = \mathbf{a}_{n-1}\mathbf{t}.\mathbf{P}_{n-1}(\mathbf{t}) - \mathbf{b}_n\mathbf{t}.\mathbf{P}_n(\mathbf{t}) \end{cases} \quad (6)$$

- We have proved that the matrix **B** is **diagonalizable** (See [1]), then**the eigenvalues** of the matrix**B**and **the eigenvectors associated** will be searched.
- So, the matrix **S**, whose columns are **the eigenvectors associated** with **the eigenvalues** of the matrix **B**, will be determined.
- Therefore, we have (See **[9]**):$\mathbf{B} = \mathbf{SDS^{-1}}$(7)

  (**D**is **a diagonal matrix** with **the proper values** of the matrix **B** on her **principal diagonal**)
- Next we will put **the following change of variable**: $\mathbf{Q}(\mathbf{t}) = \mathbf{S^{-1}P}(\mathbf{t})$(8)

So, $\mathbf{Q}'(\mathbf{t}) = \mathbf{S^{-1}P}'(\mathbf{t}) = \mathbf{S^{-1}t}.\mathbf{B}.\mathbf{P}(\mathbf{t}) = \mathbf{t}.\mathbf{S^{-1}BSQ}(\mathbf{t}) = \mathbf{t}.\mathbf{DQ}(\mathbf{t})$(Because:$\mathbf{P}(\mathbf{t}) = \mathbf{SQ}(\mathbf{t})$ (9))

- Thus we solve **first**the equation:$\mathbf{Q}'(\mathbf{t}) = \mathbf{t}.\mathbf{DQ}(\mathbf{t})$(10)
- Then we conclude**P(t)** according to the relation**:**$\mathbf{P}(\mathbf{t}) = \mathbf{SQ}(\mathbf{t})$(11)

1.2 Solving the equation:$\mathrm{P}'(\mathrm{t}) = \mathrm{t}.\mathrm{B}.\mathrm{P}(\mathrm{t})$(5)

- Thus we obtain the matrix **B** of the following form:

$$\mathbf{B} = \begin{pmatrix} -(\mathbf{a}_1 + \mathbf{b}_1) & \mathbf{b}_2 & \mathbf{0} \\ \mathbf{a}_1 & \ddots & \mathbf{b}_n \\ \mathbf{0} & \mathbf{a}_{n-1} & -(\mathbf{a}_n + \mathbf{b}_n) \end{pmatrix}$$

- We'll look for **the eigenvalues** and **the associated eigenvectors** of the matrix **B.**
- Let **α**be **an eigenvalue** of the matrix**B** and $\mathbf{x} \in \mathbb{R}^{\mathbf{n}} - (\mathbf{0}, \dots, \mathbf{0})$ **an associated eigenvector.**

Such that:$\mathbf{BX} = \boldsymbol{\alpha}\mathbf{X}$

Then, we have (12):
$$\begin{cases} -(\mathbf{a}_1 + \mathbf{b}_1)\mathbf{x}_1 + \mathbf{b}_2\mathbf{x}_2 = \boldsymbol{\alpha}\mathbf{x}_1 \\ \dots \\ \mathbf{a}_{k-1}\mathbf{x}_{k-1} - (\mathbf{a}_k + \mathbf{b}_k)\mathbf{x}_k + \mathbf{b}_{k+1}\mathbf{x}_{k+1} = \boldsymbol{\alpha}\mathbf{x}_k; \quad \mathbf{k} = \mathbf{2}, \dots, \mathbf{n} - \mathbf{1} \\ \dots \\ \mathbf{a}_{n-1}\mathbf{x}_{n-1} - (\mathbf{a}_n + \mathbf{b}_n)\mathbf{x}_n = \boldsymbol{\alpha}\mathbf{x}_n \end{cases}$$

