IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 14, Issue 1 Ver. II (Jan. - Feb. 2018), PP 16-30
www.iosrjournals.org

Asymptotic Investigation of the Buckling Of a Cubic–Quintic Nonlinear Elastic Model Structure Stressed By Static Load and A Dynamic Step Load.

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Abstract: *This paper seeks to determine the static and dynamic buckling load of a nonlinear elastic model structure having cubic – quintic nonlinearity. The associated static problem is first solved by adopting phase plane analysis to obtain the exact result followed by asymptotic and perturbation approach to obtain the approximate solution of the problem. Next, the dynamic problem after words is solved by employing phase plane technique to obtain the exact solution and later, by employing asymptotic and perturbation approach to obtain the approximate solution. In all the cases considered, the dynamic buckling load λ_D is mathematically related to the static buckling load λ_S. The adoption of asymptotic and perturbation procedure is made possible by the presence of small non-dimensional parameter on which asymptotic expansions are made possible.*

Keywords: *Dynamic buckling, Nonlinear elastic foundation, Static buckling, Step load.*

Date of Submission: 20-01-2018 Date of acceptance: 09-02-2018

I. Introduction

In this paper, we shall investigate the buckling (both static and dynamic) of an elastic model structure with cubic – quintic nonlinearity and with the single aim of determining its static and dynamic buckling loads, where, in the dynamic case, the structure is trapped by a step load.

Several investigators have previously studied elastic model structures with diverse degrees of nonlinearities and such investigations include those by [1], [2], [3], [4] and [5], among others. However, relative to the aforementioned investigations, the subject under discourse appears to present some formidable complexity judging from the degree and extent of the inherent nonlinearity characterizing the structure.

We shall tackle the problem, first, by resorting to phase plane analysis, and next, by applying asymptotic and perturbation techniques. The exact method (phase plane analysis) is suitable here because the step loading situation (in the dynamic case) presents an autonomous nonlinear and nonhomogeneous ordinary differential equation which can be solved without any much difficulty.

However, the approximate method (i.e the asymptotic and perturbation techniques) are equally necessary because we cannot use phase plane method in most equations characterizing physically – realistic engineering structures, even for a loading history that is as simple as step load. Thus, for such materials and for loading that are explicitly non – autonomous in time, asymptotic and perturbation methods are the next alternative for any analytical studies of such structures.

Buckling (and dynamic buckle in particular) presents formidable instabilities associated with engineering structures under compressive loading. While a sufficiently huge quantum of investigation has been done on static loading (under static compressive loads), the same cannot be said of dynamic loading. The search however continues. Relatively – recent studies on dynamic buckling include investigations by [6], [7],[8],[9], [10] and [11] among others.

We must however remarkthat [12] investigated the buckling and post buckling analysis of extensible beam – column by using the differential quadrature method while [13] studied a simple method to determine the critical buckling loads for axially inhomogeneous beams with elastic constraint. A similar investigation was undertaken by [14] when he investigated the stability analysis of non – uniform rectangular beams using homotopy perturbation method. In the same token, [15] studied exact solution and dynamic buckling analysis of a beam – column system having the elliptic type loading, while [16] gave a review of recent research on vibration energy harvesting via bistable systems. Pertinent here is the study by [17] who investigated piezoelectric buckled beams for random vibration energy harvesting, while [18] explored mechanical instability of thin elastic rods.

II. Formulaion Of The Problem

In its simplest form, the required non-dimensional differential equation satisfied by the displacement ξ(t) of an elastic model structure with cubic – quintic nonlinearity trapped by an arbitrary time dependent load $\lambda f(t)$ is

$$\ddot{\xi} + (1 - \lambda f(t))\xi + \alpha\xi^3 - \beta\xi^5 = \lambda\bar{\xi}f(t), t > 0 \qquad (1)$$

$$\xi(0) = \dot{\xi}(0), \quad \dot{()} \equiv \frac{d(\dots)}{dt} \qquad (2)$$

where, λ is the nondimensional load amplitude satisfying the inequality $0 < \lambda < 1$, $\bar{\xi}$ is the amplitude of imperfection and also satisfying the inequality $0 < \bar{\xi} << 1$, $f(t)$ is the explicitly time dependent load function of time t, and for our case, $f(t)$ is a step load satisfying

$$f(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases} \qquad (3)$$

whileα and β are constants to be suitably chosen so as to ensure imperfection – sensitivity of the structure.
As in equation (1), the governing equation of motion has a cubic – quintic nonlinearity and our main task is to determine analytically, the static buckling loadλ_S (in the static loading case) and the dynamic buckling load λ_D in the case where the structure is trapped by a step load.

III. Solution Of The Problem

(a) PHASE PLANE SOLUTION OF THE ASSOCIATED STATIC PROBLEM

The associated static problem is obtained from (1) by ignoring the inertia term and by setting $f(t) \equiv 1$. This yields

$$(1 - \lambda)\xi + \alpha\xi^3 - \beta\xi^5 = \lambda\bar{\xi} \qquad (4a)$$

As in [1 –2] and [4], the condition for static buckling is the maximization

$$\frac{d\lambda}{d\xi} = 0 \qquad (4b)$$

The static buckling load λ_S is here defined as the largest value of the load parameter for the solution of (4a) to remain bounded. Thus carrying out (4b) using (4a), we get

$$(1 - \lambda_S) + 3\alpha\xi_S^2 - 5\beta\xi_S^4 = 0 \qquad (5)$$

whereξ_S is the value of ξ at static buckling. Equation (5) is quartic which we solve to get

$$\xi_S^2 = -\left[\frac{-3\alpha \pm \sqrt{9\alpha^2 - 20\beta(1 - \lambda_S)}}{10\beta}\right] \qquad (6)$$

We shall however choose the negative sign in the square root and get

$$\xi_S^2 = \frac{3\alpha + 2\sqrt{5}(1 - \lambda_S)^{\frac{1}{2}}\beta^{\frac{1}{2}}}{10\beta}\left[1 + \frac{9}{20(1 - \lambda_S)}\left(\frac{\alpha^2}{\beta}\right)\right]^{\frac{1}{2}} \qquad (7)$$

Further simplification of (7) yields

$$\xi_S^2 = \frac{3\alpha + 2\sqrt{5}(1 - \lambda_S)^{\frac{1}{2}}\beta^{\frac{1}{2}}}{10\beta}r_1 \qquad (8a)$$

where,

$$r_1 = \left[1 + \frac{9}{20(1 - \lambda_S)}\left(\frac{\alpha^2}{\beta}\right)\right]^{\frac{1}{2}} \qquad (8b)$$

$$\therefore \ \xi_S = \frac{(1 - \lambda_S)^{\frac{1}{4}}r_1^{\frac{1}{2}}r_2}{(\sqrt{5})^{\frac{1}{2}}\beta^{\frac{1}{4}}} \qquad (9a)$$

where,

$$r_2 = \left[1 + \frac{3}{2\sqrt{5}(1 - \lambda_S)^{\frac{1}{2}}r_1}\left(\frac{\alpha}{\beta^{\frac{1}{2}}}\right)\right]^{\frac{1}{2}} \qquad (9b)$$

To determine the static buckling load λ_S, we have to determine (4a) at static buckling stage, i.e

$$\xi_S[(1 - \lambda_S) + \xi_S^2(\alpha - \beta\xi^2)] = \lambda_S\bar{\xi} \qquad (10a)$$

