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Matrix Computations of Corwin-Greenleaf Multiplicity Functions for Symplectic Lie Groups

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Abstract: *In this paper, Corwin-Greenleaf multiplicity functions for Symplectic Lie Group have been studied in the light of the Kirillov-Kostant Theory. This was pioneered by the famous mathematician L. Corwin and F. Greenleaf. The major objectives of the proposed research are to design and develop symplectic representation. This paper focused on the geometry of coadjoint orbits, which is predicted by recent results on (infinite dimensional) unitary representation theory, namely, the multiplicity-free theorem of branching laws with respect to reductive symmetric pairs.*

Keywords: *Branching laws, Corwin-Greenleaf multiplicity functions, orbit method, Symplectic Lie Groups.*

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I. Introduction

The fundamental matrix of a symplectic system is a curve in the symplectic group, denoted by $\mathrm{Sp}(2n, \mathbb{R})$, which is a closed subgroup of the general linear group $\mathrm{GL}(2n, \mathbb{R})$, hence it has a Lie group structure. This structure is extremely rich, due to the fact that symplectic forms on a vector space are intimately related to its complex structures, and such relation produces other invariant geometric and algebraic structures, such as inner products and Hermitian products. Symplectic Lie groups are the mathematical apparatus of such areas of physics as classical mechanics, geometrical optics and thermodynamics. Whenever the equations of a theory can be gotten out of a variation principle, it clears up and systematizes the relations between the quantities entering into the theory. Symplectic Lie groups simplifies and makes perceptible the frightening formal apparatus of Hamiltonian dynamics and the calculus of variations in the same way that the ordinary geometry of linear spaces reduces cumbersome coordinate computations to a small number of simple basic principles. In the present survey the simplest fundamental concepts of symplectic Lie groups and applications to the theory of multiplicity systems and to orbit method receive more thorough review the articles of A. A. Kirillov and of Hua-Costant, T. Kobayashi and S. P. Novikov.

The orbit method [1] pioneered by Kirillov and Kostant seeks to understand irreducible unitary representation by analogy with "quantization" procedures in mechanics. Physically, the idea of quantization is to replace a classical mechanical model (a phase space modeled by a symplectic manifold M) with a quantum mechanical model (a state space modeled by a Hilbert space $\mathcal{H}$) of the same system. The natural quantum analogue of the action of a group G on M by symplectomorphisms is a unitary representation of G on $\mathcal{H}$.

For a Lie group G coadjoint orbits are symplectic manifolds, and the philosophy of the orbit method says that there should be a method of "quantization" to pass from coadjoint orbits for G to irreducible unitary representations of G. Kirillov proved that this works perfectly for nilpotent Lie groups. But many specialists have pointed out that the orbit method does not work very well for semisimple Lie groups [1, 2, 3]. However, we can still expect an intimate relation between the unitary dual of G and the set of (integral) coadjoint orbits even for a semi simple Lie group. One of the fundamental problems in representation theory is to decompose a given representation into irreducible [2]. Branching laws are one of the most important cases. Here, by branching laws we mean the irreducible decomposition in terms of a direct integral of an irreducible unitary representation π of a group G when restricted to a subgroup H:

$$\pi|_{H} \simeq \int_{\widehat{H}}^{\oplus} m_{\pi}(v) v \, d\mu(v)$$

Such a decomposition is unique, and the multiplicity $m_\pi : \widehat{H} \to \mathbb{N} \cup \{\infty\}$ makes sense as a measurable function on the unitary dual $\widehat{H}$. There are two basic questions on multiplicities:

Problem 1.1. [4]
(a) For which (G, H, π), the restriction $\pi|_H$ is multiplicity-free?

(b) Relate quantum and classical pictures in the spirit of Kirillov-Kostant orbit method.

As for (a), T. Kobayashi [5] has established a unified theory on multiplicity-free theorem of branching laws for both finite and infinite dimensional representations in a broad setting. This theorem gives a uniform explanation for many known cases of multiplicity-free results and also presents many new cases of multiplicity-free branching laws. As for (b), it is well-known that the orbit method works well for nilpotent Lie groups, but only partially for reductive groups [1].

