IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 13, Issue 6 Ver. III (Nov. - Dec. 2017), PP 51-54
www.iosrjournals.org

The Atomic Decomposition Using only Properties of the Nontangential Maximal Functions Series u_r^* For Hardy Spaces

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Abstract: *We give an extremely easy proof of the atomic decomposition for distributions in* $H^{1-\varepsilon}(\mathbb{R}_+^2 \times \mathbb{R}_+^2), \varepsilon > 0$. *Our proof uses only properties of the nontangential maximal functions series* u_r^*. *We then a confirn our argument to give a "direct" proof of the Chang-Fefferman decomposition for* $H^{1-\varepsilon}(\mathbb{R}_+^2 \times \mathbb{R}_+^2)$.

Date of Submission: 28-11-2017 Date of acceptance: 18-12-2017

I. Introduction

Let $\mathbb{R}_+^{n+1} = \{(x,y): x \,\epsilon\, \mathbb{R}^n, y > 0\}$. Or $u_r(x,y)$ harmonic on $\mathbb{R}_+^{n+1}$ and $A > 0$ define

$$\sum_r u_r^*(x) = sup_{|x-t|<A_y} \sum_r |u_r(t,y)|.$$

We say that $u_r \epsilon\, H^{1-\varepsilon}$ if $u_r^* \,\epsilon L^{1-\varepsilon}$, for any A, and set $\|u_r\|_{H^{1-\varepsilon}} = \|u_r\|_{L^{1-\varepsilon}}$. If $u_r \epsilon H^{1-\varepsilon}$, $\varepsilon > 0$, then $\sum_r f_r = \sum_r \sum_i u_r(.,y)$ exists $(\mathcal{G}')$ and is said to be in $H^{1-\varepsilon}$. We set $\sum_r \|f_r\|_{H^{1-\varepsilon}} = \sum_r \|u_r\|_{H^{1-\varepsilon}}$ (see **[6,11]).**
For $\varepsilon \geq 0$, *dip-atom* is a functions series $a_r(x) \,\epsilon\, L^{2(1-\varepsilon)}(\mathbb{R}^n)$ satisfying:
(i) supp $a_r \subset (Q)_j, (Q)_j$ a cube.
(ii) $\|a_r\|_2 \leq |(Q)_j|^{\varepsilon/2(\varepsilon-1)}$ ($|(Q)_j|$ = the volume of $(Q)_j$).
(iii) $\int \sum_r a_r(x) x^\alpha dx = 0$ for all monomials x^α with $|\alpha| \leq [n(1-\varepsilon)^{-1} - 1)]$.
The following theorem is well known **[4,7,10,11]:**

THEOREM 1. *Let* $f_r \epsilon H^{1-\varepsilon}$, $\varepsilon \geq 0$. *There exist* $(1-\varepsilon)$*-atoms* $(a_r)_k$ *and numbers* λ_k *such that*

$$\sum_r f_r = \sum_r \sum_k \lambda_k (a_r)_k \;\; in\; \mathcal{G}' \qquad (1)$$

The λ_k satisfy $\sum_k |\lambda_k|^{\varepsilon-1} \leq C(1-\varepsilon, n) \sum_r \|f_r\|_{H^{1-\varepsilon}}^{1-\varepsilon}$. Conversely, every sum (1) satisfies

$$\sum_r |f_r|_{H^{\varepsilon-1}}^{\varepsilon-1} \leq C(\varepsilon-1, n) \sum_k |\lambda_k|^{\varepsilon-1}.$$

Now let u_r be biharmonic on $\mathbb{R}_+^2 \times \mathbb{R}_+^2$. Define

$$\sum_r (u_r)_A^*(x_1, x_2) = sup_{\substack{|x_i - t_i| < A_{y_i} \\ i=1,2}} \sum_r |u_r(t_1, y_1, t_2, y_2)|$$

