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Solutions of Klein - Gordan equations, using Finite Fourier Sine Transform

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Abstract: *In this paper the finite Fourier sine transform is presented for obtaining solutions for Klein-Gordan equations. The initial-boundary value problems for the Klein-Gordan equations are solved on the half range, using finite Fourier sine transform. Such problems posed on time-depend domain. The results reveal that the finite Fourier sine transform is very effective, simple, convenient and flexible.*
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I. Introduction

In this paper we consider two important equations of mathematical physics, the homogeneous Klein-Gordan equation[10]

$$u_{tt}(x,t) - u_{xx}(x,t) + u(x,t) = 0 \tag{1}$$

and the non-homogeneous Keilin-Gordan equation

$$u_{tt}(x,t) - u_{xx}(x,t) + u(x,t) = 0 \tag{2}$$

which appear in quantum field theory, relativistic physics, dispersive wave-phenomena, plasma physics, and applied physical sciences. Several techniques including finite difference, collocation, finite element, scattering, decomposition and variation iteration using Adomian's polynomials have been used to handle such equations [1,2,10,11]. He [4,13] developed the homotopy perturbation technique for solving such physical problems. In recent years, many research workers have been paid attention, to study the solutions of partial differential equations by using various methods. Among these are the Adonian decomposition method (ADM) [3], He's semi-inverse method [3], the tanh method, the homotopy perturbation method (HPM), the differential transform method and the variational iteration method (VIP) [5,8]. He [6,7] developed the homotopy perturbation method (HPM) by merging the standard homotopy and perturbation for solving various physical problems. Various ways have been proposed recently to deal with these partial differential equations, such as Adomian decomposition method. In this work we apply the finite Fourier sine transform method to solve homogeneous and non-homogeneous linear Klein-Gordan equations[12].

II. The finite Fourier sine transform

Definition (1):
The finite Fourier sine transform of a function $u(x,t)$ denoted $\hat{u}(n,t)$ is defined by[9]:

$$\mathcal{F}_s\big(u(x,t)\big) = \hat{u}(n,t) = \int_0^a u(x,t)\sin\left(\frac{n\pi x}{a}\right)dx \tag{3}$$

where n is an integer. The function $u(x,t)$ is then called the inverse finite Fourier transform and is given by :.

$$u(x,t) = \frac{2}{a}\sum_{n=1}^{\infty}\hat{u}(n,t)\sin\left(\frac{n\pi x}{a}\right) \tag{4}$$

Definition (2):
If u is some function of x and t, then finite Fourier of t for $0 < x < a$ and $t > 0$ is given by [9]:

$$\mathcal{F}_s\left(\frac{\partial^2 u(x,t)}{\partial x^2}\right) = \frac{-n^2\pi^2}{a^2}\hat{u}(n,t) + \frac{n\pi}{a}\,[u(0,t) - u(a,t)\cos(n\pi)] \qquad (5)$$

To illustrate the basic idea of this method, we consider a general non-homogeneous linear partial differential equation of the form:

$$Au_{tt}(x,t) + Bu_{xx}(x,t) + Cu(x,t) = h(x,t) \qquad (6)$$

with boundary conditions:

$$u(0,t) = u(a,t) = 0 \qquad (7)$$

and initial coditions:

$$u(x,0) = f(x) \qquad (8)$$

$$u_t(x,0) = g(x) \qquad (9)$$

where A, B,and C are constants. Taking the finite Fourier sine transform of both sides of Eq(6), we obtain

$$A\frac{d^2}{dt^2}\hat{u}(x,t) + \left(C - B\frac{n^2\pi^2}{a^2}\right)\hat{u}(n,t) + \frac{n\pi}{a}(u(0,t) - u(a,t)\cos(n\pi)) + C\hat{u}(n,t)$$

$$= H(n,t) \qquad (10)$$

where $H(n,t) = \mathcal{F}_s\,(h(x,t))$,using the boundary condition (7) and associating like terms, Eq(10)becomes

$$A\frac{d^2}{dt^2}\hat{u}(x,t) + \left(C - B\frac{n^2\pi^2}{a^2}\right)\hat{u}(n,t) = H(n,t) \qquad (11)$$

which is a second order ordinary differential equation, and has the following solutions:

