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Alternative Methods to Prove Theorem of Basis And Dimensions

Arpit Mishra
(Department of Mathematics)
(Hemvati Nandan Bahuguna Garhwal University, Srinagar (Garhwal), Uttarakhand,India) (A Central University)
Corresponding Author: Arpit Mishra

Abstract: *In this paper, we study about alternative methods by which we can proof the theorem,In a vector space V if $\{\alpha_1,\alpha_2,.....,\alpha_n\}$ generates V and if $\{W_1,W_2,.....W_m\}$ is linearly independent (LI), then $m \leq n$, where dim W = m and dim V = n.* *OR*
We can't have more LI vectors than the number of elements in a set of generators.
We all are familiar with the methods of proving the given theorems mentioned in books as reference books but there are also other methods by which we can prove the theorem using some theorems directly as statements.

Keywords: *Basis of A Vector Space, Dimension of A Vector Space, Linear Dependence of Vectors, Linear Independence of Vectors, Linear Combination of Vectors, Linear Span.*

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I. Introduction

1.1. Basis of A Vector Space :
If V(F) is a vector space and S is any subset of V(F), then S is called a basis for V(F) if :
1. S is LI.
2. **Every vector of V(F) is expressible as the linear combination of vectors of S uniquely**

i.e. S generates V(F) => L(S) = V(F).

1.2. Dimension of A Vector Space :
The number of vectors in the basis for a vector space V(F) is called dimension of V(F). It is denoted by dimV.

1.3 Linear Dependence of Vectors :
Let V(F) is a vector space and the set S = $\{W_1,W_2,.....W_m\}$ is finite set of vector in V(F), then S is called linearly dependent if there exists scalars $x_1,x_2,....,x_m$ not all zero such that
$x_1W_1+x_2W_2+\text{-------}+x_mW_m = 0$, briefly written as LD.

1.4Linear Independence of Vectors :
Let V(F) is a vector space and the set S = $\{W_1,W_2,.....W_m\}$ is finite set of vector in V(F), then S is called linearly dependent if there exists scalars $x_1,x_2,....,x_m$ all are zero such that
$x_1W_1+x_2W_2+\text{-------}+x_mW_m = 0$, briefly written as LI.

1.5. Linear Combination of Vectors :
Let V(F) is a vector space and $W_1,W_2,.....W_m$ be m-vectors and $x_1,x_2,....,x_m$ are m-scalars, then a vector
$W = x_1W_1+x_2W_2+\text{-------}+x_mW_m=\sum_{i=1}^{m} \sum W_m x_m$ is called Linear Combination of Vectors.

1.6. Linear Span :
If V(F) is a vector space and S is any subset of V(F), then the set of all Linear Combination of elements of S is called Linear Span of S and is denoted by L(S).
L(S) = $\{ W:W =\sum_{i=1}^{m} \sum W_m x_m , x_m \varepsilon F \text{ and } W_m \varepsilon S \}$
Here, L(S) also means that S generates.

II. Alternative Methods

2.1. Method 1
To prove this theorem, it is sufficient to show that every subset S of V which contains more than n vectors is linearly dependent (LD).
Suppose S = $\{W_1,W_2,.....W_m\}$ where m > n and all the vectors of S are distinct. Since $\{\alpha_1,\alpha_2,.....,\alpha_n\}$ generates V or span V, so that there exists scalars a_{ij} in F such that
$W_j = \sum_{i=1}^{n} \sum aij\alpha i$
For any scalars, $x_1,x_2,....,x_m$, we have
$x_1W_1+x_2W_2+\text{-------}+x_mW_m \quad = \sum_{j=1}^{m} \sum xjWj$