II. Corwin-Greenleaf Multiplicity Function

Corwin and Greenleaf have worked exclusively in the context of simply connected nilpotent Lie groups. As such, the governing principle in the description of the representation theory and harmonic analysis is the Kirillov method of coadjoint orbits. Here are the main ingredients. Let G be a simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. The real linear dual of $\mathfrak{g}$ is denoted $\mathfrak{g}^*$. For any $\phi \in \mathfrak{g}^*$, we write B_ϕ for the alternating form $B_\phi(X, Y) = \phi[X, Y]$ on $\mathfrak{g}$. Then one can always choose a real polarization for ϕ, that is a subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ which is maximally totally isotropic for the form B_ϕ. The character $\chi_\phi(\exp X) = e^{i\phi(X)}$ is then well-defined on the simply connected analytic subgroup A of G having Lie algebra $\mathfrak{a}$, and the induced representation $\pi_{\phi,\mathfrak{a}} = \mathrm{Ind}_A^G \chi_\phi$ is irreducible. Its class is independent of $\mathfrak{a}$, and so is unambiguously denoted π_ϕ. The map $\phi \to \pi_\phi$, $\mathfrak{g}^* \to \widehat{G}$ is surjective and G-equivariant; and factors to a bijection $\mathfrak{g}^*/G \to \widehat{G}$ of the coadjoint orbits onto the unitary dual. Now let H be any connected (and therefore simply connected) closed subgroup of G. Suppose that $\nu \in \widehat{H}$ and $\mathcal{O}_\nu \subset \mathfrak{h}^*$ is the corresponding coadjoint orbit. Let $p : \mathfrak{g}^* \to \mathfrak{h}^*$ be the canonical projection. Then the Corwin-Greenleaf orbital integral formula for the description of the induced representation $\mathrm{Ind}_A^G \nu$ is as follows:

$$\mathrm{Ind}_A^G \nu = \int_{p^{-1}(\mathcal{O}_\nu)/H}^{\oplus} \pi_\phi \, d\dot{\phi} = \int_{G\cdot p^{-1}(\mathcal{O}_\nu)/G}^{\oplus} n_\phi^\nu \pi_\phi \, d\ddot{\phi} \qquad (1)$$

where

$$n_\phi^\nu = \#\big(G \cdot \phi \cap p^{-1}(\mathcal{O}_\nu)\big)/H \qquad (2)$$

n_ϕ^ν is called the Corwin-Greenleaf multiplicity function [6]. The measures in (1) are push-forwards of Lebesgue measure. The reader should consult [6, 7] for these results. It is also worth noting that Corwin and Greenleaf had two powerful motivations for this result, namely the corresponding formulas derived by: Benoist [8] in the case that G/H is symmetric (and $\nu = 1$); and by Vergne [9] in the case G is exponential solvable, ν is a character, and $\mathfrak{h}$ is a real polarization for ν, not satisfying the Pukanszky condition.

III. Symplectic Groups

Definition: 1

A symplectic form [10] on a vector space V is a bilinear map $F : V \times V \to \mathbb{R}$ such that

(i) F is skew-symmetric, i.e. $F(v, w) = -F(w, v)\ \forall\, w, v \in V$

(ii) F is non-degenerate, i.e. if $F(v, w) = 0\ \forall\, w \in V$ then $v = 0$

and (V, F) is called a symplectic vector space.

Definition: 2

A symplectic manifold [10] is a smooth manifold M, equipped with a closed non-degenerate differential 2-form ω, called the symplectic form. The study of symplectic manifolds is called symplectic geometry or symplectic topology. Symplectic manifolds arise naturally in abstract formulations of classical mechanics and analytical mechanics as the cotangent bundles of manifolds.

Definition: 3

The name symplectic group [11] can refer to two different, but closely related, types of mathematical groups, denoted $Sp(2n, F)$ and $Sp(n)$. The latter is sometimes called the compact symplectic group to distinguish it from the former. Many authors prefer slightly different notations, usually differing by factors of 2. The notation used here is consistent with the size of the matrices used to represent the groups. The symplectic group of degree $2n$ over a field F, denoted $Sp(2n, F)$ is the group of $2n \times 2n$ symplectic matrices with entries in F, and with the group operation that of matrix multiplication. Since all symplectic matrices have determinant 1, the symplectic group is a subgroup of the special linear group $SL(2n, F)$.