So (13),
$$\begin{cases} -(\mathbf{a}_1 + \mathbf{b}_1 + \boldsymbol{\alpha})\mathbf{x}_1 + \mathbf{b}_2\mathbf{x}_2 = \mathbf{0} \\ \dots \\ \mathbf{a}_{k-1}\mathbf{x}_{k-1} - (\mathbf{a}_k + \mathbf{b}_k + \boldsymbol{\alpha})\mathbf{x}_k + \mathbf{b}_{k+1}\mathbf{x}_{k+1} = \mathbf{0} \ ; \quad \mathbf{k} = \mathbf{2}, \dots, \mathbf{n} - \mathbf{1} \\ \dots \\ \mathbf{a}_{n-1}\mathbf{x}_{n-1} - (\mathbf{a}_n + \mathbf{b}_n + \boldsymbol{\alpha})\mathbf{x}_n = \mathbf{0} \end{cases}$$

So (14), $\mathbf{b}_{k+1}\mathbf{x}_{k+1} - (\mathbf{a}_k + \mathbf{b}_k + \boldsymbol{\alpha})\mathbf{x}_k + \mathbf{a}_{k-1}\mathbf{x}_{k-1} = \mathbf{0} \ ; \quad \mathbf{k} = \mathbf{1}, \dots, \mathbf{n}$(See **[12]**)

We put: $\mathbf{x}_k = \mathbf{q}^{\mathbf{k}}$

So we get:$\mathbf{b}_{k+1}\mathbf{q}^{\mathbf{k}+1} - (\mathbf{a}_k + \mathbf{b}_k + \boldsymbol{\alpha})\mathbf{q}^{\mathbf{k}} + \mathbf{a}_{k-1}\mathbf{q}^{\mathbf{k}-1} = \mathbf{0}$

Thus for **k=n** we obtain: $\mathbf{b}_{n+1}\mathbf{q}^{\mathbf{n}+1} - (\mathbf{a}_n + \mathbf{b}_n + \boldsymbol{\alpha})\mathbf{q}^{\mathbf{n}} + \mathbf{a}_{n-1}\mathbf{q}^{\mathbf{n}-1} = \mathbf{0}$

Dividing the last equation by: $\mathbf{q}^{\mathbf{k}-\mathbf{n}}$

The relation becomes:$\mathbf{b}_{n+1}\mathbf{q}^{\mathbf{k}+1} - (\mathbf{a}_n + \mathbf{b}_n + \boldsymbol{\alpha})\mathbf{q}^{\mathbf{k}} + \mathbf{a}_{n-1}\mathbf{q}^{\mathbf{k}-1} = \mathbf{0}$(15)

By dividing the equation by:$\mathbf{q}^{\mathbf{k}-1}$

The **characteristic equation** becomes: $\mathbf{b}_{n+1}\mathbf{q}^2 - (\mathbf{a}_n + \mathbf{b}_n + \boldsymbol{\alpha})\mathbf{q} + \mathbf{a}_{n-1} = \mathbf{0}$(16)

Of **discriminant**: $\Delta = (\mathbf{a}_n + \mathbf{b}_n + \boldsymbol{\alpha})^2 - 4\mathbf{a}_{n-1}\mathbf{b}_{n+1}$ (17)

- We will discuss the solutions according to the sign of Δ and the values of the initials conditions:

$\mathbf{1^{st}case:}\boldsymbol{\alpha} \in \left]-\infty\,; -(\mathbf{a_n}+\mathbf{b_n}) - 2\sqrt{\mathbf{a_{n-1}b_{n+1}}}\right[\cup\left]-(\mathbf{a_n}+\mathbf{b_n}) + 2\sqrt{\mathbf{a_{n-1}b_{n+1}}}\,; +\infty\right[\Rightarrow \Delta > 0$

- Therefore **the characteristic equation** admits two **conjugate realsolutions** $\mathbf{r_-}$ and $\mathbf{r_+}$given by:

$$\mathbf{r_\pm} = \frac{\mathbf{a_n}+\mathbf{b_n}+\boldsymbol{\alpha}}{\mathbf{2b_{n+1}}} \pm \sqrt{\left(\frac{\mathbf{a_n}+\mathbf{b_n}+\boldsymbol{\alpha}}{\mathbf{2b_{n+1}}}\right)^2 - \frac{\mathbf{a_{n-1}}}{\mathbf{b_{n+1}}}}$$

