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Complex Analysis of Holomorphism over Functions, Sets and Mappings.

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Abstract: *An analysis of complex functions and mapping that are holomorphic in nature are studied and discussed in this paper through Riemann surfaces, which involves Riemann Mapping theorem and Caratheodory's theorem. Furthermore Montel's theorem, Runge's theorem and Mergelyan's theorem over the holomorphic nature is studied with its basic properties and developed in this paper. To enhance the reliability over the nature of holomorphism the metrics of Riemann surface and conformal maps of plane to disk is analyzed and studied in this paper.*

Keywords: *Holomorphic functions, Conformal mapping, Riemann mappings, Riemann surface and Holomorphic open sets.*

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I. Inroduction

Mathematical analysis that investigates functions of complex numbers is complex analysis. It is in particular the theory of conformal mappings, has many physical applications and is also used throughout analytic number theory. Complex analysis has a new boost from complex dynamics and the pictures of fractals produced by iterating holomorphic functions. Complex analysis is widely applicable to two-dimensional problems in physics, string theory which studies conformal invariants in quantum field theory.

Complex analysis is thought of as the subject that applies the theory of calculus to imaginary numbers. In sixteenth century Italian mathematicians named Scipione del Ferro (1465-1526), Nicolo Tartaglia (1500-1557), Girolamo Cardano (1501-1576), and Rafael Bombelli (1526-1572) made significant contributions leading up to the invention of complex numbers: In 1545,Girolamo Cardano published "*Ars Magna*" (The Great Art), in which he gave for the first time an algebraic solution to the general cubic equation but his computations were limited to numbers in "real domain" (also a Maple computing environment). Cardano is credited for making the following important discovery known as Cardano's Substitution.

William Rowan Hamilton (1805-65) in an 1831 memoir defined ordered pairs of real numbers (a, b) to be a couple. He defined addition and multiplication of couples: $(a, b) + (c, d) = (a + c, b + d)$ and $(a, b)(c, d) = (ac - bd, bc + ad)$. This is in fact an algebraic definition of complex numbers.

Organization of the paper is with respective sections: preliminaries, metric of Riemann surface, conformal map of plane to disk, Montel's theorem of holomorphic functions, Riemann mapping theorem, Caratheodory's theorem of conformal mapping, Runge's theorem of holomorphic open sets, Mergelyan's theorem of holomorphism and conclusion.

II. Preliminaries

Definition 2.1: A **curve** $z(t) = \big(x(t), y(t)\big) = x(t) + iy(t) \forall a \leq t \leq b$, and $z(t)$ is a parametrization for the curve C. C is a curve that goes from the initial point $z(a) = \big(x(a), y(a)\big) = x(a) + iy(a)$ to the terminal point $z(b) = \big(x(b), y(b)\big) = x(b) + iy(b)$. If we had another function whose range was the same set of points as $z(t)$ but whose initial and final points were reversed, we would indicate the curve this function defines by C.

Definition 2.2: Piecewise smooth curve: We define a curve to be the range of a continuous complex-valued function $z(t)$ defined on the interval $[a. b]$. That is, a curve C is the range of a function given by $z(t) = \big(x(t), y(t)\big) = x(t) + iy(t) \forall a \leq t \leq b$, where both $x(t) and\ y(t)$ are continuous real-valued functions. If both $x(t) and\ y(t)$ are differentiable, we say that the curve is smooth. A curve for which $x(t) and\ y(t)$ are differentiable except for a finite number of points is called piecewise smooth.

Definition 2.3: Interior Point: The point z_0 is said to be an interior point of the set S, that there exist ε, neighborhood of z_0 that contains only points of S [1].

Definition 2.4: Exterior Point: The point z_0 is called an exterior point of the set S if there exist ε, neighborhood of z_0 that contains no points of S [1].

Definition 2.5: Boundary Point: If the point z_0 is neither an interior point nor an exterior point of S, then it is called a boundary point of S and has the property that each ε neighborhood of z_0 contains both points in S and points not in S [1].

Definition 2.6: Complex functions: A complex function is a function whose domain and range are subsets of the complex plane.
For any complex function, both the independent variable and the dependent variable may be separated into real and imaginary parts: $z = x + iy$ and $\omega = f(z) = u(x, y) + iv(x, y)$ where $x, y \in R\varepsilon$ and $u(x, y) and\ v(x, y)$ are real-valued functions [2].