This simplifies to

$$\frac{(1 - \lambda_S)^{\frac{1}{4}}r_{1S}^{\frac{1}{2}}r_{2S}}{(\sqrt{5})^{\frac{1}{2}}\beta^{\frac{1}{4}}}\left[(1 - \lambda_S) + \left(\frac{(1 - \lambda_S)^{\frac{1}{2}}r_{1S}^{\frac{1}{2}}r_{2S}^2}{\sqrt{5}\beta^{\frac{1}{2}}}\right)\left\{\alpha - \frac{(1 - \lambda_S)^{\frac{1}{2}}r_{1S}^{\frac{1}{2}}r_{2S}^2}{\sqrt{5}\beta^{\frac{1}{2}}}\right\}\right] = \lambda_S\bar{\xi} \quad (10b)$$

where,

$$r_{1S} = r_1(\lambda_S), \qquad r_{2S} = r_2(\lambda_S) \qquad (10c)$$

Further simplification of (10b) yields

$$\frac{(1-\lambda_S)^{\frac{5}{4}} r_{1S}^{\frac{1}{2}} r_{2S}}{(\sqrt{5})^{\frac{1}{2}} \beta^{\frac{1}{4}}} \left[1 + \frac{r_{1S} r_{2S}}{\sqrt{5}(1-\lambda_S)^{\frac{1}{2}}} \left(\frac{\alpha}{\beta^{\frac{1}{2}}} \right) \left\{ 1 - \frac{(1-\lambda_S)^{\frac{1}{2}} r_{1S} r_{2S}}{\sqrt{5}} \left(\frac{\beta^{\frac{1}{2}}}{\alpha} \right) \right\} \right] = \lambda_S \bar{\xi} \quad (10d)$$

Final simplification of (10d) yields

$$(1-\lambda_S)^{\frac{5}{4}} = \frac{(\sqrt{5})^{\frac{1}{2}} \beta^{\frac{1}{4}} \lambda_S \bar{\xi}}{r_{1S}^{\frac{1}{2}} r_{2S}} \left[1 + \frac{r_{1S} r_{2S}^{\;2} \left(\frac{\alpha}{\beta^{\frac{1}{2}}} \right)}{\sqrt{5}(1-\lambda_S)^{\frac{1}{2}}} \left\{ 1 - \frac{(1-\lambda_S)^{\frac{1}{2}} r_{1S} r_{2S}^{\;2} \left(\frac{\beta^{\frac{1}{2}}}{\alpha} \right)}{\sqrt{5}} \right\} \right]^{-1} \qquad (10e)$$

We note that (10e) gives an implicit equation for determining the static buckling load λ_S.

(b) PHASE PLANE SOLUTION OF THE ASSOCIATED DYNAMIC PROBLEM

Here, we set $f(t) \equiv 1$ so that (1) becomes

$$\ddot{\xi} + (1-\lambda)\xi + \alpha\xi^3 - \beta\xi^5 = \lambda\bar{\xi} \qquad (11a)$$

$$\xi(0) = \dot{\xi}(0) = 0 \qquad (11b)$$

On multiplying (11a) by $\dot{\xi}$, we get

$$\dot{\xi}\ddot{\xi} + (1-\lambda)\xi\dot{\xi} + \alpha\xi^3\dot{\xi} - \beta\xi^5\dot{\xi} = \lambda\bar{\xi}\dot{\xi} \qquad (12a)$$

This implies

$$\frac{1}{2}\frac{d}{dt}(\dot{\xi}^2) + \frac{1}{2}(1-\lambda)\frac{d}{dt}(\xi^2) + \frac{1}{4}\alpha\frac{d}{dt}(\xi^4) - \frac{\beta}{5}\frac{d}{dt}(\xi^6) = \lambda\bar{\xi}\frac{d\xi}{dt} \qquad (12b)$$

Now, the condition for buckling in the dynamic case is the maximization [1]

$$\frac{d\lambda}{d\xi_a} = 0 \qquad (13)$$

where, ξ_a is the maximum of the displacement . If we integrate (12b), we get

$$\frac{1}{2}\dot{\xi}^2 + \frac{1}{2}(1-\lambda)\xi^2 + \frac{1}{4}\alpha\xi^4 - \frac{\beta}{5}(\xi^6) = \lambda\bar{\xi}\xi \qquad (14)$$

At maximum value of ξ, i.e at ξ_a, we note that $\dot{\xi}_a^2 = 0$. Thus, we get

$$\frac{1}{2}(1-\lambda)\xi_a^2 + \frac{\alpha}{4}\xi_a^4 - \frac{\beta}{5}\xi_a^6 = \lambda\bar{\xi}\xi_a$$

This implies

$$(1-\lambda)\xi_a + \frac{\alpha}{2}\xi_a^3 - \frac{2\beta}{5}\xi_a^5 = 2\lambda\bar{\xi} \quad (15)$$

To determine the dynamic buckling load λ_D, we use the condition (13) which yields

$$(1-\lambda_D) + \frac{3\alpha}{2}\xi_D^2 - 2\beta\xi_D^4 = 0 \qquad (16)$$

where, ξ_D is the value of ξ_a at dynamic buckling.This yields

$$2\beta\xi_D^4 - \frac{3\alpha}{2}\xi_D^2 - (1-\lambda_D) = 0 \qquad (17)$$

The solution of (17) is

$$\xi_D^2 = \frac{\frac{3\alpha}{2} \pm \sqrt{\frac{9\alpha^2}{4} + 2\beta(1-\lambda_D)}}{4\beta} \quad (18)$$

If we take the positive square root sign, we get

$$\xi_D^2 = \frac{\frac{3\alpha}{2} + 2\sqrt{2}\beta^{\frac{1}{2}}(1-\lambda_D)^{\frac{1}{2}} r_3}{4\beta} \quad (19a)$$

where,

$$r_3 = \sqrt{1 + \frac{9}{132(1-\lambda_D)}\left(\frac{\alpha^2}{\beta}\right)} \qquad (19b)$$

Simplifying ξ_D^2 further, we get

$$\xi_D^2 = \frac{\sqrt{2}}{2}\beta^{-\frac{1}{2}}(1-\lambda_D)^{\frac{1}{2}}r_3\left[1+\frac{3\left(\frac{\alpha}{\beta^{\frac{1}{2}}}\right)}{4\sqrt{2}(1-\lambda_D)^{\frac{1}{2}}r_3}\right] \tag{20}$$

$$\therefore\ \xi_D = \left[\frac{\sqrt{2}}{2}\beta^{-\frac{1}{2}}(1-\lambda_D)^{\frac{1}{2}}r_3\left\{1+\frac{3}{4\sqrt{3}(1-\lambda_D)^{\frac{1}{2}}r_3}\left(\frac{\alpha}{\beta^{\frac{1}{2}}}\right)\right\}\right]^{\frac{1}{2}} \tag{21}$$

Further simplification of (21) yields

$$\xi_D = \left\{\frac{\sqrt{2}}{2}\frac{(1-\lambda_D)^{\frac{1}{2}}r_3}{\beta^{\frac{1}{2}}}\right\}^{\frac{1}{2}} r_4 \tag{22a}$$

where,

$$r_4 = \left[1+\frac{3}{4\sqrt{2}(1-\lambda_D)^{\frac{1}{2}}r_3}\left(\frac{\alpha}{\beta^{\frac{1}{2}}}\right)\right]^{\frac{1}{2}} \tag{22b}$$

To determine the dynamic buckling load, λ_D we determine (15) at $\xi_a = \xi_D$ to get

$$\xi_D\left[(1-\lambda_D)+\xi_D^2\left(\frac{\alpha}{2}-\frac{2\beta}{5}\xi_D^2\right)\right] = 2\lambda_D\bar{\xi}(23)$$

This gives

$$2\lambda_D\bar{\xi} = \left\{\frac{\sqrt{2}}{2}\frac{(1-\lambda_D)^{\frac{1}{2}}r_3}{\beta^{\frac{1}{2}}}\right\}^{\frac{1}{2}} r_4\left[(1-\lambda_D)+\frac{(1-\lambda_D)^{\frac{1}{2}}r_3r_4^2}{\sqrt{2}\beta^{\frac{1}{2}}}\left(\frac{\alpha}{2}-\frac{2\beta}{5}\xi_D^2\right)\right] \tag{24}$$