- Therefore, $(\mathbf{x_k})_{1\leq k,i\leq n}$is given by:$\mathbf{x_k} = \boldsymbol{\gamma_-}\mathbf{r_-^k} + \boldsymbol{\gamma_+}\mathbf{r_+^k}$(18)
- Where the coefficients $\boldsymbol{\gamma_-}$and $\boldsymbol{\gamma_+}$are provided by the following conditions: $\mathbf{x_0} = \mathbf{x_{n+1}} = \mathbf{0}$

Thus we obtain: $\begin{cases} \boldsymbol{\gamma_-} + \boldsymbol{\gamma_+} = \mathbf{0} \\ \boldsymbol{\gamma_-}(\mathbf{r_-^{n+1}} - \mathbf{r_+^{n+1}}) = \mathbf{0} \end{cases}$

- Therefore, we have: $\boldsymbol{\gamma_-} = \boldsymbol{\gamma_+} = \mathbf{0}$so, $\mathbf{X} = (\mathbf{0})$→**Which is excluded.**

→ **Therefore, this case is empty.**

$\mathbf{2^{nd}case}$: $\boldsymbol{\alpha} = -(\mathbf{a_n}+\mathbf{b_n}) \pm 2\sqrt{\mathbf{a_{n-1}b_{n+1}}} \Rightarrow \Delta = \mathbf{0}$

- Then **the characteristic equation** admits a **double real solution, (**Nominated$\mathbf{r_0}$), such that:

$$\mathbf{x_k} = (\boldsymbol{\gamma_-} + \mathbf{k}\boldsymbol{\gamma_+})\mathbf{r_0^k} \quad ; \quad \mathbf{k=1, \ldots, n} \quad (19)$$

- The condition:$\mathbf{x_0} = \mathbf{x_{n+1}} = \mathbf{0}$give $\boldsymbol{\gamma_-} = \boldsymbol{\gamma_+} = \mathbf{0}$
- In the end: $\mathbf{X} = (\mathbf{0})$→ **which is excluded**.

→ **This second case is also empty.**

$\mathbf{3^{rd}case}$: $\boldsymbol{\alpha} \in \left]-(\mathbf{a_n}+\mathbf{b_n}) - 2\sqrt{\mathbf{a_{n-1}b_{n+1}}};\ -(\mathbf{a_n}+\mathbf{b_n}) + 2\sqrt{\mathbf{a_{n-1}b_{n+1}}}\right[\Rightarrow \Delta < \mathbf{0}$

- The solutions exist for: $\Delta < \mathbf{0}$

Thus we put: $\boldsymbol{\alpha} = -(\mathbf{a_n}+\mathbf{b_n}) + 2\sqrt{\mathbf{a_{n-1}b_{n+1}}}\cos\boldsymbol{\theta}$; $\boldsymbol{\theta} \in]\mathbf{0}, \boldsymbol{\pi}[\cup]\boldsymbol{\pi}, \mathbf{2}\boldsymbol{\pi}[$

Hence **the characteristic equation** becomes: $\mathbf{b_{n+1}q^2} - \mathbf{2q}\sqrt{\mathbf{a_{n-1}b_{n+1}}}\cos\boldsymbol{\theta} + \mathbf{a_{n-1}} = \mathbf{0}$(20)

Therefore, as $\mathbf{a_{n-1}} \neq \mathbf{0}$ ($\mathbf{k} = \mathbf{1}, \ldots, \mathbf{n}$), we have: $\frac{\mathbf{b_{n+1}}}{\mathbf{a_{n-1}}}\mathbf{q^2} + \mathbf{2q}\sqrt{\frac{\mathbf{b_{n+1}}}{\mathbf{a_{n-1}}}}\cos\boldsymbol{\theta} + \mathbf{1} = \mathbf{0}$