Typically, the field F is the field of real numbers $\mathbb{R}$, or complex numbers $\mathbb{C}$. In this case $Sp(2n, F)$ is a real or complex Lie group of real or complex dimension $n(2n + 1)$. These groups are connected but non-compact. The centre of $Sp(2n, F)$ consists of the matrices I_{2n} and $-I_{2n}$ as long as the characteristic of the field is not equal to 2. Note that I_{2n} denotes the $2n \times 2n$ identity matrix. The real rank of the Lie Algebra, and hence, the Lie Group for $\mathrm{Sp}(2n, F)$ is n. The condition that a symplectic matrix preserves the symplectic form can be written as

$A \in \mathrm{Sp}(2n, F)$ iff $A^T \Omega A = \Omega$, where A^T is the transpose of A and $\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$.

The Lie algebra of $\mathrm{Sp}(2n, F)$ is given by the set of $2n \times 2n$ matrices A (with entries in F) that satisfy $\Omega A + A^T \Omega = 0$ when $n = 1$, the symplectic condition on a matrix is satisfied if and only if the determinant is one, so that $\mathrm{Sp}(2, F) = \mathrm{SL}(2, F)$. For $n > 1$, there are additional conditions, i.e. $\mathrm{Sp}(2n, F)$ is then a proper subgroup of $\mathrm{SL}(2n, F)$, i.e. if (V, ω) be a symplectic space; the symplectic group of (V, ω) is the group of all symplectomorphisms of (V, ω) denoted by $\mathrm{Sp}(V, \omega)$. We denote by $\mathrm{Sp}(2n, \mathbb{R})$ the symplectic group of $\mathbb{R}^{2n}$ endowed with the canonical symplectic form.

IV. Statement Of Main Results

The main object of this paper is to provide a proof of Corwin-Greenleaf multiplicity function by elementary matrix representations. This will do for symplectic Lie groups $G = \mathrm{Sp}(n, \mathbb{R})$. Before going to the main result let us recall a connection between the orbit method in unitary representation of Lie groups and Corwin-Greenleaf multiplicity function. The idea behind the orbit method is the unication of harmonic analysis with symplectic geometry (and it can also be considered as a part of the more general idea of the unication of mathematics and physics). Consider an approach is based on the restriction of a representation on an over group $\tilde{G}$. For instance, consider a symmetric pair

$$(\tilde{G}, G) = (\mathrm{Sp}(n, \mathbb{R}), \mathrm{GL}(n, \mathbb{R}))$$

Then we have the natural embedding $G/K \to \tilde{G}/\tilde{K}$, namely

$$GL(n, \mathbb{R})/O(n) \to Sp(n, \mathbb{R})/U(n) \tag{3}$$

Via an embedding (3), G/K becomes a totally real submanifold in a complex manifold $\tilde{G}/\tilde{K}$. Let π be a discrete series holomorphic representation of scalar type of $\tilde{G}$. We shall replace the above pair $(\tilde{G}, G)$ by (G, H): Suppose G is a semisimple Liegroup such that G/K is a Hermitian symmetric space of non compact type. Then consider the case where G is connected and simply connected nilpotent Lie group. Then Kirillov [1] proved that the unitary dual $\tilde{G}$ is parametrized by $G^*/\mathfrak{g}$ the set of coadjoint orbits. We shall write the corresponding coadjoint orbits $\mathcal{O}_\pi \subset \mathfrak{g}^*$ for $\pi \in \tilde{G}$. Let H be a subgroup of G. Then the restriction $\pi|_H$ is decomposed into a direct integral of irreducsaible representation of H.

$$\pi|_H \cong \int_{\widehat{H}}^{\oplus} m_\pi(\tau)\tau \, d\mu(\tau) \qquad (branching\ law) \tag{4}$$

where $d\mu$ is a measure on $\widehat{H}$. Then, Corwin and Greenleaf proved that the multiplicity $m_\pi(\tau)$ is given by the "mod H" intersection number $n(\mathcal{O}_\pi^G, \mathcal{O}_v^H)$ defined as follows:

$$n(\mathcal{O}_\pi^G, \mathcal{O}_\tau^H) = \#((\mathcal{O}_\pi^G \cap pr^{-1}(\mathcal{O}_\tau^H))/H) \tag{5}$$