Definition 2.7: Holomorphic functions: Holomorphic functions are complex functions defined on an open subset of the complex plane that are differentiable. [3]

III. Metric Of Riemann Surface

A metric on the complex plane is of the form $ds^2 = \lambda^2 (z, \bar{z}) dz\, \overline{dz}$, where λ is a real, positive function of $\bar{z}$ and the length of a curve γ in the complex plane is thus given by $l(\gamma) = \int_\gamma \lambda (z, \overline{z})\, |dz|$and its area of a subset is given by $area\ (M) = \int_M \lambda^2 (z, \bar{z}) \frac{i}{2} dz^\wedge d\bar{z}$ where ^ is the exterior product used to construct the volume form. The determinant of the metric is λ^4, and its square root is λ^2 .
The **Euclidean volume** form on the plane is $dx^\wedge dy$ and so $dz^\wedge d\bar{z} = (dx + idy)^{dx-idy} = -2idx^\wedge dy.$
A function $\emptyset(z, \bar{z})$ is said to be the **potential of the metric** if $4 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} \emptyset(z, \bar{z}) = \lambda^2 (z, \bar{z})$
The Laplace–Beltrami operator is given by $\Delta = \frac{4}{\lambda^2} \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} = \frac{1}{\lambda^2} (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$
The Gaussian curvature of the metric is given by $K = -\Delta log\, \lambda$. This curvature is one-half of the Ricci scalar curvature. Isometries preserve angles and arc-lengths.

Remark: On Riemann surfaces, isometries are identical to changes of coordinate (both the Laplace–Beltrami operator and the curvature are invariant under isometries).

Example: Let S be a Riemann surface with metric $\lambda^2 (z, \bar{z}) dz\, d\bar{z}$ and let T be Riemann surface with metric$\mu^2(\omega, \bar{\omega}) d\omega d\bar{\omega}$. Then a map $f: S \to T$with $f: \omega(Z)$ is an isometric if and only if it is conformal and if $\mu^2(\omega, \bar{\omega})\ \frac{\partial\omega}{\partial z} \frac{\partial\bar{\omega}}{\partial \bar{z}} = \lambda^2 (z, \bar{z})$.Here, the requirement that the map is conformal is nothing more than the statement $\omega(z, \bar{z}) = \omega(z)$, that is, $\frac{\partial}{\partial \bar{z}} \omega(z) = 0$.

IV. Conformal Map Of Plane To Disk

The upper half plane can be mapped conformal to the unit disk with the Mobius transformation $\omega = e^{i\emptyset} \frac{z-z_0}{z-\bar{z}_0}$ [4], where *w* is the point on the unit disk that corresponds to the point *z* in the upper half plane. In this mapping, the constant z_0 can be any point in the upper half plane; it will be mapped to the center of the disk. The real axis $\Im z = 0$ maps to the edge of the unit disk $|\omega| = 1$. The constant real number $\emptyset$ can be used to rotate the disk by an arbitrary fixed amount. The canonical mapping is $\omega = \frac{iz+1}{z+i}$ [5] which takes *i* to the center of the disk, and *0* to the bottom of the disk.

V. Montel's Theorem Of Holomorphic Functions.

Montel's theorem: It refers to theorems about families of holomorphic functions. These are named after Paul Montel, and give conditions under which a family of holomorphic functions is normal. [6]

Theorem 3.1: First version of Montel's theorem: Uniformly bounded families:
Statement: A uniformly bounded family of holomorphic functions defined on an open subset of the complex numbers is normal.
Proof: The first version of Montel's theorem is a direct consequence of Marty's Theorem (which states that a family is normal if and only if the spherical derivatives are locally bounded) and Cauchy's integral formula. This theorem is also known as the Stieltjes–Osgood theorem.

Corollary 3.2: Suppose that F is a family of meromorphic functions on an open set D. If $Z_0 \in D$ is such that F is not normal at Z_0, and $U \subset D$ is a neighborhood of Z_0, then $\cup_{f\in f}^{f(u)}$ is dense in the complex plane.
Proof: Suppose that all the functions in F omit the same neighborhood of the point Z_0. By post composing with the map $Z \mapsto \frac{1}{z-z_0}$ we obtain a uniformly bounded family, which is normal by the first version of the theorem.