- Thus we have the following relation (21):

$$\frac{\mathbf{b_{n+1}}}{\mathbf{a_{n-1}}}\mathbf{q^2} + \mathbf{2q}\sqrt{\frac{\mathbf{b_{n+1}}}{\mathbf{a_{n-1}}}}\cos\boldsymbol{\theta} + \mathbf{1} = \left(\sqrt{\frac{\mathbf{b_{n+1}}}{\mathbf{a_{n-1}}}}\mathbf{q} - \mathbf{e^{i\theta}}\right)\left(\sqrt{\frac{\mathbf{b_{n+1}}}{\mathbf{a_{n-1}}}}\mathbf{q} - \mathbf{e^{-i\theta}}\right)$$

Which give (21): $\mathbf{x_k} = \boldsymbol{\rho}^{\mathbf{k}}(\boldsymbol{\gamma_-}\cos\mathbf{k}\boldsymbol{\theta} + \boldsymbol{\gamma_+}\sin\mathbf{k}\boldsymbol{\theta})$; **k=1, …, n**

With: $\boldsymbol{\rho} = |\boldsymbol{\omega}| = \sqrt{\frac{\mathbf{a_{n-1}}}{\mathbf{b_{n+1}}}}$ and $\boldsymbol{\theta} = \mathbf{arg}(\boldsymbol{\omega})$

- Thus **the sequence of recurrence of order 2** admits as solution:

$$\mathbf{x_k} = \left(\sqrt{\frac{\mathbf{a_{n-1}}}{\mathbf{b_{n+1}}}}\right)^{\mathbf{k}}(\boldsymbol{\gamma_+}\cos\mathbf{k}\boldsymbol{\theta} + \boldsymbol{\gamma_-}\sin\mathbf{k}\boldsymbol{\theta}) \quad ;\mathbf{k=1, \ldots, n}(22)$$

Using the initials conditions: $\mathbf{x_0} = \mathbf{x_{n+1}} = \mathbf{0}$

We obtain for $\mathbf{x_0} = \mathbf{0}$: $\boldsymbol{\gamma_-} = \mathbf{0}$

So (23), $\mathbf{x_k} = \boldsymbol{\gamma_+}\left(\sqrt{\frac{\mathbf{a_{n-1}}}{\mathbf{b_{n+1}}}}\right)^{\mathbf{k}}\sin\mathbf{k}\boldsymbol{\theta}$;**(k=1, …,n))**

The condition $\mathbf{x_{n+1}} = \mathbf{0}$gives $\boldsymbol{\gamma_+} = \mathbf{0}$or $\sin(\mathbf{n}+\mathbf{1})\boldsymbol{\theta} = \mathbf{0}$

If $\boldsymbol{\gamma_+} = \mathbf{0}$ then $\mathbf{X} = (\mathbf{0}) \rightarrow$ **Therefore, it is excluded.**

If $\sin(\mathbf{n}+\mathbf{1})\boldsymbol{\theta} = \mathbf{0}$then$\boldsymbol{\theta} = \frac{\mathbf{k}\boldsymbol{\pi}}{\mathbf{n+1}}$

- Thus, **the eigenvalues** of the matrix **A** (Called $\boldsymbol{\alpha}_{\mathbf{k}}$; $\mathbf{k} = \mathbf{1}, \ldots, \mathbf{n}$) are of the form:

$\forall\boldsymbol{\alpha}_{\mathbf{k}} \in \left]-(\mathbf{a_n}+\mathbf{b_n}) - 2\sqrt{\mathbf{a_{n-1}b_{n+1}}};\ -(\mathbf{a_n}+\mathbf{b_n}) + 2\sqrt{\mathbf{a_{n-1}b_{n+1}}}\right[$

$$\boldsymbol{\alpha}_{\mathbf{k}} = -(\mathbf{a_n}+\mathbf{b_n}) + 2\sqrt{\mathbf{a_{n-1}b_{n+1}}}\cos\left(\frac{k\pi}{n+1}\right) \quad (24)$$

