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NILPOTENCY: A Characterization of the finite p-groups

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Abstract *As parts of the characterizations of the finite p-groups is the fact that every finite p-group is NILPOTENT. Hence, there exists a derived series (Lower Central) which terminates at e after a finite number of steps. Suppose that G is a p-group of class at least $m \geq 3$. Then $L_{m-1}G$ is abelian and hence G possesses a characteristic abelian subgroup which is not contained in Z(G). If $L_3(G) = 1$ such that p^m is the highest order of an element of $G/L_2(G)$ (where G is any p-group) then no element of $L_2(G)$ has an order higher than p^m. [1]*

Keywords and phrases : *Nilpotency, central series, order, commutator, abelian*

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I. Introduction

Sequel to the discovery of the existence of p-groups, various work have been done while researches are continuously being carried out from day to day by a number of eminent personalities.

Joseph L. Lagrange, in 1771 had a theorem accredited to him based on finite Group. Meanwhile, he did not prove this theorems all he did, essentially, was to discuss some special cases. The first complete proof of the theorem was provided by Abbati. In 1872, the Normegran mathematician L. Sylow had a collection of theorems on finite group named after him [4], [7], [8].

Moreover, the Sylow theorems have been proved in a number of ways, and the history of the proofs themselves are the subjects of many papers including (Waterhouse 1979), (Scharlau, 1988), (Casadia \& Zappa 1990), (Gow 1994), and to some extent (Meo 2004). Wielandt (1959) used combinatorial arguments to show part of the Sylow theorems [6].

Frattini had his argument on Sylow subgroups of a normal subgroup which was slightly generalized by Burnside as Burnside's fusion theorem.

Others are Brauer, Gorenstein and J.L. Alperin. [2], [5].

A subgroup H of a p-group G is said to be characterisstic if α(H) $\leq$ H for all $\alpha \in$ Aut(G).

Definition: If the lower central series terminates after a finite number of steps (i.e. G(n) = { e } for some n), then G is said to be nilpotent.

Theorem (Berkovich): A p-group is Nilpotent. [3]

Proof: Assume G > { e } .
Then Z(G) > { e }
where Z(G) is the centre of G and Z(G) = $\{x \in G | g^{-1}xy = x, \; g \in G\}$.
Define $\bar{G}$ = G/Z(G). Then we have the following series.
$\bar{G} = \bar{G}_o \geq \ldots\ldots\ldots\ldots \geq \bar{G}_{n+1} = Z(G)$
Showing that $\bar{G}$ is nilpotent with the identity Z(G). And hence, G is nilpotent.

Other property of Nilpotent Group.

Theorem: Suppose that G is a noncyclic nilpotent group. Then

(a) If $a \in G$, then $< a^x \mid x \in G > \; < G$.
(b) G/Z(G) is noncyclic.

Proof: Since G is nilpotent, there is a lower central series:
G = $G_o \geq G_1 \geq$ …………… $G_{n-1} \geq G_n = \{\epsilon\}$
So, obviously, G $> \{\epsilon\}$
Hence, if a , x ∈ G. Then $x^{-1} \in G$ and $ax^{-1} \in G$.
So, $xax^{-1} \in G$ and $< a^x | x \in G > < G$ where $a^x = xax^{-1}$.
$Z(G) = \{ x \in G | x = g^{-1}xg \in G \}$
Suppose that G/Z(G) is cyclic, then G/Z(G) = {az|z ∈ G } is cyclic ⇒⇐
Thus G/Z(G) is noncyclic.
We use the idea of higher commutators to define a sequence of subgroups of a group G, which we call
the lower central series of G, by the rules
$L_1(G) = [G, L_2(G)] = [G,G] = G'$,…….
$L_I(G) = [L_{i-1},G]$ for > 2.
Definition: The lower central series of a group G is given by
G = $G_o \supseteq G_1 \supseteq G_2 \supseteq$ ……….. where for i>0 , $G_i = [G_{i-1} , G]$.
A group G is called nilpotent if $L_m(G) = 1$ for some m . If n+1 is the least value of m satisfying this condition, then n is called the class of G , i.e., cl(G) = n.
Theorem A.1
(i) $L_i(G) Char(G)$ for all i.
(ii) $L_{i+1}(G) \subseteq L_i(G)$ and $L_i(G)/L_{i+1}(G) \subseteq Z(G/L_{i+1}(G))$
Theorem A.2: Let x,y,z be elements of G and H,K be subgroups of G . Then,
[H, K] = [K, H] ……………………………………..[1]
Theorem (α): Let G be a p -group of class at least m ≥ 3. Then $L_{m-1}(G)$ is abelian and hence G possesses a characteristic abelian subgroup which is not contained in the centre of G.
Proof: By (A.2), $L_{m-1}(G)$ is abelian since
$L_{m-1}(G) = [L_{m-2}(G), G] = [G, L_{m-2}(G)]$
Now, by (A.1(ii)), $L_{m-1}(G)/L_m(G) \subseteq Z(G/L_m(G))$
⇒ if $L_m(G) = 1$, then, $L_{m-1}(G) \subseteq Z(G)$. But $L_m(G) \neq 1$,
⇒ $L_{m-1}(G) \supseteq Z(G)$……………………………………………■
Theorem (:} Let G be a p-group with $L_3(G) = 1$. If p^m is the highest order of an element of $G/L_2(G)$, then no element of $L_2(G)$ has an order higher than p^m .[1]
Proof: By definition,
G = $L_1(G) \supset L_2(G) \supset$ ……. $\supset L_i(G) \supset L_{i+1}(G) = 1$
⇒ G = $L_1(G) \supset L_2(G) \supset L_3(G) = 1$
∴ $G/L_2(G) = \{a | o(a) \leq p^m \}$
= $\{gL_2(G) | g \in G$ and $|gL_2(G)| \leq p^m \}$
$(G \supset L_2(G) \supset 1)$.
⇒ $|g||L_2(G)| \leq p^m$
⇒ $|L_2(G)| \leq \frac{p^m}{p^k}$, $k \geq 1$
⇒ $|L_2(G)| \leq p^{m-k} < p^m$ ……………………………■

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