As before, we say that $u_{r-1} \boldsymbol{\epsilon}\, H^{1-\varepsilon}(\mathbb{R}_+^2 \times \mathbb{R}_+^2)$ if $(u_r)_A^* \boldsymbol{\epsilon}\, L^{1-\varepsilon}(\mathbb{R}^2)$, and we set $\|u_r\|_{H^{1-\varepsilon}} = \|u_r^*\|_{L^{1-\varepsilon}}$. Such u give rise to boundary distributions f_r, which are said to be in $H^{1-\varepsilon}$. (See **[2,].)**
For $\varepsilon > 0$, a *Chang-Fefferman p-atom* is a functions series $a_r \boldsymbol{\epsilon} L^{1-\varepsilon}(\mathbb{R}^2)$ satisfying:
(a) supp $a_r \subset \Omega$, Ω open, $|\Omega| < \infty$.
(b) $\|a_r\|_2 \leq |\Omega|^{\frac{\varepsilon}{2(\varepsilon-1)}}$.
(c) $a_r = \sum_K \lambda_K (a_r)_K$, where λ_K are numbers and the $(a_r)_K$ are functions series atisfying:
(i) supp $(a_r)_K \subset \overline{K} \subset \Omega$ where $K = I \times J, I, J$ dyadic intervals, and $\overline{K}$ denotes the triple of K.
(ii)

$$\sum_r \left\| \frac{\partial^L (a_r)_K}{\partial x_1^L} \right\| \leq \frac{1}{\sqrt{|K|}|I|} \;\; and \; \sum_r \left\| \frac{\partial^L (a_r)_K}{\partial x_2^L} \right\| \leq \frac{1}{\sqrt{|K|}|J|^L}$$

for all $L \le \left[\frac{3+\varepsilon}{2(1-\varepsilon)}\right]$
(iii)

$$\int \sum_r a_r(\tilde{x}_1, x_2) x_2^k dx_2 = 0 \; and \int \sum_r a_r(x_1, \tilde{x}_2) x_1^k dx_1$$

for all $(\tilde{x}_1, x_2) \in \mathbb{R}^2$ and all $k < \left[\frac{1+3\varepsilon}{2(1-\varepsilon)}\right]$. And
If the "atoms" are Chang-Fefferman atoms, then Theorem A is true for
$f_r \epsilon H^{1-\varepsilon}(\mathbb{R}_+^2, \mathbb{R}_+^2)$[2][3].
Until now, proofs of the atomic decomposition have relied on showing that $u_r^* \epsilon L^{1-\varepsilon}$ implies that some auxiliary functions series(such as the "grand" maximal function or the S_r-functions series) is in $L^{1-\varepsilon}$. In this paper, we give proofs which get the atoms directly from $u_r^* \epsilon L^{1-\varepsilon}$.

II. The case $H^{1-\varepsilon}(\mathbb{R}_+^2)$ Let $\psi \in C^\infty(\mathbb{R})$ be real, radial, $supp\, \psi \subset \{|x| < 1\}$, ψ has the cancellation property γ), and

$$\int_0^\infty e^{-\theta} \hat{\psi}(\theta) d\theta = -1.$$

For $y > 0$, set $y^{-1}\psi(t/y) = \psi_y(t)$.
Take $f_r \epsilon L^{2(1-\varepsilon)} H^{1-\varepsilon}$, f_r real-valued, $u_r = P_y * f_r$ (the Poisson integral of f_r) . By Fourier transforms

$$\sum_r f_r = \int_{\mathbb{R}_+^2} \frac{\partial u_r}{\partial y}(t,y)\, \psi_y(x-t) dt dy \; in \; \mathcal{G}'.$$