- And **the eigenvectorsassociated** with **the eigenvalues** of the matrix **B** are:

$$(\mathbf{x_k})_{\mathbf{j}} = \boldsymbol{\gamma_+}\left(\sqrt{\frac{\mathbf{a_{n-1}}}{\mathbf{b_{n+1}}}}\right)^{\mathbf{k}}\sin\left(\mathbf{j}\frac{\mathbf{k}\boldsymbol{\pi}}{\mathbf{n+1}}\right); \quad \mathbf{1} \leq \mathbf{j}, \mathbf{k} \leq \mathbf{n} \quad (25)$$

- We return to our equation: $\mathbf{P'(t)} = \mathbf{t.B.P(t)}$(5)

- We have **S** a matrix whose columns are **the eigenvectors associated** with **the eigenvalues** of the matrix **B**, such that it is presented in the following form (26):

$$\mathbf{S}=\boldsymbol{\gamma}_{+}\begin{pmatrix}\sqrt{\frac{\mathbf{a_{n-1}}}{\mathbf{b_{n+1}}}}\,\mathbf{sin}\left(\frac{\pi}{\mathbf{n+1}}\right) & \cdots & \left(\sqrt{\frac{\mathbf{a_{n-1}}}{\mathbf{b_{n+1}}}}\right)^{\mathbf{n}}\mathbf{sin}\left(\frac{\mathbf{n}\pi}{\mathbf{n+1}}\right)\\ \vdots & \ddots & \vdots\\ \sqrt{\frac{\mathbf{a_{n-1}}}{\mathbf{b_{n+1}}}}\,\mathbf{sin}\left(\frac{\mathbf{n}\pi}{\mathbf{n+1}}\right) & \cdots & \left(\sqrt{\frac{\mathbf{a_{n-1}}}{\mathbf{b_{n+1}}}}\right)^{\mathbf{n}}\mathbf{sin}\left(\frac{\mathbf{n^2}\pi}{\mathbf{n+1}}\right)\end{pmatrix}$$

- We **first** solve the following **differential equation**: $\mathbf{Q}'(\mathbf{t})=\mathbf{t}.\mathbf{DQ}(\mathbf{t})$(27)

Thus, we have:

$$\mathbf{Q}'(\mathbf{t})=\begin{pmatrix}\left(-(\mathbf{a_n}+\mathbf{b_n})+2\sqrt{\mathbf{a_{n-1}b_{n+1}}}\,\mathbf{cos}\left(\frac{\pi}{\mathbf{n+1}}\right)\right).\mathbf{t} & \cdots & \mathbf{0}\\ \vdots & \ddots & \vdots\\ \mathbf{0} & \cdots & \left(-(\mathbf{a_n}+\mathbf{b_n})+2\sqrt{\mathbf{a_{n-1}b_{n+1}}}\,\mathbf{cos}\left(\frac{\mathbf{n}\pi}{\mathbf{n+1}}\right)\right).\mathbf{t}\end{pmatrix}\begin{pmatrix}\mathbf{Q_1}(\mathbf{t})\\ \vdots\\ \mathbf{Q_n}(\mathbf{t})\end{pmatrix}$$

So (28),

$$\begin{pmatrix}\mathbf{Q}_1'(\mathbf{t})\\ \cdots\\ \mathbf{Q}_n'(\mathbf{t})\end{pmatrix}=\begin{pmatrix}\left(-(\mathbf{a_n}+\mathbf{b_n})+2\sqrt{\mathbf{a_{n-1}b_{n+1}}}\,\mathbf{cos}\left(\frac{\pi}{\mathbf{n+1}}\right)\right).\mathbf{t}.\mathbf{Q_1}(\mathbf{t})\\ \vdots\\ \left(-(\mathbf{a_n}+\mathbf{b_n})+2\sqrt{\mathbf{a_{n-1}b_{n+1}}}\,\mathbf{cos}\left(\frac{\mathbf{n}\pi}{\mathbf{n+1}}\right)\right).\mathbf{t}.\mathbf{Q_n}(\mathbf{t})\end{pmatrix}$$