Final simplification of (24) yields

$$2\lambda_D\bar{\xi} = \frac{(1-\lambda_D)^{\frac{5}{4}}r_3^{\frac{1}{2}}r_4}{\left(\sqrt{2}\beta^{\frac{1}{2}}\right)^{\frac{1}{2}}}\left[1+\frac{r_3r_4^2}{2\sqrt{2}(1-\lambda_D)^{\frac{1}{2}}}\left(\frac{\alpha}{\beta^{\frac{1}{2}}}\right)\left(1-\frac{4}{5}\xi_D^2\right)\right] \tag{25}$$

For all the analysis so far, it is certain that we must choose $\alpha > 0,\ \beta > 0$.
From (10e) and (25), we can relate the static buckling load λ_S to the dynamic buckling load λ_D by

$$\left(\frac{1-\lambda_D}{1-\lambda_S}\right)^{\frac{5}{4}} = \left(\frac{2\sqrt{2}}{\sqrt{5}}\right)^{\frac{1}{2}}\left(\frac{r_{1S}}{r_3}\right)^{\frac{1}{2}}\left(\frac{r_{2S}}{r_4}\right)\frac{\lambda_D}{\lambda_S}\left[\frac{1+\frac{r_{1S}r_{2S}^2\left(\frac{\alpha}{\beta^{\frac{1}{2}}}\right)}{\sqrt{5}(1-\lambda_D)^{\frac{1}{2}}}\left\{1-\frac{(1-\lambda_D)^{\frac{1}{2}}r_{1S}r_{2S}\left(\frac{\beta^{\frac{1}{2}}}{\alpha}\right)}{\sqrt{5}}\right\}}{1+\frac{r_3r_4^2}{2\sqrt{2}}(1-\lambda_D)^{\frac{1}{2}}\left(\frac{\alpha}{\beta^{\frac{1}{2}}}\right)\left(1-\frac{4}{5}\xi_D^2\right)}\right] \tag{26}$$

Based on the fact that the analysis leading to the results of (10e), (25) and (26) has not encountered any form of approximation, we may regard these results as exact solutions obtained from phase plane consideration. Such a method may not, in most cases, lend itself to easy application when actual engineering structures are encountered in practice. Thus, for this other type of materials (engineering materials),we can only seek for approximate results obtained from approximate methods such as asymptotic and perturbation methods. To buttress the applicability of these alternative methods we shall use them here and obtain approximate results with a view to comparing the exact and approximate solutions obtained.

IV. Asymptotic And Perturbation Methods

(a) ASYMPTOTIC METHODS FOR OBTAINING THE STATIC BUCKLING LOAD, λ_S.

Here, we recast equation (4) as

$$(1-\lambda)\xi + \alpha\xi^3 - \beta\xi^5 = \lambda\bar{\xi} \tag{27}$$

Let

$$\xi = \sum_{i=1}^{\infty}\xi_i\bar{\xi}^i \tag{28}$$

On substituting (28) into (27) and equating the coefficients of $\bar{\xi}$, we get

$$\boldsymbol{O}(\bar{\xi}):\ (1-\lambda)\xi_1 = \lambda \tag{29}$$

$$\boldsymbol{O}(\bar{\xi}^2):\ (1-\lambda)\xi_2 = 0 \tag{30}$$
$$\boldsymbol{O}(\bar{\xi}^3):\ (1-\lambda)\xi_3 = -\alpha\xi_1^3 \tag{31}$$
$$\boldsymbol{O}(\bar{\xi}^4):\ (1-\lambda)\xi_4 = 3\alpha\xi_1^2\xi_2 \tag{32}$$
$$\boldsymbol{O}(\bar{\xi}^5):\ (1-\lambda)\xi_5 = \beta\xi_1^5 - 3\xi_1^2\xi_3 \tag{33}$$

From (29), we get

$$\xi_1 = B = \frac{\lambda}{1-\lambda} \tag{34}$$

From (30), we get

$$\xi_2 = 0 \tag{35}$$

From (31), we get

$$\xi_3 = \frac{-\alpha\xi_1^3}{1-\lambda} = \frac{-\alpha B^3}{1-\lambda} \tag{36}$$

On substituting for ξ_2 in (32), we get $\xi_4 = 0$.
We substitute for ξ_1 and ξ_3 in (33), and get

$$\xi_5 = \frac{1}{1-\lambda}(\beta\xi_1^5 - 3\alpha\xi_1^2\xi_3) = \frac{B^5\beta}{1-\lambda}\left(1+\frac{3}{1-\lambda}\frac{\alpha}{\beta}\right) \tag{37}$$

So far, we get

$$\xi = c_1\bar{\xi} + c_3\bar{\xi}^3 + c_5\bar{\xi}^5 + \cdots \tag{38a}$$

where,

$$c_1 = 2B, \quad c_3 = \frac{\alpha B^3}{1-\lambda}, \quad c_5 = \frac{B^5\beta\Omega_1}{(1-\lambda)^2} \tag{38b}$$

$$\Omega_1 = \left[(1-\lambda) + 3\left(\frac{\alpha}{\beta}\right)\right] \tag{38c}$$

As in [4], the static buckling load is obtained by first reversing the series (38a) to get

$$\bar{\xi} = d_1\xi + d_3\xi^3 + d_5\xi^5 + \cdots \tag{39}$$

By substituting in (39) for ξ and equating the coefficients of powers of $\bar{\xi}$, we get

$$d_1 = \frac{1}{c_1} = \frac{1}{B}, \qquad d_3 = \frac{-c_3}{c_1^4} = \frac{\alpha}{\lambda} \tag{40a}$$

$$d_5 = \frac{3c_3^2 - c_1c_5}{c_1^7} = \frac{-c_1c_5\left(1-\frac{3c_3^2}{c_1c_5}\right)}{c_1^7} = \frac{-\beta\Omega_1\Omega_2}{\lambda(1-\lambda)}, \quad \Omega_2 = \left[1-\frac{3}{\Omega_1}\left(\frac{\alpha^2}{\beta}\right)\right] \tag{40b}$$

The static buckling load condition (4) is now easily executed from (39) to get

$$d_1 + 3d_3\xi_S^2 + 5d_5\xi_S^4 = 0 \tag{41}$$

whereξ_S is the value of ξ at static buckling. On solving (41), we get

$$\xi_S^2 = \frac{-3d_3 \pm \sqrt{(3d_3)^2 - 20d_1d_5}}{10d_5} \tag{42a}$$

After substituting in (42a) for the relevant terms, we get

$$\xi_S^2 = -\left[\frac{-3\left(\frac{\alpha}{\lambda_S}\right) \pm \sqrt{\frac{9\alpha^2}{\lambda_S^2}+\frac{20\beta\Omega_1\Omega_2}{\lambda_S^2}}}{\frac{10\beta\Omega_1\Omega_2}{\lambda_S(1-\lambda_S)}}\right] \tag{42b}$$

By choosing the negative root sign and simplifying further, we get

$$\xi_S^2 = \frac{(1-\lambda_S)\Omega_3}{10\Omega_1\Omega_2}\left(\frac{\alpha}{\beta}\right) \tag{42c}$$

where,

$$\Omega_3 = 3 + \sqrt{9+20\left(\frac{\beta}{\alpha^2}\right)\Omega_1\Omega_2} \tag{42d}$$

and where Ω_1 and Ω_2 are here evaluated at $\lambda = \lambda_S$.

$$\therefore \quad \xi_S = \left\{\frac{(1-\lambda_S)\Omega_3}{10\Omega_1\Omega_2}\left(\frac{\alpha}{\beta}\right)\right\}^{\frac{1}{2}} \tag{42}$$

where, we have here taken the positive square root sign. To determine the static buckling load in this case, we now evaluate (39) at buckling and get

$$\bar{\xi} = \xi_S[d_1 + \xi_S^2(d_3 + d_5\xi_S^2)] \tag{43}$$

Simplifying this, we get

$$\bar{\xi} = \frac{(1-\lambda_S)^{\frac{3}{2}}\left(\frac{\alpha}{\beta}\right)^{\frac{1}{2}}\Omega_3^{\frac{1}{2}}}{\sqrt{10}(\Omega_1\Omega_2)^{\frac{1}{2}}\lambda_S}\left[1+\frac{\Omega_3\left(\frac{\alpha^2}{\beta}\right)}{10\Omega_1\Omega_2}\left(1-\frac{\Omega_3}{10}\right)\right] \qquad (44)$$

Here, we have taken $\alpha > 0$, $\beta > 0$, and all values of λ are taken as at $\lambda = \lambda_S$.