(This trick is due to A. P. Calderón.) For $k = 0, \pm 1, \pm 2, \ldots$, define

$$E^k = \{(u_r)_2^* > 2^k] = \bigcup_{j=1}^\infty I_j^k$$

where the I_j^k are component intervals. For I an interval, let

$$\hat{I} = \{(t,y) < \mathbb{R}_+^2 : (t-y, t+y) \subset I\}$$

be the " tent" region. Define $\hat{E}^k = \cup\, \hat{I}_j^k, T_j^k = \hat{I}_j^k \setminus \hat{E}^{k+1}$. Then

$$\sum_r f_r = \sum_{k,j} \sum_r \int_{T_j^k} \frac{\partial u_r}{\partial y}(t,y)(\psi_r)_y(x-t)\, dt dy = \sum_{k,j} g_j^k = \sum_{k,j} \sum_r \lambda_j^k (a_r)_j^k,$$

where $\lambda_j^k = C 2^k |I_j^k|^{\frac{1}{1-\varepsilon}}$ and the $(a_r)_j^k$ (we claim) are atoms. The $(a_r)_j^k$ inherit γ from ψ_r, and obviously $supp\,(a_r)_j^k \subset \hat{I}_j^k$. Note also that
Thus, we are done if we can show
We do this by duality. Let $h_r \epsilon L^{2(1-\varepsilon)}(\mathbb{R}), \|h_r\|_2 = 1$. Then

$$\left|\iint h_r(x)(g_r)_j^k(x) dx\right| = \left|\int_{T^k} \frac{\partial u_r}{\partial y}(t,y)\left(h_r * (\psi_r)_y(t)\right)^2 \frac{dt dy}{y}\right|$$

$$\le \left(\int_{T_j^k} y|\nabla u_r|^2\, dt\, dy\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}_+^2} \left|h_r * (\psi_r)_y\right|^2 \frac{dt\, dy}{y}\right)^{\frac{1}{2}} \le C\left(\int_{T_j^k} y|\nabla u_r|^2\, dt\, dy\right)^{\frac{1}{2}}$$

We estimate the last integral by Green's Theorem. It is bounded by

$$\left(\int_{\partial T_j^k} \left(|u_r| y \left|\frac{\partial u_r}{\partial v}\right| + \frac{1}{2}(u_r)^2 \left|\frac{\partial y}{\partial v}\right|\right) ds\right)$$

($\frac{\partial}{\partial v}$ is outward normal; ∂T_j^k is just smooth enough to let us use Green's
Theorem). Because of the "2" (in $(u_r)_2^*$), both $|u_r|$ *and* $y|\nabla u_r|$ are bounded by $C2^k$ on ∂T_j^k. Since $\left|\frac{\partial y}{\partial v}\right| < 1$ and $|\partial T_j^k| < C|I_j^k|$, the last term is no
larger than $C2^k |I_j^k|^{\frac{1}{2}}$.

III. The case $H^{1-\varepsilon}(\mathbb{R}_2^{n+1})$. Let ψ_r be as in **II,** except now $\psi_r \epsilon C^\infty(\mathbb{R}^n)$. Let $f_r \epsilon H^{1-\varepsilon} \cap L^{2(1-\varepsilon)}$ and u be as before. Define
where the Ω_j^k are Whitney cubes (for the definition see [9]). For Ω a cube in $\mathbb{R}^n$, define

$$\hat{\Omega} = \{(t,y): t \epsilon \Omega, 0 < j > v < l(\Omega)\}$$

where $l(\Omega) = sidelength\ of\ \Omega$. Define
With these modifications, the preceding argument goes over practically verbatim; the details are left to the reader.

IV. **The case** $H^{1-\varepsilon}(\mathbb{R}_+^2 \times \mathbb{R}_+^2)$. We first show that the proof in II yields a Chang-Fefferman decomposition for $\mathbb{R}_+^2$. For $I \subset \mathbb{R}$ a dyadic interval, let

$$I^+ = \{(t,y): t\epsilon I, |I|/2 < y < l|I|\}.$$

Define

$$\mathcal{G}_j^k = \{I^+ \cap T_j^k\}, \qquad g_Q = \int_Q \frac{\partial u_r}{\partial y}(t,y)(\psi_r)_y(x-t)dt\, dy = \lambda_j^k \lambda_Q (a_r)_Q \quad for\ Q\epsilon\, \mathcal{G}_j^k$$

where we set
Then it is easily verified that the $(a_r)_Q$ have the right cancellation, support and smoothness properties for elementary particles. And obviously

$$(a_r)_j^k = \sum_{Q\epsilon \mathcal{G}_j^k} \lambda_Q\ (a_r)_Q,$$

$$\left(\sum_{Q\epsilon \mathcal{G}_j^k} \lambda_Q^2\right)^{\frac{1}{2}} \leq |\hat{I}_j^k|^{\frac{1-2\varepsilon}{2(1-\varepsilon)}}.$$