Thus, $$\mathbf{Q_k}(\mathbf{t})=\boldsymbol{\delta}_{\mathbf{k}}\mathbf{e}^{\left(-(\mathbf{a_n}+\mathbf{b_n})+2\sqrt{\mathbf{a_{n-1}b_{n+1}}}\,\mathbf{cos}\left(\frac{\mathbf{n}\pi}{\mathbf{n+1}}\right)\right)\int_0^{\mathbf{t}}\mathbf{t}.\mathbf{dt}}\quad;\quad(\mathbf{k}=\mathbf{1},\ldots,\mathbf{n})$$

- So(29), $$\mathbf{Q_k}(\mathbf{t})=\boldsymbol{\delta}_{\mathbf{k}}\mathbf{e}^{\left(-(\mathbf{a_n}+\mathbf{b_n})+2\sqrt{\mathbf{a_{n-1}b_{n+1}}}\,\mathbf{cos}\left(\frac{\mathbf{k}\pi}{\mathbf{n+1}}\right)\right)\frac{\mathbf{t^2}}{\mathbf{2}}}\quad;\quad(\mathbf{k}=\mathbf{1},\ldots,\mathbf{n})$$

With $\boldsymbol{\delta}_{\mathbf{k}}$ is a constant to be determined, if one has an initial condition.

- Therefore, and according to the following relation: $\mathbf{P}(\mathbf{t})=\mathbf{SQ}(\mathbf{t})$(11)

We have (30):

$$\mathbf{P}(\mathbf{t})=\boldsymbol{\gamma}_{+}\begin{pmatrix}\sqrt{\frac{\mathbf{a_{n-1}}}{\mathbf{b_{n+1}}}}\,\mathbf{sin}\left(\frac{\pi}{\mathbf{n+1}}\right) & \cdots & \left(\sqrt{\frac{\mathbf{a_{n-1}}}{\mathbf{b_{n+1}}}}\right)^{\mathbf{n}}\mathbf{sin}\left(\frac{\mathbf{n}\pi}{\mathbf{n+1}}\right)\\ \vdots & \ddots & \vdots\\ \sqrt{\frac{\mathbf{a_{n-1}}}{\mathbf{b_{n+1}}}}\,\mathbf{sin}\left(\frac{\mathbf{n}\pi}{\mathbf{n+1}}\right) & \cdots & \left(\sqrt{\frac{\mathbf{a_{n-1}}}{\mathbf{b_{n+1}}}}\right)^{\mathbf{n}}\mathbf{sin}\left(\frac{\mathbf{n^2}\pi}{\mathbf{n+1}}\right)\end{pmatrix}\begin{pmatrix}\boldsymbol{\delta}_{\mathbf{1}}\mathbf{e}^{\left(-(\mathbf{a_n}+\mathbf{b_n})+2\sqrt{\mathbf{a_{n-1}b_{n+1}}}\,\mathbf{cos}\left(\frac{\pi}{\mathbf{n+1}}\right)\right)\frac{t^2}{2}}\\ \vdots\\ \boldsymbol{\delta}_{\mathbf{n}}\mathbf{e}^{\left(-(\mathbf{a_n}+\mathbf{b_n})+2\sqrt{\mathbf{a_{n-1}b_{n+1}}}\,\mathbf{cos}\left(\frac{\mathbf{n}\pi}{\mathbf{n+1}}\right)\right)\frac{t^2}{2}}\end{pmatrix}$$

- Finally, for **j** between **1** and **n**, $\mathbf{P_j}(\mathbf{t})$ is in the following form (31):

$$\mathbf{P_j}(\mathbf{t})=\boldsymbol{\gamma}_{+}\sum_{\mathbf{k=1}}^{\mathbf{n}}\boldsymbol{\delta}_{\mathbf{k}}\left(\sqrt{\frac{\mathbf{a_{n-1}}}{\mathbf{b_{n+1}}}}\right)^{\mathbf{k}}\mathbf{sin}\left(\frac{\mathbf{jk}\pi}{\mathbf{n+1}}\right)\mathbf{e}^{\left(-(\mathbf{a_n}+\mathbf{b_n})+2\sqrt{\mathbf{a_{n-1}b_{n+1}}}\,\mathbf{cos}\left(\frac{\mathbf{k}\pi}{\mathbf{n+1}}\right)\right)\frac{t^2}{2}}$$

