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Fractional heat conduction in a conducting magneto-thermoelastic the half-space in contact with vacuum

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Abstract: *In this paper, we introduce a fractional heat conduction equation with a time-fractional derivative based on using the fractional heat Fourier law, in a half space in contact with vacuum perfectly-conductive half space. The Caputo fractional derivative is used to study the interaction between elastic, thermal, and magnetic fields. The solutions for displacement, temperature, stress and perturbed magnetic fields both in the vacuum and in the half-space are derived. The solutions are first obtained in terms of Laplace transform. The inverse of the Laplace transform is done numerically using a method based on Fourier expansion techniques. Numerical results for a copper-like material are presented graphically and discussed.*
Keywords *-Magneto-thermoelasticity; half space; Fractional heat conduction; Laplace transform.*

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I. Introduction

The study of electro-magneto-thermoelastic interactions which deals with the interactions among strain, temperature and electromagnetic fields in a thermoelastic material is of great practical importance due to its extensive uses in diverse field, such as geophysics (for understanding the effect of the Earth's magnetic field on seismic waves), damping of acoustic waves in a magnetic field, designing machine elements like heat exchangers, boiler tubes where temperature induced elastic deformation occurs, biomedical engineering (problems involving thermal stress), emissions of electromagnetic radiations from nuclear devices, development of a highly sensitive super conducting magnetometer, electrical power engineering, plasma physics, etc. [1, 2].

In the field of magneto elasticity or Magneto-thermo-elasticity, many studies have employed an analytical treatment of the interaction between elastic, electromagnetic and temperature fields. A generalized thermoelastic problem for an infinitely long hollow cylinder for short times was presented in [3]. The problem of Magneto-thermoelasticity of an infinite cylindrical region was solved by Dhaliwal and Singh [4]. Authors [5] gave a combined formulation of the two theories of generalized thermoelasticity to discuss the problem of an infinite solid with a cylindrical or spherical hole. In [6], authors applied a generalized elasticity theory to solve the problem of magneto-thermoelastic waves produced by thermal shock in an infinite elastic solid with a cylindrical cavity using the theories described in [3]. Sherief et al. [7] used the Laplace transform technique to find the distribution of thermal stresses and temperature in a generally thermoelastic electrically conducting half-space under sudden thermal shock and permeated by a primary uniform magnetic field. Solodyak and Gachkevich [8] presented an analytical method for obtaining electromagnetic and temperature fields as well as mechanical stresses in a ferromagnetic solid subjected to a harmonic electromagnetic field at the frequency usually used in industry. In [9], stresses and strains in the mid-plane of a pulsed magnetic field were calculated by solving the system of equations describing the displacement in each layer of the coil.

Fractional calculus is a natural extension of the classical mathematics. In fact, since the foundation of the differential calculus the generalization of the concept of derivative and integral to a non-integer order has been the subject of distinct approaches. Due to this reason there are several definitions [10, 11, 12] which are proved to be equivalent. Fractional calculus has been applied in many fields, ranging from statistical physics, chemistry to biological sciences and economics. In recent years, there has been a great deal of interest in fractional differential equations. Several definitions of the fractional derivative have been proposed. The history and classic transform rules of this subject are well covered in the monograph by Podlubny [13].

During recent years, fractional calculus has also been introduced in the field of thermoelasticity. Povstenko [14] has constructed a quasi-static uncoupled thermoelasticity model based on the heat conduction equation with a fractional order time derivative. He used the Caputo fractional derivative [15] and obtained the stress components corresponding to the fundamental solution of a Cauchy problem for the fractional order heat conduction equation in both the one-dimensional and two-dimensional cases. In 2010, a new theory of generalized thermoelasticity in the context of a new consideration of heat conduction with a fractional order has been proposed by Youssef [16]. In the same year, Sherief et al. [17] has constructed another model in generalized thermoelasticity theory by using fractional time derivatives. Abouelregal [18] used the generalized thermoelasticity theory that, based on a fractional order model, to solve a one-dimensional boundary value problem of a semi-infinite piezoelectric medium.