Theorem 3.3: Second Version: Stronger version of Montel's Theorem (The Fundamental Normality Test):
Statement: A family of holomorphic functions ⇔ all of which omit the same two values$a, b \in C$, is normal.
Sufficient Condition: The conditions in the above theorems are sufficient, but not necessary for normality
Necessity Condition: The family $Z \mapsto Z + a: a \in C$ is normal, but does not omit any complex value.
Proof: The second version of Montel's theorem can be deduced from the first by using the fact that there exists a holomorphic universal covering from the unit disk to the twice punctured plane$\frac{C}{\{a,b\}}$. (Such a covering is given by the elliptic modular function).This version of Montel's theorem can be also derived from Picard's theorem, by using Zalcman's lemma.

VI. Riemann Mapping Theorem

Uniqueness of the Riemann mapping 4.1:

1. Simple Riemann mappings have no explicit formula using only elementary functions.
2. Simply connected open sets in the plane can be highly complicated.
3. The analog of the Riemann mapping theorem for more complicated domains is not true.
4. Any doubly connected domain except for the punctured disk and the punctured plane is conformally equivalent to some annulus $\{z : r < |z| < 1\}$ with $0 < r < 1$.
5. The analogue of the Riemann mapping theorem in three or more real dimensions is not true.
6. Even if arbitrary homeomorphisms in higher dimensions are permitted, contractible manifolds can be found that are not holomorphic to the ball.
7. The Riemann mapping theorem is the easiest way to prove that any two simply connected domains in the plane are homomorphism.

Theorem 4.2:
Statement: If *U* is a non-empty simply connected open subset of the complex number plane **C** which is not all of **C**, then there exists a biholomorphic (bijective and holomorphic) mapping *f* from *U* onto the open unit disk $D = \{Z \in C: |Z| < 1\}$.This mapping is known as a Riemann mapping. [5]
Proof: Given *U* and z_0, we want to construct a function *f* which maps *U* to the unit disk and z_0 to 0. Assume that *U* is bounded and its boundary is smooth $f(z) = (z - z_0)e^{g(z)}$, where $g = u + iv$, is some (to be determined) holomorphic function, then z_0 is the only zero of *f*. We require $|f(z)| = 1$ for $z \in \partial z_0$, so we need, $u(z) = -log|z - z_0|$, on the boundary. Since *u* is the real part of a holomorphic function, we know that *u* is necessarily a harmonic function; i.e., it satisfies Laplace's equation. Once the existence of u has been established, the Cauchy-Riemann equations for the holomorphic function *g* allow us to find *v* (this argument depends on the assumption that U be simply connected). Once *u* and *v* have been constructed, one has to check that the resulting function *f* does indeed have all the required properties.
Uniformization theorem: The Riemann mapping theorem can be generalized to Riemann surfaces. If *U* is a simply-connected open subset of a Riemann surface, then U is biholomorphic to one of the following: the Riemann sphere, **C** or *D*. This is known as the Uniformization theorem.
Smooth Riemann mapping theorem: In the case of a simply connected bounded domain with smooth boundary, the Riemann mapping function and all its derivatives extend by continuity to the closure of the domain. This can be proved using regularity properties of solutions of the Dirichlet boundary value problem, which follow either from the theory of Sobolev spaces for planar domains or from classical potential theory. Other methods for proving the smooth Riemann mapping theorem include the theory of kernel functions or the Beltrami equation.

VII. Caratheodory's Theorem Of Conformal Mapping

Statement: Form 1: If U is a simply connected open subset of the complex plane **C**, whose boundary is a Jordan curve Γ then the Riemann map f: U → D from U to the unit disk D extends continuously to the boundary, giving a homeomorphism $F : \Gamma \rightarrow S^1$.Such a region is called a Jordan domain. [8]

Statement: Form 2: U there is a homeomorphism $F : cl(U) \rightarrow cl(D)$ from the closure of U to the closed unit disk $cl(D)$ whose restriction to the interior is a Riemann map, i.e. it is a bijective holomorphic conformal map.