Here $\mathcal{O}_\pi^G \subset \mathfrak{g}^*$ and $\mathcal{O}_\tau^H \subset \mathfrak{h}^*$ are the coadjoint orbits corresponding to $\pi \in \tilde{G}$ and $\tau \in \widehat{H}$, respectively, under the Kirillov correspondence, and $pr : \mathfrak{g}^* \to \mathfrak{h}^*$ is the canonical projection. Here $n(\mathcal{O}_\pi^G, \mathcal{O}_v^H)$ is referred as the Corwin-Greenleaf multiplicity function.

Contrary to nilpotent Lie groups, there is no reasonable bijection between $\hat{G}$ and $\mathfrak{g}^*/G$. Therefore it is not obvious if an analogue statement of Corwin-Greenleaf's theorem makes sense for a semisimple lie group G. But, the orbit method still gives a good approximation of the unitary dual $\hat{G}$. For example, to an 'integral' elliptic coadjoint orbit $\mathcal{O}_\lambda^G = Ad^*(G)\lambda \subset \mathfrak{g}^*$, one can associate a unitary representation, denoted by π_λ ,of G as a generalization of the Borel-Weil-Bolt theorem due to Schmid and Wong, combined with a unitarization theorem of Vogan and Wallach. Furthermore, π_λ is nonzero and irreducible for 'most' λ [2, 3] of both geometric and algebraic results in this direction. Namely, to such a adjoint orbit $\mathcal{O}_\lambda^G$, one can naturally attach an irreducible unitary representation $\pi_\lambda \in \hat{G}$.

Let τ be an involutive automorphism of G. Put $G^\tau \coloneqq \{g \in G | \tau g = g\}$. We write G_0^τ for the identity component of G^τ. Throughout this paper, we shall identify $\mathfrak{g}^*$ with $\mathfrak{g}$ via the trace form (or via the Killing form) of $\mathfrak{g}$ defined. Correspondingly, $\mathfrak{k}^*$ is defined with $\mathfrak{k}$.
Here, we are ready to state the main result.

Conjecture: 1

Let (G, H) be a reducible symmetric pair [12]. Then under certain condition:

$$n(\mathcal{O}_\pi^G, \mathcal{O}_v^H) = \#(Ad^*(H) - orbit\ in\ \mathcal{O}_\pi^G \cap pr^{-1}(\mathcal{O}_v^H)) \leq 1.$$

Already conjecture 1 has been proved for an arbitrary Riemannian symmetric pair [13].

We recall that if G be a simple non-compact Lie group, then G is Hermitian [14] if the corresponding Riemannian symmetric space G/K is a complex bounded symmetric domain. In terms of group theory, this is equivalent to saying that the center $\mathfrak{c}(\mathfrak{k})$ of $\mathfrak{k}$ is non-trivial. Moreover, it is known [15] that, dim $\mathfrak{c}(\mathfrak{k}) = 1$ and every $ad\ z|\mathfrak{p}, z \in \mathfrak{c}(\mathfrak{k}) \setminus \{0\}$ is regular. We denote by $\mathfrak{c}(\mathfrak{k})^*$ for the one dimensional subspace of $\mathfrak{k}^*$, which corresponds to $\mathfrak{c}(\mathfrak{k})$ in $\mathfrak{k}$ under the bijection $\mathfrak{k} = \mathfrak{k}^*$. Then Conjecture 4.1 can be rewritten precisely as follows:

Conjecture: 2

Let G be Hermitian and (G, H) a symmetric pair. We assume a coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$ meets $\mathfrak{c}(\mathfrak{k})^*$.Then for any $\mu \in \mathfrak{g}^*$,

$$\#\left(\frac{\mathcal{O} \cap (\mathfrak{h}^* + \mu)}{Ad^*(H)}\right) \leq 1.$$

That is, $\mathcal{O} \cap (\mathfrak{h}^* + \mu)$ is a single H-orbit, provided it is not empty.