A flux rate term was incorporated into the Fourier law of heat conduction by Lord and Shulman [19], who formulated a generalized theory, which involves a hyperbolic heat transport equation admitting finite speed, though large for thermal signals. Green and Lindsay [20], by including temperature-rate among the constitutive variables, developed a temperature-rate-dependent thermoelasticity that does not violate the classical Fourier law of heat conduction for bodies having a center of symmetry. This theory also predicts a finite speed for heat propagation. The theory of thermoelasticity without energy dissipation is another generalized theory and was formulated by Green and Naghdi [21]. It includes the "thermal displacement gradient" among its independent constitutive variables, and differs from the previous theories in that it does not accommodate dissipation of thermal energy [22].

In his paper, the partial derivatives in generalized heat conduction equation were replaced by derivatives of non-integer order. The one-dimensional problem of distribution of thermal stresses and temperature is consider in a thermoelastic electrically conducting half-space permeated by a primary uniform magnetic field when the bounding plane is suddenly heated to a constant temperature. The conducting medium is also subjected to a Lorentz force. Thus, two kinds of stress arise: thermal stress caused by eddy current loss magnetic stress caused by the Lorentz force. The solutions valid for the deformation, stress, temperature distribution and perturbed magnetic field in the half-space as well as in the vacuum are derived. The Laplace transform technique is used to solve the problem. The theories of coupled magneto-thermoelasticity, generalized magneto-thermoelasticity with one relaxation time and of generalized magneto-thermoelasticity without energy dissipation follow as limited cases.

II. Fractional Integral And Fractional Derivatives

In this section, we will recall the main ideas of the fractional derivatives. Among several definitions for the fractional derivative, the Riemann-Liouville derivative and the Caputo derivative are often used. Let us consider causal functions, namely complex or real valued functions $f(t)$ of a real variable t that are vanishing for $t<0$. According to the Riemann--Liouville approach to fractional calculus the notion of fractional integral of order $\alpha(\alpha>0)$ for a causal function $f(t)$, sufficiently well-behaved, is a natural analogue of the well-known formula (usually attributed to Cauchy), but probably due to Dirichlet, which reduces the calculation of the $\alpha-$fold primitive of a function $f(t)$ to a single integral of convolution type. The Cauchy formula reads for $t>0$:

$$I^{\alpha} f(t) := \frac{1}{(\alpha-1)!}\int_0^t \frac{1}{(t-\tau)^{1-\alpha}} f(\tau)d\tau; \quad \alpha\in \mathrm{N},\ t\in \mathrm{R}^{+}, \qquad (1)$$

where I^{α} is the $\alpha-$fold integral operator with $I^0 f(t)=f(t)$, N is the set of positive integers, and R is the set of positive reals. Riemann--Liouville analytically continued Cauchy's result by replacing $(\alpha-1)!$ with Euler's continuous gamma function $\Gamma(\alpha)$, thereby producing

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{1}{(t-\tau)^{1-\alpha}} f(\tau)d\tau; \quad \alpha,t\in \mathrm{R}^{+}, \qquad (2)$$

where I^{α} is the Riemann--Liouville integral operator of order α. Equation (2) is the cornerstone of the fractional calculus, although it may vary in its assignment of limits of integration. In this report we take the lower limit to be zero and upper limit to be some positive real. Actually, α can be complex [11], but for our purposes we only need it to be real.

It is well known that the relationship between the heat flow and the temperature gradient, based on experimental observations, is given by Fourier, which is named as the Fourier law. The law is given in the form of

$$q_i = -k_{ij}\theta_j \qquad (3)$$

The classical Fourier's law is replaced by Maxwell-Cattaneo's equation

$$q_i + \tau_0 \frac{\partial}{\partial t} q_i = -k_{ij}\theta_j, \qquad (4)$$

where θ is the temperature-change above a uniform reference temperature T_0, assumed to be such that $|(T-T_0)/T_0| = 1$, k_{ij} and τ_0 are positive constants, $\mathbf{q}$ is the heat flux vector and τ_0, being the time required to establish steady-state heat conduction in a volume element after a temperature gradient has suddenly been imposed across the element, is called relaxation time parameter. In the absence of internal heat sources, Cattaneo's equation results in the linear hyperbolic heat equation

$$\frac{\partial}{\partial x_i}\left(K_{ij}\theta_j\right)=\rho C_E\left(1+\tau_0\frac{\partial}{\partial t}\right)\frac{\partial\theta}{\partial t}+\gamma_{ij}T_0\left(1+\tau_0\frac{\partial}{\partial t}\right)\frac{\partial e_{ij}}{\partial t} \qquad (5)$$

Cattaneo's equation is the most obvious and simple generalization of Classical Fourier's heat law that gives rise to a finite propagation speed for thermal disturbances. In Lord and Shulman model, the heat conduction equation is of hyperbolic type and is closely connected with the theories of second sound.

