IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 13, Issue 6 Ver. III (Nov. - Dec. 2017), PP 45-47
www.iosrjournals.org

On Closed Subsets Of Free Groups

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Abstract: *We give two examples of a finitely generated subgroup of a free group and a subset, closed in the prfinite topology of a free group, such that their product is not closed in the profinite topology of a free group.*
Keywords: *Free group, Profinite topology, Closed set.*

Date of Submission: 24-11-2017 Date of acceptance: 22-12-2017

I. Introduction

A theorem of M. Hall, proved in [4], states that any finitely generated subgroup of a free group is closed in the profinite topology. This result has been generalized by many researchers.

The authors proved in [2] and [3] that the product of two finitely generated subgroups of a free group is closed in the profinite topology of the free group. The first published proof of that theorem is due to G.A. Niblo, cf. [6].

Denote the profinite topology on a free group F by $PT(F)$.

A more general result saying that for any finitely generated subgroups $H_1, \cdots, H_n$ of a free group F the set $H_1 \cdots H_n$ is closed in $PT(F)$ was obtained by L. Ribes and P.A. Zalesskii in [7], by K. Henckell, S.T. Margolis, J.E. Pin, and J. Rhodes in [5], and by B. Steinberg in [8].

T. Coulbois in [1] proved that property RZ_n is closed under free products, where a group G is said to have property RZ_n if for any n finitely generated subgroups $H_1, \cdots H_n$ of G, the set $H_1 \cdots H_n$ is closed in $PT(G)$.

The aforementioned results lead to the following question: is it true that for any finitely generated subgroup H of a free group F and for any subset S of F which is closed in $PT(F)$, the product SH is closed in $PT(F)$.

In this paper we provide a negative answer to this question by constructing two counterexamples.

The First Example

Let F be a finitely generated free group. The profinite topology on F is defined by proclaiming all subgroups of finite index of F and their cosets to be basic open sets. An open set in $PT(F)$ is a (possibly infinite) union of cosets of various subgroups of finite index and the closed sets in $PT(F)$ are the complements of such unions in F.

The following example describes a set S, closed in $PT(F)$, such that its product with a free factor of F is not closed in $PT(F)$.

Example 1.

Let $F =< a, b >$ be a free group of rank two. Consider an infinite sequence $A = \{a, a^{2!}, a^{3!}, \cdots, a^{k!}, \cdots\} \subset F$. Note that A converges to 1_F. Indeed, let N be a normal subgroup of finite index m in F. If $k \geq m$ then $a^{k!}$ is contained in N. Hence any open neighborhood of 1_F in F contains all, but finitely many elements of A, therefore A converges to 1_F. Note that $1_F \notin A$, so A is not closed in $PT(F)$.

Let $m_k, k \geq 1$ be integers such that $m_k \to m_0 \in \hat{Z} \setminus Z$ in $PT(\hat{Z})$, where $\hat{Z}$ is the completion of Z in $PT(Z)$. Then $a^k b^{m_k} \to a^0 b^{m_0} \in \hat{F} \setminus F$, where $\hat{F}$ is the completion of F in $PT(F)$. Hence the sequence $a^k b^{m_k}$ has no other limit points. In particular, it has no limit points in F. Therefore for every $w \in F$ with $w \neq a^k b^{m_k}$ for all $k \geq 1$, there exists an open neighborhood U of w such that $a^k b^{m_k} \notin U$, for all $k \geq 1$. It follows that the set $S = \{ab^{m_1}, a^{2!}b^{m_2}, \cdots, a^{k!}b^{m_k}, \cdots\}$ is closed in $PT(F)$.

Note that $1_F \notin S < b >$, however $A \subseteq S < b >$, so $1_F \in \bar{A} \subseteq \overline{S < b >}$. We conclude that $S < b >$ is not closed in $PT(F)$.

The Second Example

The example in the previous section raises the following question: is it possible to impose some restrictions on a set S, closed in $PT(F)$, such that the product of S with a free factor of F would be closed in $PT(F)$.

The following example demonstrates that such restrictions on S should be severe.

Let F be a free group on free generators $K \cup L$, with $|K| = k$, $|L| = l$, and $F =< K > * < L >$. We describe a discrete set S, closed in $PT(F)$, such that $S < K >$ is not closed in $PT(F)$ and the last syllable of all elements of S is in $< L >$.

Example 2.

Construct by induction a sequence of normal subgroups of finite index $G_1 > G_2 > \cdots > G_m > \cdots$, elements $r_m \in < K >$ and $s_m \in F$, and an increasing function $f(m)$ satisfying the following conditions:

(1) $G_m \cdot r_m$ is at distance greater than $f(m)$ from $G_m \cdot 1$ in F/G_m.

(2) $G_m s_m = G_m r_m$.

(3) The last syllable of all s_m is in $< L >$.

(4) For all $k > m, G_m r_k \neq G_m r_m$.

Let $f(1)$ be an arbitrary integer. Choose G_1 such that the index of $H_1 = G_1 \cap < K >$ in $< K >$ exceeds $2k(2k-1)^{f(1)-1}$, which is the upper bound on the number of elements in a ball of radius $f(1)$ in $< K > /H_1$ around 1. Then we can choose $r_1 \in < K >$ such that the distance between $G_1 \cdot r_1$ and $G_1 \cdot 1$ in F/G_1 is bigger than $f(1)$. Choose $s_1 \in F$ such that $G_1 r_1 = G_1 s_1$ and the last syllable of s_1 is in $< L >$.