(b) ASYMPTOTIC SOLUTION OF THE DYNAMIC PROBLEM (11a, b)

Here, we recast (11a, b) in full as

$$\ddot{\xi} + (1-\lambda)\xi + \alpha\xi^3 - \beta\xi^5 = \lambda\bar{\xi} \qquad (45a)$$
$$\xi(0) = \dot{\xi}(0) = 0 \qquad (45b)$$

Let

$$\hat{t} = (1-\lambda)^{\frac{1}{2}}\left(1+\mu_2\bar{\xi}^2+\mu_3\bar{\xi}^3+\mu_4\bar{\xi}^4+\cdots\right)^{\frac{1}{2}}t \qquad (46)$$

$$\therefore\ \frac{d\xi}{dt} = \frac{d\xi}{d\hat{t}}\frac{d\hat{t}}{dt} = (1-\lambda)^{\frac{1}{2}}\left(1+\mu_2\bar{\xi}^2+\mu_3\bar{\xi}^3+\mu_4\bar{\xi}^4+\cdots\right)^{\frac{1}{2}}\frac{d\xi}{d\hat{t}} \qquad (47a)$$

$$\frac{d^2\xi}{dt^2} = (1-\lambda)\left(1+\mu_2\bar{\xi}^2+\mu_3\bar{\xi}^3+\mu_4\bar{\xi}^4+\cdots\right)\frac{d^2\xi}{d\hat{t}^2} \qquad (47b)$$

We now substitute (46) – (47b) into (45a, b) and use the asymptotic series

$$\xi = \sum_{i=1}^{\infty} A^{(i)}(\hat{t})\bar{\xi}^i \qquad (48)$$

The following are the sequence of equations obtained.

$$\boldsymbol{O}(\bar{\xi}):\ \frac{d^2A^{(1)}}{d\hat{t}^2} + A^{(1)} = B = \frac{\lambda}{1-\lambda} \qquad (49)$$

$$\boldsymbol{O}(\bar{\xi}^2):\ \frac{d^2A^{(2)}}{d\hat{t}^2} + A^{(2)} = 0 \qquad (50)$$

$$\boldsymbol{O}(\bar{\xi}^3):\ \frac{d^2A^{(3)}}{d\hat{t}^2} + A^{(3)} = \frac{-\alpha\left(A^{(1)}\right)^3}{1-\lambda} - \mu_2\frac{d^2A^{(1)}}{d\hat{t}^2} \qquad (51)$$

$$\boldsymbol{O}(\bar{\xi}^4):\ \frac{d^2A^{(4)}}{d\hat{t}^2} + A^{(4)} = \frac{-3\alpha\left(A^{(1)}\right)^2A^{(2)}}{1-\lambda} - \mu_2\frac{d^2A^{(2)}}{d\hat{t}^2} - \mu_3\frac{d^2A^{(1)}}{d\hat{t}^2} \qquad (52)$$

$$\boldsymbol{O}(\bar{\xi}^5):\ \frac{d^2A^{(5)}}{d\hat{t}^2} + A^{(5)}$$
$$= \frac{-3\alpha}{1-\lambda}\left(\left(A^{(1)}\right)^2A^{(3)} + A^{(1)}A^{(2)^2}\right) + \frac{\beta\left(A^{(1)}\right)^5}{1-\lambda} - \mu_2\frac{d^2A^{(3)}}{d\hat{t}^2} - \mu_3\frac{d^2A^{(2)}}{d\hat{t}^2}$$
$$+ \mu_4\frac{d^2A^{(1)}}{d\hat{t}^2} \qquad (53)$$

etc.

The associated initial conditions are

$$A^{(i)}(0) = 0, \quad \frac{dA^{(i)}}{d\hat{t}}(0) = 0, \quad i = 1, 2, 3, \ldots \qquad (54)$$

On solving (49), we get (using I = 1 in (54))

$$A^{(1)}(\hat{t}) = B(\cos\hat{t} - 1) \qquad (55)$$

From (50), for $i = 2$ in (54), we get

$$A^{(2)}(\hat{t}) = 0 \qquad (56)$$

We next substitute into (51) and to ensure a uniformly valid solution in $\hat{t}$, we equate to zero the coefficient of $\cos\hat{t}$ and get

$$\mu_2 = \frac{15\alpha B^2}{4(1-\lambda)} \qquad (57)$$

The remaining equation in the substitution into (51) is

$$\frac{d^2A^{(3)}}{d\hat{t}^2} + A^{(3)} = \frac{-\alpha B^3}{(1-\lambda)}\left[-\frac{5}{2} - \frac{B\cos 2\hat{t}}{2} + \frac{\cos 3\hat{t}}{4}\right] \qquad (58a)$$

$$A^{(3)}(0) = 0, \quad \frac{dA^{(3)}}{d\hat{t}}(0) = 0 \qquad (58b)$$

On solving (58a, b), we get

$$A^{(3)}(\hat{t}) = a_3 \cos \hat{t} + \frac{\alpha B^3}{(1-\lambda)}\left[\frac{5}{2} - \frac{\cos 2\hat{t}}{2} + \frac{\cos 3\hat{t}}{32}\right] \qquad (59a)$$

where,

$$a_3 = \frac{-65\alpha B^3}{32(1-\lambda)} \qquad (59b)$$

On substituting (55) and (56) into (52), we observe that to ensure a uniformly valid solution in $\hat{t}$, we must choose $\mu_3 = 0$. On solving the remaining equation in (52), we get

$$A^{(4)}(\hat{t}) = 0$$

The following simplifications are necessary in the substitutions which soon follow:

$$\begin{aligned}\left(A^{(1)}\right)^2 A^{(3)} = &\left[\frac{29\alpha B^5}{8(1-\lambda)}\right. \\ &+ \left(\frac{7a_3}{4} - \frac{2875B^3}{640}\right)B^2 \cos \hat{t} \\ &+ B^2\left(-a_3 + \frac{15\alpha B^3}{32(1-\lambda)}\right)\cos 2\hat{t} \\ &+ B^2\left(\frac{a_3}{4} + \frac{35\alpha B^3}{64(1-\lambda)}\right)\cos 3\hat{t} \\ &\left.- \frac{5\alpha B^5}{32(1-\lambda)}\cos 4\hat{t} + \frac{\alpha B^5}{128(1-\lambda)}\cos 5\hat{t}\right] \qquad (60)\end{aligned}$$

$$\left(A^{(1)}\right)^5 = B^5(\cos \hat{t} - 1)^5 = B^5(\cos^5 \hat{t} - 5\cos^4 \hat{t} + 10\cos^3 \hat{t} - 10\cos^2 \hat{t} + 5\cos \hat{t} - 1) \qquad (61a)$$

Further expansion of (61a) gives

$$\left(A^{(1)}\right)^5 = B^5\left[-\frac{27}{8} + \frac{105\cos \hat{t}}{8} - 3\cos 2\hat{t} + \frac{45\cos 3\hat{t}}{16} - \frac{5\cos 4\hat{t}}{8} + \frac{\cos 5\hat{t}}{16}\right] \qquad (61b)$$