Recently, some interesting models have been proposed successfully by applying the fractional calculus to study the physical processes particularly in the area of mechanics of solids, control theory, electricity, heat conduction, diffusion problems and viscoelasticity etc. It has been verified/examined that the use of fractional order derivatives/integrals leads to the formulation of certain physical problems which is more economical and useful than the classical approach. There are some materials (e.g., porous materials, man-made and biological materials/polymers and colloids, glassy, etc.) and physical situations (like low temperature, amorphous media and transient loading, etc.) where the CTE theory based on the classical Fourier's law is unsuitable. In such cases it is better to use a generalized thermoelasticity (and more generally thermo-viscoelasticity) theory based on an anomalous heat conduction theory involving fractional time derivatives (see [22]).

The time-nonlocal dependence between the heat flux vector and the temperature gradient with the " long-tale" power kernel can be interpreted in terms of fractional integrals and derivatives [17]

$$q_i+\tau_0\frac{\partial^\alpha}{\partial t^\alpha}q_i=-k_{ij}\theta_j, \qquad (6)$$

where α is a constant such that $0<\alpha\le 1$.

$$\frac{\partial^\alpha}{\partial t^\alpha}f(x,t)=\begin{cases} f(x,t)-f(x,0), & \alpha\to 0\\ I^{1-\alpha}\dfrac{\partial}{\partial t}f(x,t), & 0<\alpha<1\\ \dfrac{\partial}{\partial t}f(x,t), & \alpha\to 1\end{cases} \qquad (7)$$

In the limit as α tends to one, Eq. (6) reduces to the well-known Cattaneo law used by Lord and Shulman [19] to derive the equation of the generalized theory of thermoelasticity with one relaxation time.

Taking divergence of both sides of Eq. (6), we get

$$q_{i,i}+\tau_0\frac{\partial^\alpha}{\partial t^\alpha}q_{i,i}=-\left(K_{ij}\theta_j\right)_{,i} \qquad (8)$$

Then the time-fractional heat conduction equation take the form [17]

$$\frac{\partial}{\partial x_i}\left(K_{ij}\theta_j\right)=\rho C_E\left(1+\tau_0\frac{\partial^\alpha}{\partial t^\alpha}\right)\frac{\partial\theta}{\partial t}+\gamma_{ij}T_0\left(1+\tau_0\frac{\partial^\alpha}{\partial t^\alpha}\right)\frac{\partial e_{ij}}{\partial t}-\left(1+\tau_0\frac{\partial^\alpha}{\partial t^\alpha}\right)Q \qquad (9)$$

III. Basic Equations In Thermoelasticity And Magneto-Thermoelasticity

In the context of the fractional order theory of generalized thermoelasticity [17], the field equations for a linear, homogeneous and isotropic magneto-thermoelastic material, in the absence of the body force, take the following form:

Equations of motion have the form taking into account the Lorentz force has the form

$$\mu\nabla^2\mathbf{u}+(\lambda+\mu)\nabla div\mathbf{u}-\gamma\nabla\theta+\mathbf{F}=\rho\ddot{\mathbf{u}}, \qquad (10)$$

where $\mathbf{F}=\frac{1}{c}(\mathbf{J}\times\mathbf{B})$ are the components of Lorentz force

Since, in generalized thermoelasticity only the infinitesimal temperature derivations from the reference temperature are considered, therefore in the absence of the body force and internal heat source, the Duhamel-Neumann constitutive equations are

$$\boldsymbol{\sigma}=\lambda(div\mathbf{u})\mathbf{I}+\mu\left(\nabla\mathbf{u}+\nabla\mathbf{u}^T\right)-\gamma\theta\mathbf{I} \quad (11)$$

In these equations $\boldsymbol{\sigma}$ is the stress tensor, c is the speed of light in vacuum, $\mathbf{B}$ is the magnetic induction vector, $\mathbf{J}$ denotes the current density vector, $\mathbf{u}$ is the displacement vector, $\mathbf{I}$ being the identity tensor, λ and μ are the Lame' constants, $\gamma=(3\lambda+2\mu)\alpha_t$, α_t is the coefficient of volume expansion, and ρ is the mass density. We take a homogeneous, isotropic and thermally conducting medium, then the time-fractional heat conduction equation take the form [17]

$$K\nabla^2\theta = \rho C_E\left(\delta+\tau_0\frac{\partial^\alpha}{\partial t^\alpha}\right)\frac{\partial\theta}{\partial t}+\gamma T_0\left(\delta+\tau_0\frac{\partial^\alpha}{\partial t^\alpha}\right)\frac{\partial e}{\partial t} \quad (12)$$

where e is the cubical dilatation, C_E is the specific heat and K is the thermal conductivity.