Assume that for some $n > 1$ we have constructed the normal subgroups $G_1 > G_2 > \cdots > G_n$ of finite index in F, elements $r_1, \cdots, r_n$ of $< K >$ and $s_1, \cdots, s_n$ of F, and $f(1) < f(2) < \cdots < f(n)$ satisfying conditions 1, 2, 3, and 4 for any $m \leq n$.

Let $H_m = G_m \cap < K >, m = 1, \cdots, n$. Let $\phi_{i,j} : F/G_i \to F/G_j$ for $i > j$ be the natural homomorphisms, and let $\psi_{i,j} :< K > /H_i \to < K > /H_j$ be the corresponding natural homomorphisms.

We want to define G_{n+1}, $r_{n+1} \in < K >$, $s_{n+1} \in F$, and $f(n+1)$.

In order to satisfy condition 4, we need $G_{n+1} r_{n+1}$ to be distinct from the cosets $\phi^{-1}_{n+1,1}(G_1 r_1), \phi^{-1}_{n+1,2}(G_2 r_2), \cdots, \phi^{-1}_{n+1,n}(G_n r_n)$.

On Closed Subsets Of Free Groups

In order to choose $G_{n+1}r_{n+1}$ such that $r_{n+1} \in < K >$, we need to consider the subgroups $H_i = G_i \cap < K >$ and the preimages

$\psi^{-1}_{n+1,1}(H_1 r_1)$, $\psi^{-1}_{n+1,2}(H_2 r_2), \cdots, \psi^{-1}_{n+1,n}(H_n r_n)$.

Note that the preimage of $H_m r_m$ in $< K > /H_{n+1}$ contains $[H_m : H_{n+1}]$ elements. Therefore, $H_{n+1}r_{n+1}$ should be different from $[H_1 : H_{n+1}] + [H_2 : H_{n+1}] + \cdots + [H_n : H_{n+1}]$ out of the total $[< K >: H_{n+1}]$ elements. In addition, in order to satisfy condition 1, the coset $H_{n+1}r_{n+1}$ should lie outside the ball of radius $f(n+1)$ around 1 in $< K > /H_{n+1}$. Note that the number of elements in this ball does not exceed $2k(2k-1)^{f(n+1)-1}$.

Also note that $[H_1 : H_{n+1}] + [H_2 : H_{n+1}] + \cdots + [H_n : H_{n+1}] =$

$= [< K >: H_{n+1}](\frac{1}{[<K>:H_1]} + \frac{1}{[<K>:H_2]} + \cdots + \frac{1}{[<K>:H_n]})$.

Assuming that the sequence of indices $[< K >: H_1]$, $[< K >: H_2], \cdots$, increases rapidly enough, we may assume that for all $n \geq 1$ the quantity

$\frac{1}{[<K>:H_1]} + \frac{1}{[<K>:H_2]} + \cdots + \frac{1}{[<K>:H_n]}$ is smaller than $\frac{1}{2}$.

Now we can choose G_{n+1} such that the index $[F : G_{n+1}]$ is big enough and for $H_{n+1} =< K > \cap G_{n+1}$ the index $[< K >: H_{n+1}]$ is big enough.

We choose an element $r_{n+1} \in < K >$ such that $G_{n+1}r_{n+1}$ is outside the ball of radius $f(n+1)$ in F/G_{n+1} around $G_{n+1} \cdot 1$ and such that $G_{n+1}r_{n+1}$ is different from all the elements of the preimages $\phi^{-1}_{n+1,1}(G_1 r_1), \phi^{-1}_{n+1,2}(G_2 r_2), \cdots, \phi^{-1}_{n+1,n}(G_n r_n)$.

Choose $s_{n+1} \in F$ such that $G_{n+1}s_{n+1} = G_{n+1}r_{n+1}$ and the last syllable of s_{n+1} is in $< L >$.

Hence, by induction, we have satisfied conditions 1, 2, 3, and 4.

Let $S = \{s_1, s_2, \cdots\}$. We claim that S is discrete in $PT(F)$. Indeed, for each n the coset $G_n r_n = G_n s_n$ does not contain any $G_n r_k = G_n s_k$ for $k > n$, so we have found an open neighborhood of s_n containing at most n elements of S. As the profinite topology on a free group is Hausdorff, it follows that S is discrete.

Note that S does not have limit points in F. Indeed, consider $x \in F$. For any $n \geq 1$ the coset $G_n \cdot x$ is an open neighborhood of x in $PT(F)$ and the distance between $G_n \cdot 1$ and $G_n \cdot x$ is bounded by the length of x.

By definition of S, the distance between $G_n \cdot 1$ and $G_n \cdot s_n$ is greater than $f(n)$ for almost all $n \geq 1$, hence the intersection $S \cap G_n \cdot x$ is finite.

It follows that x is not a limit of S, therefore, S is closed in $PT(F)$.

Note that $1 \in G_n s_n r_n^{-1} \in G_n S < K >$ for all $n \geq 1$, so $1 \in \overline{S < K >}$, but $1 \notin S < K >$ because $S \cap < K >= \emptyset$. Therefore, $S < K >$ is not closed in $PT(F)$.

Acknowledgment

The first author would like to thank the Institute of Mathematics of the Hebrew University for generous support.

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