We now substitute (61a, b) into (53) and get

$$\begin{aligned}\frac{d^2A^{(5)}}{d\hat{t}^2} + A^{(5)} = &-\mu_2\left[-a_3 cos\hat{t} + \frac{\alpha B^3}{1-\lambda}\left(2cos2\hat{t} - \frac{9cos3\hat{t}}{32}\right)\right] + \mu_4 B cos\hat{t} \\ &- \frac{3\alpha}{1-\lambda}\left[\frac{29\alpha B^5}{8(1-\lambda)} + \left(\frac{7a_3}{4} - \frac{2875B^3}{640}\right)cos\hat{t} + B^2\left(\frac{15\alpha B^3}{32(1-\lambda)} - a_3\right)cos2\hat{t}\right. \\ &\left.+ B^2\left(\frac{a_3}{4} + \frac{15\alpha B^3}{32(1-\lambda)}\right)cos3\hat{t} - \frac{5\alpha B^5}{32(1-\lambda)}\cos 4\hat{t} + \frac{\alpha B^5}{128(1-\lambda)}\cos 5\hat{t}\right] \\ &+ \frac{\beta B^5}{1-\lambda}\left[-\frac{27}{8} + \frac{105\cos \hat{t}}{8} - 3\cos 2\hat{t} + \frac{45\cos 3\hat{t}}{16} - \frac{5\cos 4\hat{t}}{8} + \frac{\cos 5\hat{t}}{16}\right] \qquad (62)\end{aligned}$$

To ensure a uniformly valid solution in $\hat{t}$, we equate to zero in (62), the coefficient of $\cos \hat{t}$and get

$$\mu_4 = -\frac{1}{B}\left[\frac{3\alpha}{1-\lambda}\left(\frac{7a_3B^2}{4} - \frac{2875B^5}{640}\right) + \frac{105\beta B^5}{8(1-\lambda)} + \mu_2 a_3\right] \qquad (63)$$

The remaining equation in (62) is

$$\frac{d^2A^{(5)}}{d\hat{t}^2} + A^{(5)} = q_1 + q_2 cos2\hat{t} + q_3 cos3\hat{t} + q_4 cos4\hat{t} + q_5 cos5\hat{t} \qquad (64a)$$

$$A^{(5)}(0) = 0, \quad \frac{dA^{(5)}}{d\hat{t}}(0) = 0 \qquad (64b)$$

where,

$$q_1 = \frac{87\alpha^2 B^5}{8(1-\lambda)^2} - \frac{27\beta B^5}{8(1-\lambda)} = q_1(0) \qquad (64c)$$

$$q_2 = \left[\frac{3\beta B^5}{(1-\lambda)} - \frac{3\alpha\mu_2 B^3}{1-\lambda} - \frac{3\alpha B^2}{1-\lambda}\left(a_3 - \frac{15\alpha B^2}{32(1-\lambda)}\right)\right] \qquad (64d)$$

$$q_3 = \left[\frac{9\alpha\mu_2 B^3}{32(1-\lambda)} - \frac{3\alpha B^2}{1-\lambda}\left(\frac{a_3}{4} + \frac{35\alpha B^3}{64(1-\lambda)}\right) + \frac{45\beta B^5}{16(1-\lambda)}\right] \qquad (64e)$$

$$q_4 = \frac{15\alpha^2 B^5}{32(1-\lambda)^2} - \frac{5\beta B^5}{8(1-\lambda)} = q_4(0) \qquad (64f)$$

$$q_5 = \frac{\beta B^5}{16(1-\lambda)} - \frac{3\alpha^2 B^5}{128(1-\lambda)^2} = q_5(0) \qquad (64g)$$

where,

$$q_2(0) = \frac{3\beta B^5}{(1-\lambda)} - \frac{15\alpha^2 B^5}{(1-\lambda)^2} \quad (64h)$$

$$q_3(0) = \frac{45\beta B^5}{16(1-\lambda)} + \frac{15\alpha^2 B^5}{16(1-\lambda)^2} \quad (64i)$$

On solving (64a, b), using (64c – i), we obtain

$$A^{(5)} = a_5 cos\hat{t} + q_1 - \frac{q_2 cos2\hat{t}}{3} - \frac{q_3 cos3\hat{t}}{8} - \frac{q_4 cos4\hat{t}}{15} - \frac{q_5 cos5\hat{t}}{24} \quad (65a)$$

where,

$$a_5 = -q_1 + \frac{q_2}{3} + \frac{q_3}{8} + \frac{q_4}{15} + \frac{q_5}{24} = \frac{904\beta B^5}{197(1-\lambda)} - \frac{16105\alpha^2 B^5}{1024(1-\lambda)^2} \quad (65b)$$

Thus far, we can write

$$\xi = A^{(1)}\bar{\xi} + A^{(3)}\bar{\xi}^3 + A^{(5)}\bar{\xi}^5 + \cdots \quad (66)$$

We next determine the maximum displacement ξ_a subsequent upon which we shall apply the maximization (13) to determine the dynamic buckling load,λ_D. The usual condition for maximum is $\frac{d\xi}{d\hat{t}} = 0$ and this gives the least value of $\hat{t}$ at maximum displacement as

$$\hat{t}_a = \pi \quad (67)$$

By evaluating (66) at this maximum value, we get

$$\xi_a = e_1\bar{\xi} + e_3\bar{\xi}^3 + e_5\bar{\xi}^5 + \cdots \quad (68)$$

where,

$$e_1 = 2B, \quad e_3 = \frac{33\alpha B^3}{8(1-\lambda)}, \quad e_5 = \frac{1275\alpha^2 B^5}{16(1-\lambda)^2}\varphi_1 \quad (69a)$$

where,

$$\varphi_1 = \left(1 - \frac{494(1-\lambda)}{381}\left(\frac{\beta}{\alpha^2}\right)\right) \quad (69b)$$

We next reverse the series (69a) as

$$\bar{\xi} = f_1\,\xi_a + f_3\xi_a^3 + f_5\xi_a^5 + \cdots \quad (70)$$

By substituting for ξ_a from (68) in (70) and equating the coefficients of $\bar{\xi}$, we get

$$f_1 = \frac{1}{e_1}, \quad f_3 = -\frac{e_3}{e_1^4}, \quad f_5 = \frac{3e_3^2 - e_1e_2}{e_1^7} \quad (71a)$$

This gives

$$f_1 = \frac{1}{2B}, \quad f_3 = -\frac{33\alpha}{128\lambda}, \quad f_5 = \frac{-127\alpha^2}{1024\lambda(1-\lambda)}\varphi_1\varphi_2 \quad (71b)$$

where,

$$\varphi_2 = \left(1 - \frac{326\lambda^6}{64(1-\lambda)^8}\right) \quad (71c)$$

The maximization (13) to determine the dynamic buckling load λ_D is better achieved using (70) to obtain

$$f_1 + 3f_3\xi_{aD}^2 + 5f_5\xi_{aD}^4 = 0 \quad (72)$$

where, ξ_{aD} is the value of ξ_a at dynamic buckling and where f_i, $i = 1, 3, 5$, in (72) are determined at $\lambda = \lambda_D$. On solving (72), we get

$$\xi_{aD}^2 = \frac{-3f_3 \pm \sqrt{9f_3^2 - 20f_1f_5}}{10f_5} \quad (73)$$

By taking the negative root sign and simplifying (73) a number of times, we get, for $\alpha > 0, \beta > 0$

$$\xi_{aD}^2 = \frac{10{,}197}{20{,}320\varphi_1\varphi_2}\left(\frac{1+\varphi_3}{\alpha\beta}\right) \quad (74)$$

where,

$$\varphi_3 = \left[1 + \left(\frac{520{,}665}{2{,}509{,}056}\right)\left(\frac{\varphi_1\varphi_2\beta^2}{\lambda_D^2}\right)\right]^{\frac{1}{2}} \quad (75)$$

Thus, we have

$$\xi_{aD} = \sqrt{\frac{10{,}197}{20{,}320\varphi_1\varphi_2}\left(\frac{1+\varphi_3}{\alpha\beta}\right)} \quad (76)$$