The simplified linearized equations of electrodynamics of slowly moving continuous media having perfect electrical conductivity are (Maxwell's electromagnetic field equations, neglecting the charge density)

$$curl(\mathbf{h}) = \frac{4\pi}{c}\mathbf{J}, \quad (13)$$

$$curl(\mathbf{E}) = -\frac{\mu_0}{c}\frac{\partial\mathbf{h}}{\partial t}, \quad (14)$$

$$\nabla.\mathbf{h} = 0, \quad (15)$$

$$\mathbf{E} = -\frac{\mu_0}{c}\left(\frac{\partial\mathbf{u}}{\partial t}\times\mathbf{H}_0\right), \quad (16)$$

where $\mathbf{E}$ denotes the electric field, $\mathbf{h}$ is the induced magnetic, $\mathbf{H}_0$ is the initial constant magnetic field and μ_0 is the magnetic permeability.

Maxwell stress components are given by

$$T_{ij} = \mu_0\left(H_i h_j + H_j h_i - H_k h_k \delta_{ij}\right) \quad (17)$$

After linearization,

$$\mathbf{J}\times\mathbf{B} = \mathbf{J}\times\mu_0(\mathbf{H}_0+\mathbf{h}) \cong \mu_0(\mathbf{J}\times\mathbf{H}_0) = \frac{c\mu_0}{4\pi}\left[(\nabla\times\mathbf{h})\times\mathbf{H}_0\right] \quad (18)$$

Equation (1), then, after linearization, reduces to

$$\mu\nabla^2\mathbf{u}+(\lambda+\mu)\nabla div\mathbf{u}-\gamma\nabla\theta+\frac{\mu_0}{4\pi}\left[(\nabla\times\mathbf{h})\times\mathbf{H}_0\right] = \rho\ddot{\mathbf{u}} \quad (19)$$

Since, the elastic medium is in contact with the vacuum, equations (13)-(16) have to be supplemented by the electrodynamic equations in vacuum. The system of Maxwell's equations in vacuum is expressed as

$$curl(\mathbf{E}^0) = -\frac{1}{c}\frac{\partial\mathbf{h}^0}{\partial t}, \qquad curl(\mathbf{h}^0) = \frac{1}{c}\frac{\partial\mathbf{E}^0}{\partial t}, \quad (20)$$

where $\mathbf{h}^0$ and $\mathbf{E}^0$ are the perturbed magnetic field and the electric field in vacuum.

These equations yield the following equations satisfied by $\mathbf{E}^0$ and $\mathbf{h}^0$:

$$\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{h}^0 = 0 \quad (21)$$

$$\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{E}^0 = 0 \quad (22)$$

Equations (12) and (19) are the field equations of the generalized linear magneto-thermoelasticity with fractional order heat conduction, applicable to the coupled theory and two generalizations, as follows:

The equations of the coupled thermoelasticity CTE theory, when $\tau_0 = 0$, $\delta = 1$.

Lord and Shulman's theory LS, when $\tau_0 > 0$, $\delta = \alpha = 1$.

Green and Naghdi's theory GN, when $\tau_0 = 1$, $\delta = 0$, $\alpha = 1$.

The correspondent equations for the generalized thermoelasticity without magnetic field results from the above mentioned cases by taking $H_0 = 0$ and $\alpha = 1$.