We should point out that an analogous statement fails if we drop one of the assumptions. Also the above conjecture implies that $\mathcal{O} \cap (\mathfrak{h}^* + \mu)$ is connected because H can be taken to be connected. Such topological results will be new (even in compact cases).

Example: 1

Suppose $G = Sp(2, \mathbb{R}), SU(2)$ respectively and $H = K = SO(2)$. Let X be a central element in $\mathfrak{k}$. Then in each case, for any $y \in pr(\mathcal{O}_X^H)$ we have

$$\mathcal{O}_X^G \cap pr^{-1}(y) = circle = single\ SO(2) - orbit.$$

A sufficient condition for the proof of Conjecture 2 in the Riemannian symmetric pair has been proved for classical cases [13]. In my paper I have given an alternative and independent proof by elementary matrix computations.

The main result this thesis paper can be stated briefly as:

Theorem: 1

Conjecture 1(as well as Conjecture 2) is true when $G = Sp(n, \mathbb{R})$.

V. Matrix Computations Of Corwin-Greenleaf Multiplicity Functions For Symplectic Lie Groups

In this section we obtain a general formula which is mainly the key computation in proving our result Theorem 1 by elementary matrix computations. This will done using symplectic groups.

Let G/K is Hermitian, associated to an (integral) coadjoint orbit that goes through $([\mathfrak{k}, \mathfrak{k}] + \mathfrak{p})^\perp \subset \mathfrak{g}^*$, the corresponding unitary representation becomes a highest weight module of scalar type. By the identification $\mathfrak{g} \simeq \mathfrak{g}^*$, the coadjoint orbit

$$\mathcal{O}_\lambda^G = Ad^*(G)\lambda \subset (\mathfrak{g}^*)$$

corresponds to the adjoint orbit given by $\mathcal{O}_z^G = Ad(G).z \subset \mathfrak{g}$, where z is a non-zero central element in $\mathfrak{k}$. We also write $pr : \mathfrak{g} \to \mathfrak{h}$ for the projection instead of $pr : \mathfrak{g}^* \to \mathfrak{h}^*$.

Now, we consider the projection $pr^\theta : \mathfrak{g} \to \mathfrak{k}$ or simply, denoted by $pr : \mathfrak{g} \to \mathfrak{k}$. Then the pullback $(pr^\theta)^{-1}(y)$ $(y \in pr^\theta(\mathcal{O}^K))$ is $Ad(K)$ −stable. Our main result Theorem 1 can be rewritten precisely as follows:

Theorem: 2

Let G/K be a Hermitian symmetric space of noncompact type, and a central element in $\mathfrak{k}$. Then the intersection

$$\mathcal{O}_X^G \cap pr^{-1}(\mathcal{O}^K)$$

is a single K orbit for any adjoint orbit $\mathcal{O}^K \subset \mathfrak{k}$, whenever it is non-empty.

We have the global Cartan [12] decomposition: $G = K\ exp\ \mathfrak{p}$. We take a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$, and a positive system $\sum^+(\mathfrak{g}, \mathfrak{a})$ of the restricted root system $\sum(\mathfrak{g}, \mathfrak{a})$. Let $\{\alpha_1, \dots, \alpha_r\}$ be the set of simple roots and we define the corresponding Weyl chamber by

$$\mathfrak{a}_+ = \{H \in \mathfrak{a} : \alpha_1(H) \geq 0, \dots \alpha_r(H) \geq 0\}.$$

Then we have a Cartan decomposition $G = KA_+K$, where $A_+ = exp(\mathfrak{a}_+)$.

A key result in proving our main theorem is:

Proposition: 1

For any $a, a' \in A_+$, $pr^\theta(Ad(a)z)$ and $pr^\theta(Ad(a')z)$ are conjugate under $Ad(K)$ if and only if $a = a'$.