Case1: if $C > B\frac{n^2\pi^2}{a^2}$ then, solution of Eq(11)is:

$$\hat{u}(n,t) = c_1\cos(\beta_0 t) + c_2\sin(\beta_0 t) + \lambda(t) \qquad (12)$$

Case2: if $C < B\frac{n^2\pi^2}{a^2}$ then, solution of Eq(11) is:

$$\hat{u}(n,t) = c_1\cosh(\beta_0 t) + c_2\sin h(\beta_0 t) + \lambda(t) \qquad (13)$$

where $\beta_0 = \sqrt{\frac{Ca^2 - Bn^2\pi^2}{Aa^2}}$ and $\lambda(t)$ is the particular solution of Eq(11).
It is easy to show that $c_1 = \lambda(0) + f(x)$ and $c_2 = \frac{\lambda(0)+g(x)}{\beta_n}$ by applying the finite Fourier transform to the initial conditions (8) and (9).

Case3: if $C = B\frac{n^2\pi^2}{a^2}$, then the solution of Eq(11) is:

$$\hat{u}(n,t) = f(x) + tg(x)\frac{1}{A}\int_0^t (t-\tau)H(n,t)dt \qquad (14)$$

Taking the inverse finite Fourier sine transform to get the final solution using Eq(4).

III. Applications

The finite Fourier transforms are used to solve differential equations arising in boundary value problems of physics and mechanics [9]. In this section we will apply the finite Fourier sine transform to solve homogeneous and non-homogeneous linear Klein-Gordan equations [12]:

3.1 Illustrative example

Consider the following boundary value problem [9]

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 t}{\partial x^2} \qquad (15)$$

with boundary conditions:

$$u(0,t) = u(5,t) = 0 \quad , \qquad (16)$$

and initial condition:

$$u(x,0) = 10\sin(4\pi x) \quad , \qquad 0 < x < 5 \qquad (17)$$

Taking finite Fourier sine transform of both sides of Eq(15), and using Eq(5) (with $a = 5$), we obtain

$$\mathcal{F}_s\left(\frac{\partial u(x,t)}{\partial t}\right) = \frac{-2n^2\pi^2}{25}\hat{u}(n,t) + \frac{2n\pi}{5}\,[u(0,t) - u(5,t)\cos(n\pi)] \qquad (18)$$

then using the boundary conditions (16), Eq(18) becomes

$$\frac{d}{dt}\hat{u}(n,t) = \frac{-2n^2\pi^2}{25}\hat{u}(n,t) \qquad (19)$$

which is a separable ordinary differential equation, and has a solution

$$\hat{u}(n,t) = Ce^{\frac{-n^2\pi^2}{25}t} \qquad (20)$$

Taking finite Fourier sine transform of the initial condition (17),combining with Eq(20), we have

$$\hat{u}(0,t) = C = \int_0^5 10\sin\left(\frac{n\pi}{5}x\right)dx = \begin{cases} 25 \ , & for \ n = 20 \\ 0 \ , & for\ all\ other\ values\ of\ n \end{cases} \qquad (21)$$

Hence, Eq(20) becomes

$$\hat{u}(n,t) = 25e^{\frac{-n^2\pi^2}{25}t} \qquad (22)$$

Taking inverse Fourier sine transform of Eq(22), we obtain

$$\hat{u}(n,t) = 10e^{-2n^2t}\sin(n\pi x) \qquad (23)$$

which is the required solution.