IV. Formulation Of The Problem

We assume that a magneto-thermoelastic wave is produced in an elastic half-space $x \geq 0$ at a uniform reference temperature T_0 in contact with a vacuum $x < 0$. The magneto-thermoelastic wave propagated in the medium $x > 0$ is assumed to depend on x and time t and that the displacement vector has components $(u(x,t),0,0)$. A magnetic field with constant intensity, namely $\mathbf{H}_0 = (0,0,H_0)$ where H_0 is a constant acts parallel to the bounding plane (take as the direction of the $z-$axis). Due to the application of initial magnetic field $\mathbf{H}$, there results an induced magnetic field $\mathbf{h}$ and an induced electric field $\mathbf{E}$. The simplified linear equations of electrodynamics of a slowly moving medium for a homogeneous, isotropic and thermally conducting elastic solid (13)-(16) are

$$\mathbf{J} = -\frac{c}{4\pi}\left(0, \frac{\partial h_z}{\partial x}, 0\right), \qquad (23)$$

$$\mathbf{h} = -H_0\left(0, 0, \frac{\partial u}{\partial x}\right), \qquad (24)$$

$$\mathbf{E} = \frac{\mu_0 H_0}{c}\left(0, \frac{\partial u}{\partial t}, 0\right), \qquad (25)$$

Equations (12) and (19), in one-dimensional case, for a perfect conductor simplify to

$$\left(\lambda + 2\mu + \frac{\mu_e H_0^2}{4\pi}\right)\frac{\partial^2 u}{\partial x^2} - \gamma\frac{\partial T}{\partial x} - \frac{\partial h_3}{\partial x} = \rho\frac{\partial^2 u}{\partial t^2} \qquad (26)$$

$$K\frac{\partial^2 \theta}{\partial x^2} = \rho C_E\left(\delta + \tau_0\frac{\partial^\alpha}{\partial t^\alpha}\right)\frac{\partial \theta}{\partial t} + \gamma T_0\left(\delta + \tau_0\frac{\partial^\alpha}{\partial t^\alpha}\right)\frac{\partial^2 u}{\partial t \partial x} \qquad (27)$$

In vacuum the system of equations of electrodynamics (18) and (19) reduce to

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\overset{0}{h}_3 = 0 \qquad (28)$$

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\overset{0}{E}_2 = 0 \qquad (29)$$

The components of Maxwell's stress tensor in the elastic medium T_{11} and in vacuum $\overset{0}{T}_{11}$ are given by

$$T_{11} = -\frac{\mu_0 H_0}{4\pi}h_z \qquad (30)$$

$$\overset{0}{T}_{11} = -\frac{\overset{0}{H_0}}{4\pi}\overset{0}{h}_z \qquad (31)$$

The normal stress in the elastic medium is obtained as

$$\sigma_{xx} = (2\mu + \lambda)\frac{\partial u}{\partial x} - \gamma\theta \qquad (32)$$

The total stress σ_T in the half-space is composed of Hook's mechanical stress and Maxwell's stress. Thus, the total stress in the half-space is

$$\sigma_T = \sigma_{xx} + T_{11} = \left(\lambda + 2\mu + \frac{\mu_0 H_0^2}{4\pi}\right)\frac{\partial u}{\partial x} - \gamma\theta. \quad (33)$$

V. Initial And Boundary Conditions

We assume that the medium is initially at rest and the undisturbed state is maintained at uniform reference temperature. Then we have

$$u(x,t)\big|_{t=0} = \left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = 0,\ x > 0,\ \theta(x,t)\big|_{t=0} = \left.\frac{\partial \theta(x,t)}{\partial t}\right|_{t=0} = 0,\ x > 0, \quad (34)$$

The regularity boundary conditions are $T(x,t), u(x,t)$ and $\sigma_{xx}(x,t) \to 0$ as $x \to \infty$.

The continuity of total stress composed of thermoelastic and electromagnetic stress across the boundary $x = 0$ yields

$$\sigma_T = \overset{0}{T}_{11} \qquad \text{on} \qquad x = 0 \qquad (35)$$

The tangential component of $E-$ field is continuous across $x = 0$, which leads to

$$E_2 = \overset{0}{E}_2, \qquad h_3 = \overset{0}{h}_3 \qquad \text{on} \qquad x = 0 \qquad (36)$$

The thermal boundary condition on

$$\theta(x,t) = H(t)\theta_1 \qquad \text{on} \qquad x = 0, \qquad (37)$$

where θ_1 is a constant and $H(t)$ is the Heaviside unit function.