The reduction of the main result can easily as follows:

Proof: It is obvious that for any $a \in A_+$ $pr^\theta(Ad(a)z) \in pr^\theta(\mathcal{O}_z^G)(\because Ad(a)z \in \mathcal{O}_z^G)$. Then for any $y \in \mathcal{O}^K$ we have,

$$\mathcal{O}_z^G \cap (pr^\theta)^{-1}(y) = Ad(K)Ad(A_+)z \cap (pr^\theta)^{-1}(y).$$

This implies that for any $x, x' \in \mathcal{O}_z^G \cap (pr^\theta)^{-1}(y)$, there exists $a, a' \in A_+$ and $k_1 \in K$ such that

$$x = Ad(k_1)Ad(a)z, \text{ and } x' = Ad(k_1)Ad(a')z.$$

Now, $x \sim x'$ under $Ad(K)$ implies that there exists some $k_2 \in K$ satisfying $Ad(k_2)x = x'$. Substituting the value of x and x' the claim easily follows.

Now we state our key computations. We consider our case where $G = Sp(n, \mathbb{R})$. Put $n = p + q$. If I_n denotes the unit matrix of order n, we put

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

Therefore, we realize the symplectic group $Sp(n, \mathbb{R})$ in $GL(2n, \mathbb{R})$ as follows:

$$G = Sp(n, \mathbb{R}) = \{g \in GL(2n, \mathbb{R}) : g^t J_g = J\}$$

$$\mathfrak{g} = \{X \in M(2n, \mathbb{R}) : X^t J + JX = 0\}$$

Then the Lie algebra $\mathfrak{g}$ of G will be identified with

$$\mathfrak{g} = \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \middle| A \in M(n, \mathbb{R}),\ B = B^t,\ C = C^t \right\}$$

We write $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ for the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G, corresponding to the Cartan involution θ (which can be defined by $\theta(X) = I_n X I_n$).

Here

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \middle| B = B^t,\ A = -A^t \right\} \simeq \mathfrak{u}(n) \in A + \sqrt{-1}B$$

$$\mathfrak{p} = \left\{ \begin{pmatrix} X & Y \\ Y & -X \end{pmatrix} \middle| B = B^t, A = A^t \right\}.$$

The decomposition: $\mathfrak{h} = \mathfrak{g} \cap (2p, 2q) \simeq \mathfrak{u}(p, q)$.

$$= \left\{ \begin{pmatrix} A_1 & B_1 & A_2 & B_2 \\ B_1^t & C_1 & B_3 & C_2 \\ -A_2^t & B_3^t & A_4 & B_4 \\ B_2^t & -C_2^t & B_4^t & C_4 \end{pmatrix} \cap \mathfrak{g} : A_1 = -A_1^t,\ A_4 = -A_4^t,\ C_1 = -C_1^t,\ C_4 = -C_4^t \right\}$$

$$= \left\{ \begin{pmatrix} A_1 & B_1 & A_2 & B_2 \\ -B_1^t & C_1 & B_3 & C_2 \\ A_2^t & -B_3^t & A_4 & B_4 \\ -B_2^t & C_2^t & -B_4^t & C_4 \end{pmatrix} \cap \mathfrak{g} : A_1 = A_2^t,\ A_4 = A_4^t,\ C_1 = C_1^t,\ C_4 = C_4^t \right\}$$

$$= \left\{ \begin{pmatrix} A_1 & 0 & A_2 & 0 \\ 0 & C_1 & 0 & C_2 \\ A_2^t & 0 & A_4 & 0 \\ 0 & C_2^t & 0 & C_4 \end{pmatrix} \cap \mathfrak{g} : A_1 = -A_4,\ C_1 = -C_4 \right\}$$

Therefore

$$\mathfrak{q} \cap \mathfrak{p} = \left\{ \begin{pmatrix} A_1 & 0 & A_2 & 0 \\ 0 & C_1 & 0 & C_2 \\ A_2 & 0 & -A_1 & 0 \\ 0 & C_2 & 0 & -C_1 \end{pmatrix} : A_1 = A_1^t,\ C_1 = C_1^t,\ A_2 = A_2^t, C_2 = C_2^t \right\}$$

A maximal compact subgroup K is given by $K = U(n)$. Then we have

$$H \cap K = O(n) \text{ and } \mathfrak{p} \cap \mathfrak{q} = \left\{ \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix} : B = B^t \right\}$$

We define the matrix unit $E_{i,j}$ to be 1 in the $(i, j)^{th}$ place and 0 elsewhere. Then, a maximal abelian subspace of $\mathfrak{p} \cap \mathfrak{q}$ is given by

$$\mathfrak{a} = \left\{ \sum_{i=1}^{n} t_i (E_{i,i} - E_{n+i,n+i}) : t_1, \dots . t_n \in \mathbb{R} \right\}$$