3.2 The homogeneous linear Klein-Gordan equation

we next investigate the Klein-Gordan equation[12]:

$$u_{tt}(x,t) - u_{xx}(x,t) + u(x,t) = 0 \qquad (24)$$

with boundary conditions:

$$u(0,t) = u(a,t) = 0 \tag{25}$$

and initial condition:

$$u(x,0) = 0 \tag{26}$$

$$u_t(x,0) = x \tag{27}$$

Taking finite Fourier sine transform of both sides of Eq(24), and using Eq(5), we obtain

$$\mathcal{F}_s(u_{tt}(x,t)) = \frac{-n^2\pi^2}{a^2}\hat{u}(n,t) + \frac{n\pi}{a}\,[u(0,t) - u(a,t)\cos(n\pi)] + \hat{u}(n,t) = 0 \tag{28}$$

Then using the boundary conditions (25) and associating like terms, we get

$$\frac{d^2}{dt^2}\hat{u}(n,t) + \beta_0^2\hat{u}(n,t) = 0 \tag{29}$$

which is a second order homogeneous ordinary differential equations, and has a solution:

$$\hat{u}(n,t) = c_1\cos(\beta_n t) + c_2\sin(\beta_n t) \tag{30}$$

where

$$\beta_n = \frac{\sqrt{n^2\pi^2 + a^2}}{a} \tag{31}$$

c_1 and c_2 are arbitrary constants of integration, taking finite Fourier sine transform of the initial conditions (26), we get

$$c_1 = 0 \tag{32}$$

then Eq(30) becomes:

$$\hat{u}(n,t) = c_2\sin(\beta_n t) \tag{33}$$

taking finite Fourier sine transform of the initial conditions (27), we obtain

$$c_2 = \frac{-a^2}{n\pi\beta_n}\cos(n\pi) \tag{34}$$

then Eq(33) becomes

$$\hat{u}(n,t) = \frac{-a^2}{n\pi\beta_n}\cos(n\pi)\;\;\sin(\beta_n t) \tag{35}$$

Taking inverse Fourier sine transform of Eq(35), we get

$$u(x,t) = \frac{2}{a}\sum_{n=0}^{\infty}\frac{-a^2}{n\pi\beta_n}\cos(n\pi)\;\;\sin(\beta_n t)\sin\left(\frac{n\pi}{a}x\right) \tag{36}$$

Substituting β_n from Eq(31), we obtain

$$u(x,t) = \frac{-2a^2}{\pi}\sum_{n=0}^{\infty}\frac{\cos(n\pi)}{n\sqrt{n^2\pi^2 + a^2}}\;\;\sin\left(\frac{\sqrt{n^2\pi^2 + a^2}}{a}t\right)\sin\left(\frac{n\pi}{a}x\right) \tag{37}$$

which is the required solution.

3.3 The in-homogeneous linear Klein-Gordan equation

We next consider the in-homogeneous linear Klein-Godan equation [12]:

$$u_{tt}(x,t) - u_{xx}(x,t) + u(x,t) = 2\sin x \tag{38}$$

with boundary conditions:

$$u(0,t) = u(a,t) = 0 \tag{39}$$

and initial conditions:

$$u(x,0) = \sin x \quad , \quad 0 < x < a \tag{40}$$

$$u_t(x,0) = 1 \quad , \quad 0 < x < a \tag{41}$$

Taking finite Fourier sine transform of both sides of Eq(38), and using Eq(5), we obtain

$$\begin{aligned}\mathcal{F}_s(u_{tt}(x,t)) = \frac{-n^2\pi^2}{a^2}\hat{u}(n,t) + \frac{n\pi}{a}\,[u(0,t) - u(a,t)\cos(n\pi)] + \hat{u}(n,t)\\ = \mathcal{F}_s(2\sin x\,)\end{aligned} \tag{42}$$