To determine the dynamic buckling load λ_D, we have to determine (70) at dynamic buckling stage, and get

$$\bar{\xi} = \xi_{aD}[f_1 + \xi_{aD}^2(f_3 + f_5\xi_{aD}^2)] \quad (77)$$

This gives, using the positive square root sign,

$$2\bar{\xi} = \left(\frac{10{,}197}{20{,}320\varphi_1\varphi_2}\left(\frac{1+\varphi_3}{\alpha\beta}\right)\left(\frac{1-\lambda_D}{\lambda_D}\right)\right)\left[1 - \frac{2\lambda_D\xi_{aD}^2}{(1-\lambda_D)}\left\{\frac{33\alpha}{128\lambda_D} + \frac{127\alpha^2(\varphi_1\varphi_2)\xi_{aD}^2}{1024\lambda_D(1-\lambda_D)}\right\}\right] \quad (78)$$

Equation (78) is an implicit expression for determining the dynamic buckling load λ_D. We remark that all functions of λ in (78) are to be evaluated at $\lambda = \lambda_D$. Using (78) and (44), we can relate the dynamic buckling load λ_D to its static equivalent λ_S as shown below.

$$2 = \frac{\left(\frac{10{,}197}{20{,}320\varphi_1\varphi_2}\left(\frac{1+\varphi_3}{\alpha\beta}\right)\left(\frac{1-\lambda_D}{\lambda_D}\right)\right)\left[1 - \frac{2\lambda_D\xi_{aD}^2}{(1-\lambda_D)}\left\{\frac{33\alpha}{128\lambda_D} + \frac{127\alpha^2(\varphi_1\varphi_2)\xi_{aD}^2}{1024\lambda_D(1-\lambda_D)}\right\}\right]}{\frac{(1-\lambda_S)^{\frac{3}{2}}\left(\frac{\alpha}{\beta}\right)^{\frac{1}{2}}\Omega_3^{\frac{1}{2}}}{\sqrt{10}(\Omega_1\Omega_2)^{\frac{1}{2}}\lambda_S}\left[1 + \frac{\Omega_3\left(\frac{\alpha^2}{\beta}\right)}{10\Omega_1\Omega_2}\left(1 - \frac{\Omega_3}{10}\right)\right]} \quad (79)$$

V. Analysis Of Result

The results (10e), (25) and (26) were derived using phase plane analysis and to the extent that no approximations were encountered in their derivation, they could be said to be exact results. However, the results (44), (75) and (79) were derived using asymptotic and perturbation procedures. The analysis is such that in both exact and approximate results, we were able to relate the dynamic buckling load to its static equivalent showing that if, say, the dynamic load is already known, then the static buckling load canin principle, be easily calculated without the labour of carrying out the tedious calculation all over for the other buckling load. Using the above results, simple programming codes written in Q – Basic, are used to generate the tables given in Table 1 to Table 6 and the resultant graphical plots in Figure 1 to Figure 9.The results obtained from asymptotic and perturbation methods are strictly asymptotic in nature and are valid as the small parameter $\bar{\xi}$ becomes strictly less than unity.

VI. Conclusion

While phase plane technique was easily available to analyse the problem and the results obtained proved excellent and easy to use, the perturbation and asymptotic approaches on the other hand could be tedious and in some cases cumbersome in most applications. However, most equations of motions charactererising actual real – life engineering structures are ridden with excessive spatial nonlinearities to the extent that phase plane method cannot possibly encompass. In such cases, asymptotic and perturbation methods are likely to hold sway. Thus, any real – life engineering structures with cubic – quintic nonlinearity may be easily analyzed by using perturbation and asymptotic procedures with the aim that the small parameter $\bar{\xi}$ be extremely less than unity.

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Table 1: Relationship between the imperfection parameter $\bar{\xi}$ and the static buckling load λ_s for the case of phase plane method.

IMPERFECTION PARAMETER $\bar{\xi}$	STATIC BUCKLING LOAD λ_S
0.01	0.551108
0.02	0.491879
0.03	0.455144
0.04	0.428261
0.05	0.407003
0.06	0.389409
0.07	0.374403
0.08	0.361323
0.09	0.349736

Table 2. Relationship Between The Imperfection Parameter $\bar{\xi}$ And The Dynamic Buckling Loadλ_DFor The Case Of Phase Plane Method.

IMPERFECTION PARAMETER $\bar{\xi}$	DYNAMIC BUCKLING LOAD λ_D
0.01	0.464539
0.02	0.393835
0.03	0.351511
0.04	0.321366
0.05	0.298069
0.06	0.279174
0.07	0.263349
0.08	0.249789
0.09	0.237966

Table 3. Relationship Between The Static Buckling Loadλ_SAnd The Dynamic Buckling Load λ_DFor The Case Of Phase Plane Method.

STATIC BUCKLING LOAD λ_S	DYNAMIC BUCKLING LOAD λ_D
0.1	0.050845
0.15	0.081813
0.2	0.117159
0.25	0.152179
0.3	0.190639
0.35	0.230778
0.4	0.272314
0.45	0.315031
0.5	0.358778
0.55	0.403469
0.6	0.449098
0.65	0.475758

0.7	0.543681
0.75	0.593331
0.8	0.645553
0.85	0.701925
0.9	0.765648
0.95	0.844905

Table 4: Relationship Between the Imperfection Parameter $\bar{\xi}$ And the Static Buckling Load λ_S for the Case Of Asymptotic Method.

IMPERFECTION PARAMETER $\bar{\xi}$	**STATIC BUCKLING LOAD λ_S**
0.01	0.936227
0.02	0.894295
0.03	0.861736
0.04	0.834695
0.05	0.811379
0.06	0.790786
0.07	0.772287
0.08	0.755459
0.09	0.740004

Table 5. Relationship Between The Imperfection Parameter $\bar{\xi}$ And The Dynamic Buckling Load λ_D For The Case Of Asymptotic Method.

IMPERFECTION PARAMETER $\bar{\xi}$	**DYNAMIC BUCKLING LOAD λ_D**
0.01	0.678867
0.02	0.659641
0.03	0.636488
0.04	0.611767
0.05	0.558079
0.06	0.566591
0.07	0.547347
0.08	0.529969
0.09	0.513981

Table 6.Relationship Between The Static Buckling Load And The Dynamic Buckling Load For The Case Of Asymptotic Method.

STATIC BUCKLING LOAD λ_S	**DYNAMIC BUCKLING LOAD λ_D**
0.1	0.071238
0.15	0.071163
0.2	0.071127
0.25	0.071106
0.3	0.071092
0.35	0.071083
0.4	0.071076
0.45	0.071071
0.5	0.071068
0.55	0.071065

0.6	0.071063
0.65	0.071061
0.7	0.071059
0.75	0.071058
0.8	0.071057
0.85	0.071056
0.9	0.071056
0.95	0.071055

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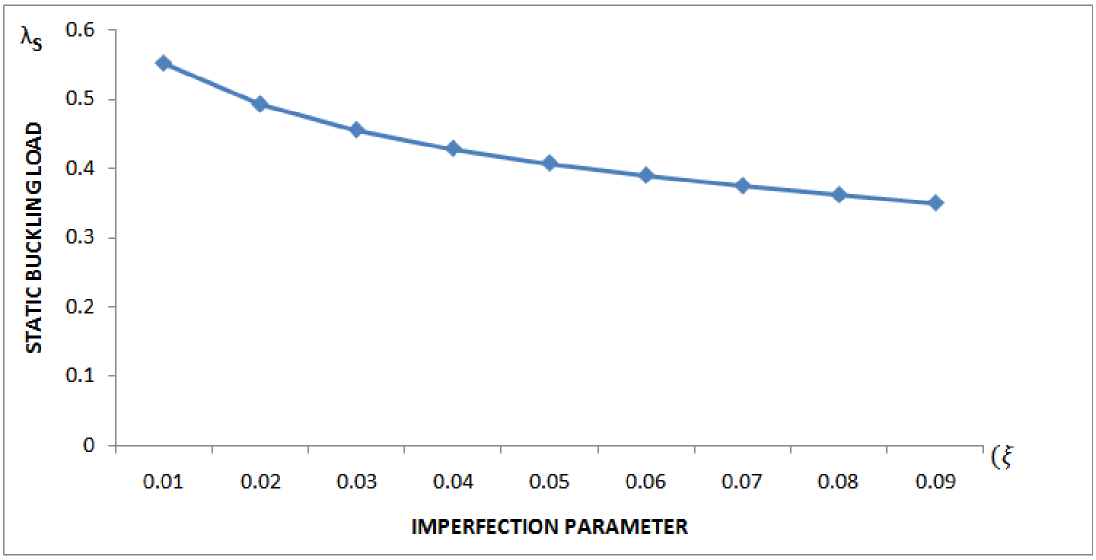

Figure 1. Graphical PlotOf Table 1.(Using Eqn. (10e)).