Note that the constant $\boldsymbol{\gamma}_{+}$is obtained if an initial condition exists (Example: $(\mathbf{X_k})_{\mathbf{0}}=\mathbf{cst}_{\mathrm{k}};\ \mathbf{k}=\mathbf{1},\ldots,\mathbf{n}$)
Note also that the constant $\boldsymbol{\delta}_{\mathbf{k}}$is obtained if an initial condition exists (Example: $\mathbf{Q_k}(\mathbf{0})=\mathbf{cst}_{\mathrm{k}};\ \mathbf{k}=\mathbf{1},\ldots,\mathbf{n}$)

III. Conclusion

- It now needed a wide variety of possibly transformable patterns one in the other, according to a combinatorial procedure, to find the one that suited a reality which, in turn, was always made up of several different realities, in time as in the space. Calvino, Palomar.
- It is extremely difficult to obtain general results for arbitrary forms of the birth and death rates and therefore we must content ourselves in obtaining various types of approximations.
- The stochastic models developed here are complementary to deterministic models. The latter can be used to understand average or long-term behaviors as they explore more extreme behaviors, for example: to understand events leading to extinction, to introduce age structures to take into account phenomena of aging or maturation,

look at sexual and diploid populations, understand what happens in the case where mutations are rare or of low amplitude ...

- Thanks to this law, several problems of changing size of any type of population can be solved, used more particularly in biology, demography, physics, sociology, statistics ... etc.

.

Acknowledgements

- I would like to express my sincere gratitude to my advisor **DrM.ElMerouani** for the continuous support and for his patience.

References

[1]. M.Eloussati, A.Arbai and M.Benslimane,Birth and Death Processes with Finished Number of States,*International Journal of Mathematics and Statistics Invention (IJMSI),*12 August 2017.

[2]. M.Eloussati, A.Arbai, Birth and Death Processes_Generalcase,*International Journal of Mathematics and Statistics Invention (IJMSI),*12 October 2017.

[3]. A.Zeifman, S.Leorato, E. Orsingher, Ya. Satin, G.Shilova,Some universal limits for nonhomogeneous birth and death processes*, Springer, 16 August 2005.*

[4]. Jan van den Broek, Hans Heesterbeek,Nonhomogeneous birth and death models for epidemic outbreak data,*Biostatistics, Volume 8, Issue 2, 1 April 2007, Pages 453–467.*Published: 06 September 2006.

[5]. A.I. Zeifman, Vologda State Pedagogical Institute, Vologda, S.Orlova, Russia,Upper and lower bounds on the rate of convergence for nonhomogeneous birth and death processes,*Stochastic Processes and their Applications 59 (1995) 157-173.*

[6]. Forrest W. Crawford and Marc A. Suchardy, Birth-death processes*, July 28, 2014, cite as: arXiv: 1301.1305v2 [stat.ME],*25 Jul, 2014.

[7]. Forrest W. Crawford and Marc A. Suchard,Transition probabilities for general birth-death processes with applications in ecology,genetics, and evolution,*Typeset on September 26, 2011.*

[8]. Patrice Wira, Recall on the Matrix,*Sciences and technologies Faculty (2000-2001), University of Haute Alsace.*Martin Otto, Linear Algebra II, *Summer Term 2011*.

[9]. E. Kowalski, kowalski@math.ethz.ch, Linear Algebra,*ETH Zurich-Fall2015 andSpring 2016, Version of September 15, 2016.*

[10]. AntoninaMitrofanova, NYU, Continuous times Markov Chains, Poisson Process, Birth and Death process,*December 18, 2007, department of Computer Science.*

[11]. Recurrences, *MATH 579 Spring 2012 Supplement*.

M. EL OUSSATI1"Non-Homogeneousbirth And Death Processes (Particular Case)" IOSR Journal of Mathematics (IOSR-JM) 14.1 (2018): 58-63.