Statement: Form 3 : For any pair of simply connected open sets U and V bounded by Jordan curves $\Gamma 1$ and $\Gamma 2$, a conformal map $f : U \rightarrow V$ extends to a homeomorphism $F: \Gamma 1 \rightarrow \Gamma 2$. Let $g : D \longrightarrow U$ be the inverse of the Riemann map, where $D \subset \boldsymbol{C}$ is the unit disk, and $U \subset \boldsymbol{C}$ is a simply connected domain. Then g extends continuously to $G : cl(D) \rightarrow cl(U)$ if and only if the boundary of U is locally connected.

Context of Caratheodory's theorem: Compared to general simply connected open sets in the complex plane **C**, those bounded by Jordan curves are particularly well-behaved. It is a study of boundary behavior of conformal maps. Jordan curve boundary is sufficient for such an extension to exist; it is by no means necessary. [9]

Example : The map $f(z) = z^2$ from the upper half-plane **H** to the open set G that is the complement of the positive real axis is holomorphic and conformal, and it extends to a continuous map from the real line **R** to the positive real axis $\mathbf{R}^+$; however, the set G is not bounded by a Jordan curve.

VIII. Runge's Theorem Of Holomorphic Open Sets

Statement: If A to be a subset of the Riemann sphere $\boldsymbol{C} \cup \{\infty\}$ and requires that *A* intersect also the unbounded connected component of *K* (which now contains ∞). That is, the rational functions may turn out to have a pole at infinity, in general formulation the pole can be chosen instead anywhere in the unbounded connected component of *K*.

Proof:

Let C denote the set of complex numbers and *K* be a compact subset of **C**, f be a function which is holomorphic on an open set containing *K*. If *A* is a set containing at least one complex number from every bounded connected component of $\boldsymbol{C}\backslash K$ then there exists a sequence $(r)_n, n \in N$ of rational functions which converges uniformly to *f* on *K* and such that all the poles of the functions $(r)_n, n \in N$ are in *A*. Not every complex number in *A* needs to be a pole of every rational function of the sequence $(r)_n, n \in N$, that do have poles, those poles lie in *A*. Choose any complex numbers from the bounded connected components of $\boldsymbol{C}\backslash K$ and the existence of a sequence of rational functions with poles only amongst those chosen numbers. For the special case in which **C***K* is a connected set (or equivalently that *K* is simply-connected), the set *A* in the theorem will clearly be empty. Since rational functions with no poles are simply polynomials, we get the following corollary: If *K* is a compact subset of **C** such that **C***K* is a connected set, and *f* is a holomorphic function on *K*, then there exists a sequence of polynomials (P_n) that approaches *f* uniformly on *K*. There is a closed piecewise-linear contour Γ in the open set, containing *K* in its interior. By Cauchy's integral theorem$f(\omega) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)dz}{z-\omega}$ for *w* in *K*. Riemann approximating sums can be used to approximate the contour integral uniformly over *K*. Each term in the sum is a scalar multiple of $(z - w)^{-1}$ for some point *z* on the contour. Thus an uniform approximation by a rational function with poles on Γ.

Take poles at specified points in each component of the complement of *K*.

To check: terms of the form $(z - w)^{-1}$.

If z_0 is the point in the same component as *z*, take a piecewise-linear path from *z* to z_0. If two points are sufficiently close on the path, any rational function with poles only at the first point can be expanded as a Laurent series about the second point. That Laurent series can be truncated to give a rational function with poles only at the second point uniformly close to the original function on *K*. Proceeding by steps along the path from *z* to z_0 the original function $(z - w)^{-1}$. can be successively modified to give a rational function with poles only at z_0.

If z_0 is the point at infinity, then by the above procedure the rational function $(z - w)^{-1}$ can first be approximated by a rational function *g* with poles at $R > 0$ where *R* is so large that *K* lies in $w < R$. The Taylor series expansion of *g* about 0 can then be truncated to give a polynomial approximation on *K*.

IX. Mergelyan's Therom Of Holomorphism

Mergelyan's theorem ([10], [11]), is the ultimate development and generalization of the Weierstrass approximation theorem and Runge's theorem. It gives the complete solution of the classical problem of approximation by polynomials

Statement: Let K be a compact subset of the complex plane **C** such that $\frac{C}{K}$ is connected. Then, every continuous function $f: K \rightarrow C$, such that the restriction $\frac{f}{int\,(k)}$ is holomorphic, can be approximated uniformly on K with polynomials (Here int(K) denotes the interior of K.)