Then using the boundary conditions (39) and associating like terms, we get

$$\frac{d^2}{dt^2}\hat{u}(n,t) + \beta_0^2\hat{u}(n,t) = \frac{n\pi}{a}\xi_0\cos(n\pi)\sin(a) \tag{43}$$

which is a second order non-homogeneous ordinary differential equations, where

$$\xi_0 = \frac{2a^2}{n^2\pi^2 - a^2} \quad , \quad n^2\pi^2 \neq a^2 \tag{44}$$

Eq(43) has a solution

$$\hat{u}(n,t) = c_1\cos(\beta_n t) + c_2\sin(\beta_n t) + \lambda_n \tag{45}$$

where β_n is as defined in Eq(31), and

$$\lambda_n = \frac{\xi_0}{\beta_n^2}\cos(n\pi)\sin(a) \tag{46}$$

Upon using the initial condition (40) gives

$$c_1 = \frac{\lambda_n}{\xi_0} \tag{47}$$

Then, using the initial condition (41), we have

$$c_2 = \frac{-a}{n\pi\beta_n}\,(\cos(n\pi) - 1) \tag{48}$$

Therefore Eq(43) becomes

$$u(x,t) = \sum_{n=1}^{\infty} \left(\frac{\lambda_n}{\xi_0} \cos(\beta_n t) + \frac{-a}{n\pi\beta_n} \ (\cos(n\pi) - 1) \sin(\beta_n t) + \lambda_n \right) \sin\left(\frac{n\pi}{a} x\right) \quad (49)$$

which is the required solution.

IV. Conclusion

After the direct application of finite Fourier sine transform method and from the results obtained, we can say that this method is easy to implement and effective. As a result, the conclusion that comes through this work, is that the finite Fourier sine transform method can be applied to other partial differential equation, due to the efficiency in the application to get the possible results.

References

[1]. S. Abbasbandy. Numerical solutions of nonlinear Klein-Gordon equation by variational iteration method. Internat.

[2]. A.M. Wazwaz. The modified decomposition method for analytic treatment of differential equations. Appl. Math. Comput. 2006, pp. 165-176.

[3]. G. Adomian. Solving Frontier Problems of Physics: The Decomposition Method. Boston: Kluwer Acad. Publ.1994.

[4]. J.H. He. Homotopy perturbation technique. Computer Methods in Applied Mechanics and Engineering. 1999.

[5]. J.H. He. Variational iteration method a kind of nonlinear analytical technique: some examples. International Journal of Nonlinear Mechanics. 1999.

[6]. J.H. He. Homotopy perturbation method: a new nonlinear analytical technique. Applied Mathematics and Computation. 2003.

[7]. E. Hesameddini, H. Latifizadeh. A new vision of the He’s homotopy perturbation method. International Journal of Nonlinear Sciences and Numerical Simulation, 2009.

[8]. E. Hesameddini, H. Latifizadeh. Reconstruction of variational iteration algorithms using the Laplace transform. International Journal of Nonlinear Sciences and Numerical Simulation. 2009.

[9]. S.L.Kakani, C.Hemajani, Mathematical physics,CBS Publisher and Distributors, New Dalhi (India),2004.

[10]. S.T. Mohyud-Din, M.A. Noor, K.I. Noor. some relatively new techniques for nonlinear problems. Math. Porb.Eng. Article ID 234849, 25 pages, doi: 10.1155/2009/234849, 2009.

[11]. S.T. Mohyud-Din, A. Yildirim. Variational iteration method for solving Klein- Gordon equations. Journal of Applied Mathematics, Statistics and Informatics, 2010.

[12]. Jagdev Singh, Devendra Kumar and Sushila Rathore, Application of Homotopy Perturbation Transform Method for Solving Linear and Nonlinea Klein- Gordon Equations, Journal of Information and Computing

[13]. Science,ISS 1746-765 England, UK,Vol. 7, No. 2, 2012, pp. 131-139.

[14]. A. Yildirim. An Algorithm for Solving the Fractional Nonlinear Schröndinger Equation by Means of the Homotopy Perturbation Method. International Journal of Nonlinear Science and Numerical Simulation. 2009.

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