Then

$$\mathfrak{a}_+ \leftrightarrow \{t_i \in \mathbb{R} : t_1 \geq \cdots \geq t_p \geq 0, t_{p+1} \geq \cdots \geq t_n \geq 0\}$$

$$A_+ = \{a(t) : t = (t_1, \cdots, t_n), t_1 \geq \cdots \geq t_p \geq 0, t_{p+1} \geq \cdots \geq t_n \geq 0\}$$

$$= \left\{ exp\left(\sum_{i=1}^{n} t_i (E_{i,i} - E_{n+i,n+i}) \right) : t_1 \geq \cdots . \geq t_n \geq 0 \right\}$$

$$= \{ \sum_{i=1}^{n} (e^{t_i} E_{i,i} + e^{-t_i} E_{n+i,n+i}) \}$$

Since the center $\mathfrak{c}(\mathfrak{k})$ is one dimensional. Let the center of $\mathfrak{k}$ be defined by $\mathfrak{c}(\mathfrak{k}) = \mathbb{R}J$, under the isomorphism $\mathfrak{k} \simeq \mathfrak{u}(n)$, J corresponds to $\sqrt{-1}\left(\sum_{i=1}^{n} E_{i,i}\right)$ shows that $\mathfrak{k}$ is isomorphic to the sum : $\mathfrak{c}(\mathfrak{k}) \oplus \mathfrak{u}(n)$, where $\mathfrak{c}(\mathfrak{k})$ is the center of $\mathfrak{k}$.

Now we see that $\mathfrak{g} = \mathfrak{k} + \mathfrak{p} = \left\{X \middle| X = \frac{1}{2}(X - X^t) + \frac{1}{2}(X + X^t)\right\}$

The Cartan projection $pr^{\theta} : \mathfrak{g} \to \mathfrak{h}$ is given by matrix as follows:

$$\mathfrak{g} \to \mathfrak{k}, \quad \begin{pmatrix} A & B \\ B^* & c \end{pmatrix} \mapsto \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix}$$

Then we get the projection

$$pr^{\theta} : \mathfrak{g} \to \mathfrak{h}, \; X \mapsto \frac{1}{2}(X - X^t) \tag{6}$$

Now, for $a(t) \in A_+$ and for $z \in \mathfrak{c}(k)$ for $t_1, \ldots, t_n \in \mathbb{R}$, we take

$$Ad(a(t))z = \sum_{i=1}^{n} e^{2t_i}(E_{i,n+i} - e^{2t_i}E_{n+i,i})$$

Remark: 1

The above normalization of the centre element z is different from the previous sections.

And thus under the projection (6) we have

$$pr^{\theta}\big(Ad\big(a(t)\big)z\big) = \left(\sum_{i=1}^{n} cosh2t_i \left(E_{i,n+i} - E_{n+i,i}\right)\right)$$

This corresponds to

$$\left\{\sqrt{-1}\begin{pmatrix} cosh2t_1 & & & 0 \\ & cosh2t_2 & & \\ & & \ddots & \\ 0 & & & cosh2t_n \end{pmatrix} \in \mathfrak{k} \simeq \mathfrak{u}(n)\right\} \tag{7}$$

Therefore we see that $pr^{\theta}(Ad(a(t))z) \in \mathfrak{h} \cap \mathfrak{k}$.

Likewise, for $t_1^{'}, \ldots, t_n^{'} \in \mathbb{R}$ we take $a^{'} \in \mathbb{R}$ such that we obtain

$$pr^{\theta}\left(Ad\left(a^{'}(t)\right)z = \left(\sum_{i=1}^{n} cosh2t'_i \,(E_{i,n+i} - E_{n+i,i}\right)\right)$$

$$= \left\{\sqrt{-1}\begin{pmatrix} cosh2t'_1 & & & 0 \\ & cosh2t'_2 & & \\ & & \ddots & \\ 0 & & & cosh2t'_n \end{pmatrix} : t_1, \ldots, t_n \in \mathbb{R}\right\} \tag{8}$$