$= \sum_{j=1}^{m} \sum xj(\sum_{i=1}^{n} \sum aij\alpha i)$, (Since, $W_j = \sum_{i=1}^{n} \sum aij\alpha i$)
$= \sum_{j=1}^{m} \sum \sum_{i=1}^{m} \sum (aijxj)\ \alpha i$
$= \sum_{i=1}^{m} \sum (\sum_{j=1}^{m} \sum\ aijxj)\ \alpha i$
Since, we know that if A is a n×n matrix and n×m then the homogeneous system of linear equation AX = 0 has non- trivial solution.
Hence, for m > n, implies that there exists scslars $x_1, x_2, \ldots, x_m$ not all zero such that $\sum_{j=1}^{m} \sum\ aijxj = 0$, $1 \leq i \leq n$.
Hence, $x_1W_1+x_2W_2+\text{-------}+x_mW_m = 0$. This shows that $S = \{W_1, W_2, \ldots W_m\}$ m > n is linearly dependent (LD) set which contradicts the hypothesis that S is linearly independent (LI).
Hence, $m \ngtr n$ i.e. $m \leq n$.
Thus , We can't have more LI vectors than the number of elements in a set of generators.

2.2. Method 2
Given dim W = m and dim V = n .
So let set $S = \{ x_1, x_2, \ldots, x_n\}$ is a basis for V(F) and also we have L(S) = V(F).
Therefore, every element of V(F) be a linear combination of elements of S.
Also, W is given subspace of V(F) so clearly $W \subset V$.
Therefore, every element of W be also a linear combination of elements of S.
Here, S is linearly independent (LI).
Therefore, either S is a basis of W or any subset of S be a basis for W.
Thus, basis of W cannot contain more than n- elements.
Hence, dim W ≤ dim V or m ≤ n.

2.3. Method 3
If $A = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ generates a span V and if $S = \{W_1, W_2, \ldots W_m\}$ is LI, then we have to show that $m \leq n$.
If $W_1 \varepsilon V(F)$ then W_1 is a linear combination of the α_i's since $A = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ generates a span V.
So, $W_1 \varepsilon L(A)$ i.e. for some scalars $a_1, a_2, \ldots, a_n$.
$W_1 = a_1\alpha_1+a_2\alpha_2+\ldots+a_n\alpha_n$.
Since, S is LI and $W \varepsilon S$ then $W_1 \neq 0$, hence not all the a_i's are zero.
Therefore, let at least one $a_i \neq 0$, say $a_1 \neq 0$.
Hence, $\alpha_1 = a_1^{-1}W_1+(a_1^{-1}a_2)\alpha_2+(-a_1^{-1}a_2)a_3+\ldots+(-a_1^{-1}a_n\alpha_n)$.
This relation shows that any vector which is expressible as a linear combination of $\alpha_1, \alpha_2, \ldots, \alpha_n$ can be expressed as a linear combination of the vectors $W_1, \alpha_2, \alpha_3 \ldots, \alpha_n$ i.e.
$L (\{W_1, \alpha_2, \alpha_3 \ldots, \alpha_n\}) = L (\{\alpha_1, \alpha_2 \ldots, \alpha_n \}) = L(A) = V$, (Since , A generates V).
We can now repeat the above process of replacement with the vector W_2 and the generating set $\{W_1, \alpha_2, \alpha_3 \ldots, \alpha_n \}$instead of W_1 and the generating set $\{\alpha_1, \alpha_2 \ldots, \alpha_n \}$. This process would yield the relation
$V = \{(W_1, W_2, \alpha_2, \alpha_3 \ldots, \alpha_n)\}$.
We repeat this process with W_3 and so on . At each step we are able to add one W's and delete one of the α's in generating set.
If $m \leq n$, then we finally obtain a generating set or spanning set of the form $\{W_1, W_2, \ldots W_m, \alpha_j, \alpha_{j+1}, \ldots, \alpha_{n-m}\}$.
Lastly, we show that m > n is not possible. Otherwise, after n of the above steps, we obtain the generating sets $\{W_1, W_2, \ldots W_n\} \Rightarrow W_{n+1}$ is a linear combination of $W_1, W_2, \ldots W_n$ i.e. for scalars $c_1, c_2, \ldots, c_n$ such that
$W_{n+1} = c_1W_1+c_2W_2+\text{-------}+c_nW_n$
So the set $\{W_1, W_2, \ldots W_n, W_{n+1}\}$ and $\{W_1, W_2, \ldots W_m\}$ is LD which contradicts the hypothesis that $\{W_1, W_2, \ldots W_m\}$ is LI.
Hence, $m \leq n$.

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