Proof: In the case that $\frac{C}{K}$ is *not* connected, in the initial approximation problem the polynomials have to be replaced by rational functions. An important step of the solution of this further rational approximation problem was also suggested by Mergelyan in 1952. Further deep results on rational approximation are due to, in particular, A. G. Vitushkin.

Weierstrass and Runge's theorems were put forward in 1885, while Mergelyan's theorem dates from 1951. This rather large time difference is not surprising, as the proof of Mergelyan's theorem is based on a new powerful method created by Mergelyan. After Weierstrass and Runge, many mathematicians (in particular Walsh, Keldysh, and Lavrentyev) had been working on the same problem. The method of the proof suggested by Mergelyan's is constructive, and remains the only known constructive proof of the result.

X. Conclusion

In the nineteenth century there were many contributions. The French mathematician Augustin Louis Cauchy (1789-1857) contributed theorems that are part of the body of complex analysis. The German mathematician Johann Carl Friedrich Gauss (1777-1855) reinforced the utility of complex numbers by using them in several proofs of the Fundamental Theorem of Algebra. In an 1831 paper, he produced a clear geometric representation of by identifying it with the point in the coordinate plane. He also described the arithmetic operations with these new complex numbers. It would be a mistake, however, to conclude that in 1831 complex numbers were transformed into legitimacy. In that same year the prolific logician Augustus De Morgan (1806-1871) commented in his book, On the Study and Difficulties of Mathematics." The remarkable behavior of holomorphic functions near essential singularities is described by Picard's Theorem. Functions that have only poles but no essential singularities are called meromorphic. Laurent series are similar to Taylor series but can be used to study the behavior of functions near singularities.

A bounded function that is holomorphic in the entire complex plane must be constant; this is Liouville's theorem. It can be used to provide a natural and short proof for the fundamental theorem of algebra which states that the field of complex numbers is algebraically closed. If a function is holomorphic throughout a connected domain then its values are fully determined by its values on any smaller sub domain. The function on the larger domain is said to be analytically continued from its values on the smaller domain. This allows the extension of the definition of functions, such as the Riemann zeta function, which are initially defined in terms of infinite sums that converge only on limited domains to almost the entire complex plane. Sometimes, as in the case of the natural logarithm, it is impossible to analytically continue a holomorphic function to a non-simply connected domain in the complex plane but it is possible to extend it to a holomorphic function on a closely related surface known as a Riemann surface.

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REFERENCES

[1]. Carl Friedrich Gauss (1831) "The geometric representation of complex numbers since 1796, submitted his ideas to the Royal Society of Gottingen.
[2]. Carathéodory, C. (1954), Theory of functions of a complex variable, Vol. 2, translated by F. Steinhardt, Chelsea
[3]. John B. Conway (1978). Functions of One Complex Variable I. Springer-Verlag. ISBN 0-387-90328-3.
[4]. Carathéodory, C. (1998), Conformal representation (reprint of the 1952 second edition), Dover, ISBN 0-486-40028-X.
[5]. Pommerenke, C. (1992), Boundary behaviour of conformal maps, Grundlehren der Mathematischen Wissenschaften, **299**, Springer, ISBN 3-540-54751-7
[6]. Hazewinkel, Michiel, ed. (2001) [1994], "Montel theorem", Kluwer Academic Publishers, ISBN 978-1-55608-010-4
[7]. Richard M. Timoney (2004), Riemann mapping theorem from Trinity College, Dublin
[8]. Ahlfors, Lars V. (2010), Conformal invariants: topics in geometric function theory, AMS Chelsea Publishing, ISBN 978-0-8218-5270-5
[9]. Shields, Allen (1988), "Carathéodory and conformal mapping", The Mathematical Intelligencer, **10** (1): 18–22, doi:10.1007/BF03023846, ISSN 0343-6993, MR 0918659
[10]. Dieter Gaier, Lectures on Complex Approximation, Birkhäuser Boston, Inc. (1987), ISBN 0-8176-3147-X.
[11]. A. G. Vitushkin, Half a century as one day, Mathematical events of the twentieth century, 449–473, Springer, Berlin, (2006), ISBN 3-540-23235-4.

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