We can define a positive system $\sum^{+}(\mathfrak{g}, \mathfrak{a})$ such that the corresponding dominant Weylchamber is given by

$$\mathfrak{a}_+ = \left\{\sum_{i=1}^{n} t_i(E_{i,m+i} + E_{m+i.i}) : \; t_1 \geq \cdots \geq t_n \geq 0\right\}$$

We put $A_+ = exp(\mathfrak{a}_+)$. Now assume that $pr^{\theta}(Ad(a^{'})z \in Ad(K)pr^{\theta}(Ad(a)z)$ for $a, a^{'} \in A_+$.

Remark: 2

If $\mathfrak{h} = \mathfrak{g}^{\tau} = \mathfrak{u}(p,q)$, then we have $\mathfrak{h}^{a} = \mathfrak{g}^{\tau\theta} = \mathfrak{sp}(p,\mathbb{R}) \oplus \mathfrak{sp}(q,\mathbb{R})$. In fact in this case $\mathfrak{h}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}}$, where $\mathfrak{c}(\mathfrak{k})$ is the center of $\mathfrak{k}$.

Now we recall that if the Lie algebra $\mathfrak{g}$ is Hermitian, then $\mathfrak{g}$ and $\mathfrak{k}$ have the same rank. That is, there exists a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{k}$. We choose a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{k}$ of $\mathfrak{g}$. Then we have the following:

Lemma: 1

For any $X, X^{'} \in \mathfrak{t}$, the following are equivalent.

(i) X and X' are conjugate under G

(ii) X and X' are conjugate under K

(iii) X and X' are conjugate under the Weyl group [13] is given by $W(\mathfrak{h}, \mathfrak{t}) \simeq S_p \times S_q$.

And also we observe that $pr^{\theta}(Ad(a(t))z) \in \mathfrak{t}$. Therefore, under the permutation group S_p we have

$$pr^{\theta}(Ad(a(t))z) \sim pr^{\theta}(Ad(a(t'))z)$$

and under the permutation group S_q we have

$$pr^{\theta}(Ad(a(t))z) \sim pr^{\theta}(Ad(a(t'))z)$$

Proof: The Carten subalgebra $\mathfrak{k}$ in $\mathfrak{g}$ can be taken as

$$\mathfrak{k} = \sqrt{-1}\left\{\sum_{i=1}^{n} \mathbb{R}E_{i,i}\right\} = \sqrt{-1}\left\{\begin{pmatrix} a_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{m+n}\end{pmatrix} : a_1, \dots, a_{m+n} \in \mathbb{R}\right\}$$

Then in the computations (7), (8) we observe that

$$pr^{\theta} Ad(a(t))z, pr^{\theta}(Ad(a'(t))z \in \mathfrak{k}$$

Whence, the lemma easily follows.

Now we complete the proof of Proposition 1.

Assume $pr^{\theta}(Ad(a(t))z)$ and $pr^{\theta}(Ad(a'(t))z)$ are conjugate by $K \simeq U(n)$. Then following the remark 2 in this case means that

$$pr^{\theta}(Ad(a(t))z) \sim pr^{\theta}(Ad(a'(t))z)$$

Then Lemma 1 in this special case amounts that there exists an element $\sigma \in S_n$, such that $t_j = t_{\sigma(i)} = t_i'$, for $1 \leq i \leq j \leq n$.

Again it follows from our assumption: $t_1 \geq \cdots \geq t_p \geq 0$, $t_1' \geq \cdots \geq t_p' \geq 0$. Whence it is easily proved that $a = a'$ for any $a, a' \in A_+$. Therefore we have proved Proposition 1 in the case where $G = Sp(n, \mathbb{R})$.

VI. Conclusion

Many experiments and theoretical observations have been made to understand how orbit method, multiplicity function are applied to another fields of science. This paper has made and attempt to describe first: the essence of the Symplectic Lie Groups for non-experts and second: to attract the younger generation of mathematicians to some old and still unsolved problems in multiplicity function where we believe the orbit method could be helpful. This paper could be a guideline of studying of modern approach in this area of matrix computations of Corwin Greenleaf multiplicity function for Symplectic Lie Groups.

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