In order to do our proof in $\mathbb{R}_+^2 \times \mathbb{R}_+^2$, we need tents, and we need a way to do Green's Theorem. For these, we need some notation.
For $(t,y) = (t_1,y_1,t_2,y_2) \epsilon (\mathbb{R}^2)^2$, let $K_{t,y}$ be the rectangle with sides parallel to the coordinate axes, centered at $(t_1,t_2) \epsilon \mathbb{R}^2$, and with dimensions $2y_1 \times 2y_2$.
Take $f_r \epsilon \cap L^{2(1-\varepsilon)}, H^{1-\varepsilon} = P_{y_1}.P_{y_2} * f_r$(the double Poisson integral of f_r).
Let ψ_r be as in II but with cancellation corresponding to (iii). Then

$$\sum_r f_r = \sum_r \int_{(\mathbb{R}_+^2)^2} \frac{\partial^2 u_r}{\partial y_1 \partial y_2}(t,y)(\psi_r)_{y_1}(x_1-t_1)(\psi_r)_{y_2}(x_2-t_2)dt\, dy\ in\ \mathcal{G}'$$

Let M be the strong maximal functions series. Let $\delta > 0$ be small, to be chosen later. Define

$$E^k = \{u_{100}^* > 2^k), F^k = \{M\chi_{E^k} > \delta).$$

It is a fact that $|F^k| \leq C_\delta |E^k|$. Set

$$\hat{F}^k = \{(t,y) : K_{t,y} \subset F^k\},$$
$$T^k = F^k \backslash \hat{F}^{k+1}$$

$$\sum_r (g_r)^k = \sum_r \int_{T^K} \frac{\partial^2 u_r}{\partial y_1 \partial y_2}(t,y)(\psi_r)_{y_1}(x_1-t_1)(\psi_r)_{y_2}(x_2-t_2)dt\, dy = \sum_r \sum_k \lambda_k (a_r)_k$$

where we set $\lambda_k = C2^k|E^k|^{\frac{1}{1-\varepsilon}}$.
For $K = I \times J, I, J$ dyadic intervals, let $K^+ = I^+ \times J^+ \subset \mathbb{R}_+^2 \times \mathbb{R}_+^2$. Set
$\mathcal{G}_k = \{\boldsymbol{Q} = \boldsymbol{K}^+ \times \boldsymbol{T^k}\}$,

$$\sum_r (\boldsymbol{g}_r)_{\boldsymbol{Q}}$$
$$= \sum_r \int_{T^K} \frac{\partial^2 u_r}{\partial y_1 \partial y_2}(t,y)(\psi_r)_{y_1}(x_1-t_1)(\psi_r)_{y_2}(x_2-t_2)dt\, dy = \sum_r \sum_k \lambda_k \lambda_Q (a_r)_k \quad (Q\epsilon\, \mathcal{G}_k)$$

where we set

$\lambda_Q = \sum_k \sum_r C(\lambda_K^{-1}) \left(\int_Q y_1 y_2 |\nabla_1 \nabla_2 u_r|^2 dt\, dy\right)^{\frac{1}{2}}$ **with** $|\nabla_1 \nabla_2 u_r|^2 = \left|\frac{\partial^2 u_r}{\partial x_1 \partial x_2}\right|^2 + \left|\frac{\partial^2 u_r}{\partial x_1 \partial y_2}\right|^2 + \left|\frac{\partial^2 u_r}{\partial y_1 \partial x_2}\right|^2 + \left|\frac{\partial^2 u_r}{\partial y_1 \partial y_2}\right|^2.$

Then, in exact analogy to case II, everything will be done once we show

$$\sum_r \int_{T^k} y_1 y_2 |\nabla_1 \nabla_2 u_r|^2 dt\, dy \leq C 2^{2k} |E^k|$$

For this we need a lemma of Merryfield. The lemma requires a little more notation.