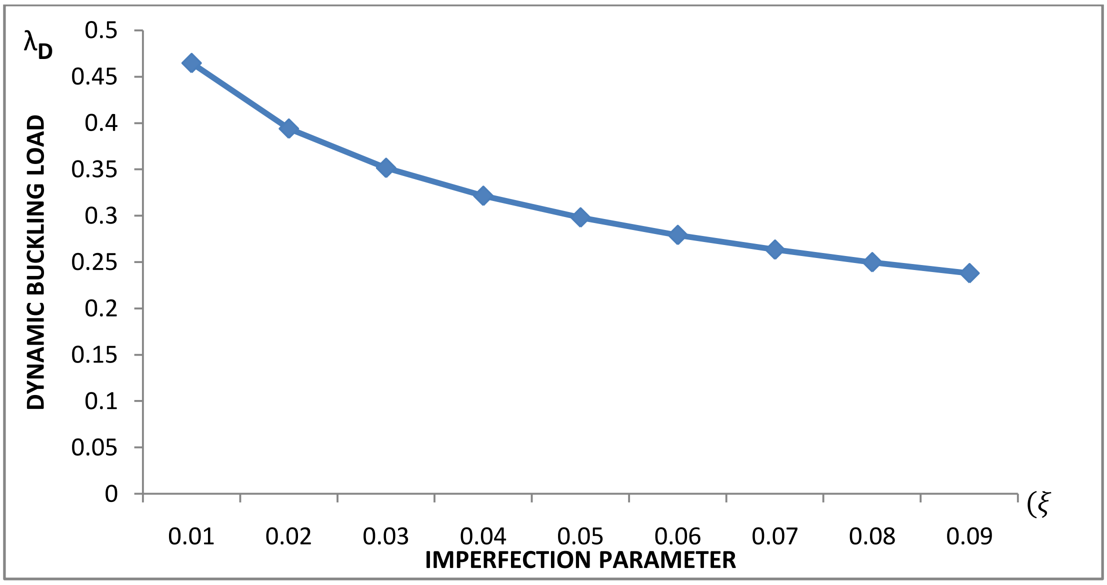

Figure 2. Graphical Plot Of Table 2.(Using Eqn. (25))

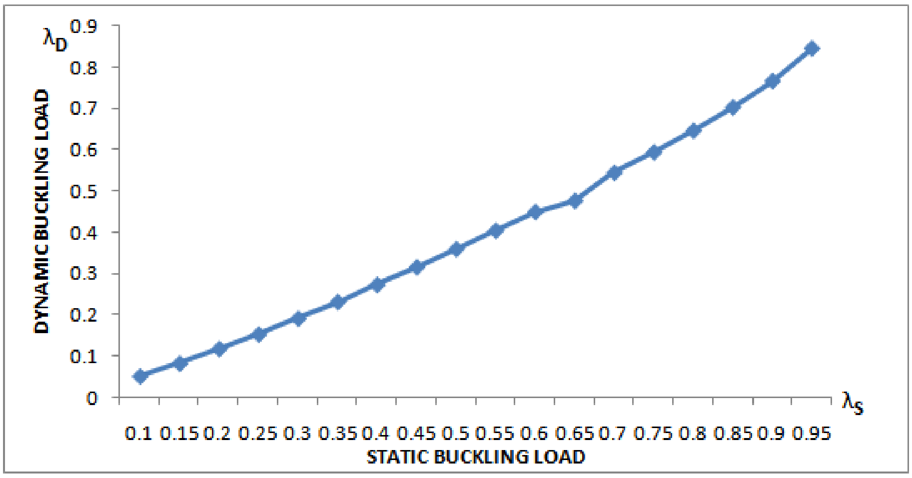

Figure 3. Graphical PlotOf Table 3. (Using Eqn. (26))

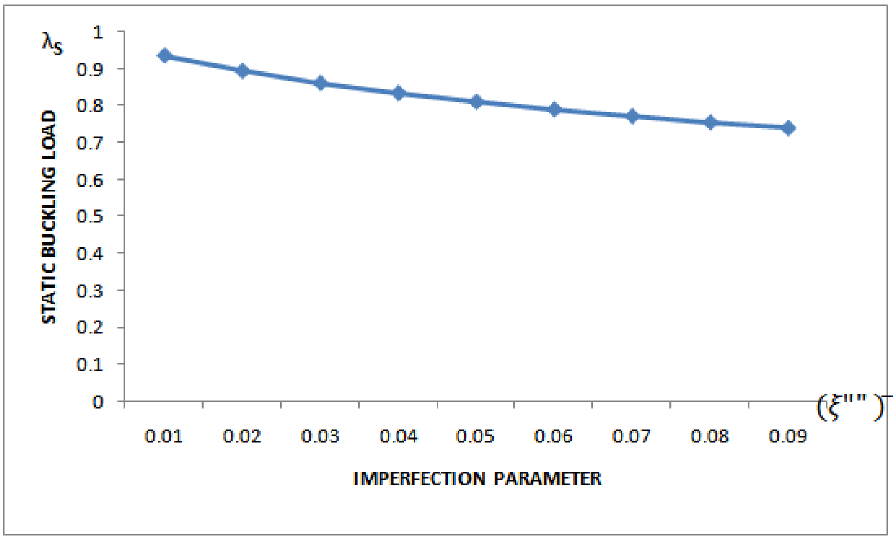

Figure 4. Graphical PlotOf Table 4.(Using Eqn. (44)).

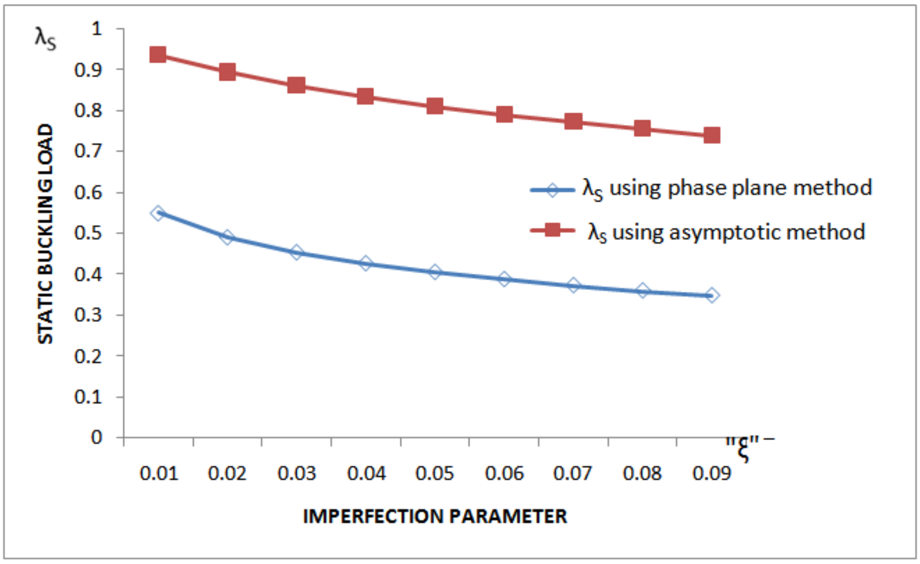

Figure 5.Graphical Plot Comparing The Static Buckling Loads Using Phase Plane Method And Asymptotic Method For Various Values Of The Imperfection Parameter (Using Eqn. (10e) Andeqn. (44)).

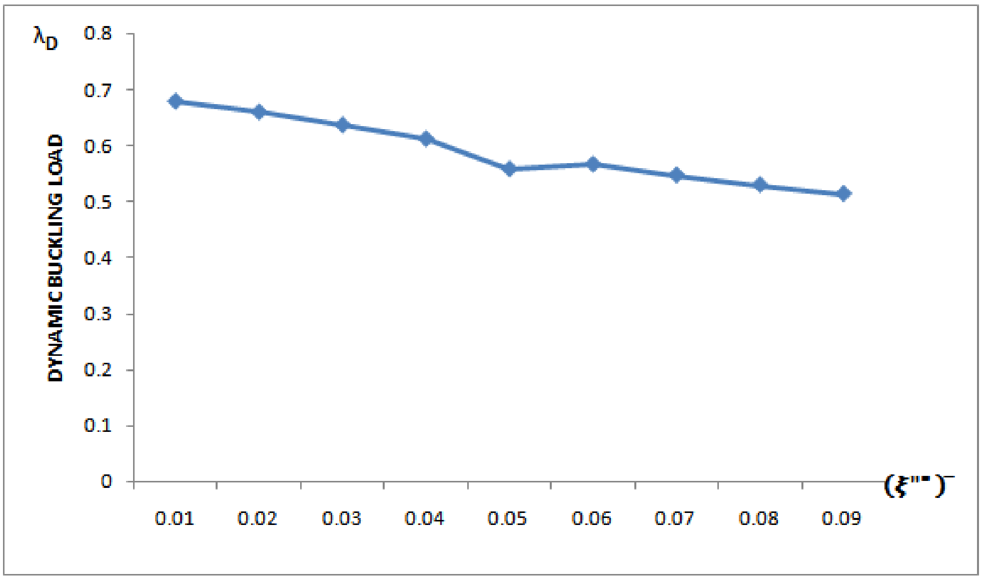

Figure 6. Graph Plot Of Table 5, (Using Eqn. (78)).

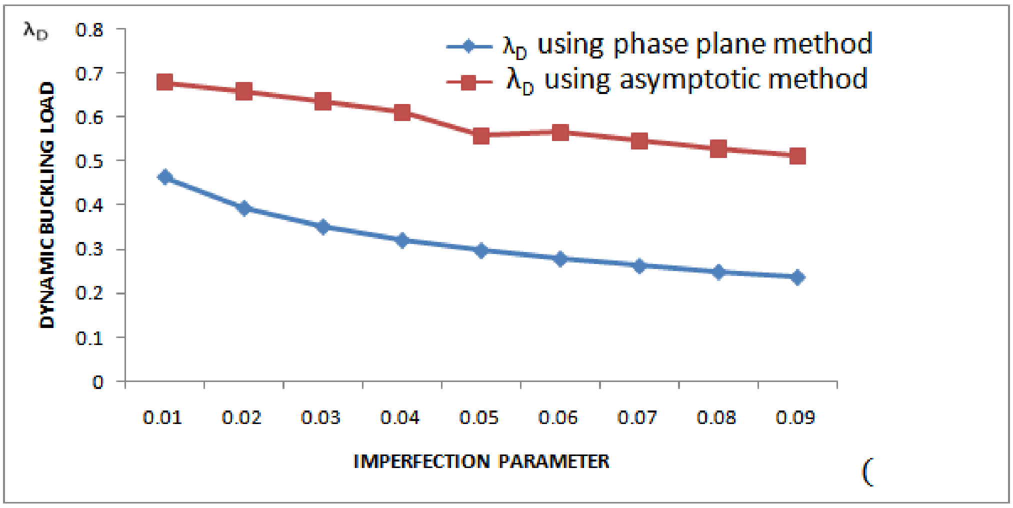

Figure 7. Graphical PlotComparing The Dynamic Buckling Loads Using Phase Plane Method And Asymptotic Method For Various Values Of The Imperfection Parameter, Using Eqn. (25) And Eqn. (78).

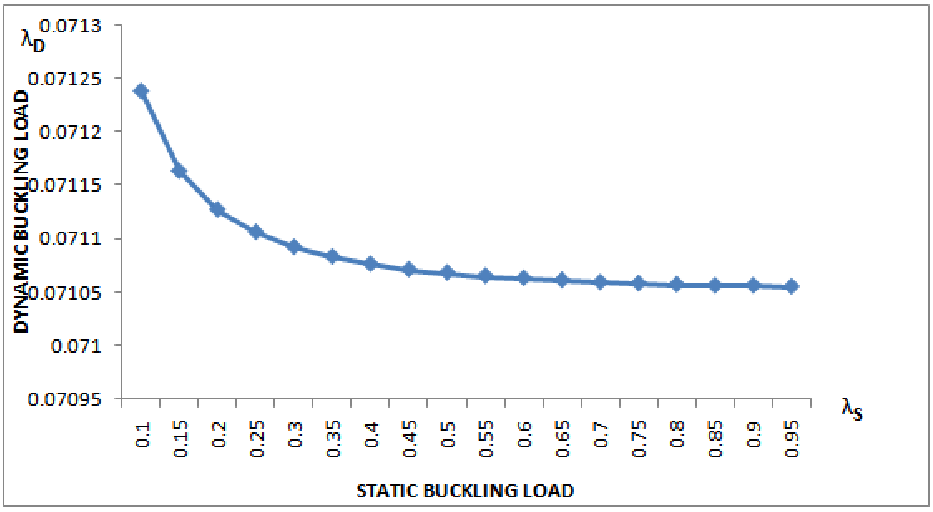

Figure 8. Graphical Plot Of Table 6, Using Eqn. (79).

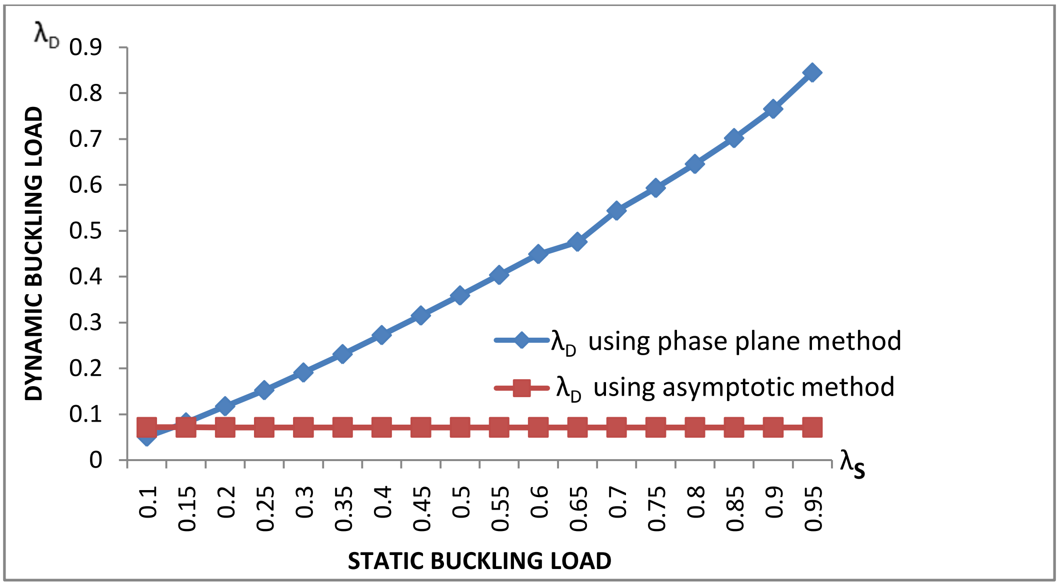

Figure 9. Comparing the Dynamic Buckling Loads Using Phase Plane Method And Asymptotic Method For Various Values Of the Static Loads, Using Eqn. (26) and Eqn. (79).

J. U. Chukwuchekwa "Asymptotic Investigation of the Buckling Of a Cubic–Quintic Nonlinear Elastic Model Structure Stressed By Static Load and A Dynamic Step Load.." IOSR Journal of Mathematics (IOSR-JM) 14.1 (2018): PP 16-30.