Let $\eta \,\epsilon\, C^\infty(\mathbb{R}), \eta \geq 0, supp\, \eta\, c\, [-1,1], \eta \geq \frac{1}{2} on\, [-\frac{1}{2}, \frac{1}{2}]$ and $\int \eta = 1$.

Define For $E \epsilon \mathbb{R}^2$, set

Now, $V_E(t,y)$ is essentially the density of E in $K_{t,y}$, r In particular, if this density is greater than $1 — \delta, \delta$ small, then $V_E(t,y) > 10^{-6}$.

Merryfield's lemma is [8]:

$$\sum_r \int_{(\mathbb{R}_+^2)^2} y_1 y_2 |\nabla_1 \nabla_2 u_r|^2 V_E^2(t,y) dt\, dy \leq C\lambda^2 |E|$$

(Note: Merryfield states this for E open, but openness, as his proof shows, is not required.)

Let us set $G^k = F^k \backslash E^{k+1}$. Merryfield's lemma says that

$$\sum_r \int_{\mathbb{R}_+^2} y_1 y_2 |\nabla_1 \nabla_2 u_r| V_{G^k}^2(t,y) dtdy \leq C 2^{2k} |G^k| \leq C 2^{2k} |E^k|$$

Therefore, we will have (2) (and be done) if we can show

$$V_{G^k} > 10^{-6}\, on\, T^k.$$

Take $(t,y) \epsilon\, T^k$. Then $K_{t,y} \subset F^k$ but $K_{t,y} \not\subset F^{k+1}$. So there is an $x\, \epsilon K_{t,y} \cap (F^k \backslash F^{k+1})$. Since $x \notin F^{k+1}, M\chi_{E^{k+1}}(x) < \delta$. From the definition of M, this implies

$$|K_{t,y} \cap E^{k+1}| / |K_{t,y}| \leq \delta.$$

Since $K_{t,y} \subset F^k$,

$$|K_{t,y} \cap (F^k \backslash E^{k+1})| / |K_{t,y}| \geq 1 - \delta.$$

But $F^k \backslash E^{k+1} = G^k$, and this imphes that $V_{G^k}(t,y) > 10^{-6}$, for δ small.

References

[1]. A. P. Calderón, An atomic decomposition of distributions in parabolic Hp spaces, Adv. in Math., 25 (1977), 216-225.
[2]. S.-Y. Chang and R. Feffeπnan, A continuous version of duality of Hι with BMO on the bidisc, Ann. of Math., **112** (1980), 179-201.
[3]. The Calderón-Zygmund decomposition on product domains, preprint.
[4]. R. R. Coifman, A real variable characterization of Hp, Studia Math., 51 (1974), 269-74.
[5]. R. Coifman, Y. Meyer and E. M. Stein, Un nouvel espace fonctionnel adapte a Γ etude des operateurs definiepar des integrates singulieres, preprint.
[6]. C. Fefferman and E. M. Stein, Hp spaces of several variables, A cta Math., **129** (1972), 137-193.
[7]. R. H. Latter, A characterization of $H^p(\mathbb{R}^p)$ in terms of atoms, Studia Math., 62 (1978), 93-101.
[8]. K. G. Merryfield, Hp spaces in poly-half spaces, Ph.D. Thesis, University of Chicago, 1980.
[9]. E. M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton University Press, 1970.
[10]. J. M. Wilson, A simple proof of the atomic decomposition for $H^p(\mathbb{R}^n), 0 < p < 1$, Studia Math., 74 (1982), 25-33.
[11]. J. MICHAEL WILSON, On The Atomic Decomposition for Hardy Spaces, Pacific Journal of Mathematics Vol. 116, No. 1, 1985

Ahmed Abdallatif "The Atomic Decomposition Using only Properties of the Nontangential Maximal Functions Series u_r^* For Hardy Spaces." IOSR Journal of Mathematics (IOSR-JM) 13.6 (2017): 51-54.