VI. Solution Of The Proplem

To find the solution of the problem we introduce the following notations and non-dimensional variables

$$X = c_0\eta x, \qquad U = \frac{\rho c_0^2\eta}{\gamma T_0}u, \qquad \tau = c_0^2\eta\tau,$$

$$\Theta = \frac{\theta}{T_0}, \quad h_z^{'} = \frac{\rho c_0^2}{H_0\gamma T_0}h_z, \quad \sigma_{11} = \frac{\sigma_T}{\gamma T_0} \quad \eta = \frac{\rho C_E}{K}, \quad (38)$$

$$c_0^2 = c_1^2 + a_0^2, \quad c_1^2 = \frac{\lambda + 2\mu}{\rho}, \qquad a_0^2 = \frac{\mu_0 H_0^2}{4\pi\rho}.$$

where c_1 is the dilatational wave velocity in the half-space and a_0 is the Alfven wave velocity of the medium. In terms of the non-dimensional quantities defined in Eq. (34), the governing equations (23) and (24) will be in the forms

$$\frac{\partial^2 U}{\partial X^2} - \frac{\partial\Theta}{\partial X} = \frac{\partial^2 U}{\partial\tau^2}, \qquad (39)$$

$$C_T^2\frac{\partial^2\Theta}{\partial X^2} = \left(\delta + \tau_0\frac{\partial^\alpha}{\partial\tau^\alpha}\right)\frac{\partial\Theta}{\partial\tau} + \varepsilon\left(\delta + \tau_0\frac{\partial^\alpha}{\partial\tau^\alpha}\right)\frac{\partial^2 U}{\partial\tau\partial X}, \quad (40)$$

where $\varepsilon = \dfrac{\gamma^2 T_0}{\rho^2 C_v c_0^2}$ is the magneto-thermoelastic coupling, which reduces to ε the thermoelastic coupling constant for $H_0 = 0$. Here $C_T^2 = c_3^2/c_0^2$, where $c_3^2 = K/\rho^2 C_E$ and C_T is the non-dimensional finite thermal wave speed corresponding to c_3. Equations (25) and (26) are reduce to

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{a_0^2}\frac{\partial^2}{\partial\tau^2}\right)\overset{0}{h}_3 = 0 \qquad (41)$$

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{a_0^2}\frac{\partial^2}{\partial\tau^2}\right)\overset{0}{E}_2 = 0 \qquad (42)$$

Further, the boundary condition for continuity of total stress across $x = 0$ in nondimensional form takes the form

$$\frac{\partial^2 U}{\partial X^2} - \frac{\partial\Theta}{\partial X} + \beta_1\frac{\partial h_z^0}{\partial X} = 0 \qquad \text{on} \qquad x = 0, \quad (43)$$

where $\beta_1 = H_0/(4\pi\gamma T_0)$.

The condition of continuity of $E-$ field across $x = 0$ reduces to $E_2 = E_2^0$ which, by the help of (20)-(22) and (37) and (38), yields, in non-dimensional form, the following equation:

$$\beta_2\frac{\partial^2 U}{\partial\tau^2} - \frac{\partial h_z^0}{\partial X} = 0 \qquad \text{on} \qquad x = 0, \qquad (44)$$

where $\beta_2 = \dfrac{\gamma T_0\mu_e H_0}{\rho c^2}$.

Lastly, the thermal boundary condition gives

$$\Theta(x,\tau) = H(\tau)\theta_2 \qquad \text{on} \qquad x = 0, \qquad (45)$$

Where $\theta_2 = \theta_1 T_0$.

The non-dimensional total stress in the half-space is obtained from (30) as

$$\Sigma = \frac{\sigma_T}{\gamma T_0} = \frac{\partial U}{\partial X} - \Theta \qquad (46)$$

The perturbed magnetic field in the half-space in non-dimensional form reduces to

$$h_z^{'} = -\frac{\partial U}{\partial X} \qquad (47)$$

It may be observed that (35) and (37) are fully hyperbolic but the mixed derivative term on the right-hand side of (36) destroys the wave structure. In fact, this equation predicts a non-wave-like heat conduction different from the usual diffusion equation predicted by conventional parabolic heat equation. It is therefore expected that the solutions of the coupled equations (35) and (36) with coupled boundary conditions (39) and

(40) should be composed of a wave part (dilatational wave) and a diffusive part due to the presence of the thermal damping term in the heat transport equation. As (37) (hyperbolic type) uncouples from the system, its solution will be composed of a wave part traveling with Alfv'en acoustic wave speed in vacuum.
Introducing the thermoelastic potential function ϕ defined by the relation

$$\phi = \frac{\partial U}{\partial X} \quad (48)$$

Then (35), on integrating with respect to X, yields

$$\left(\frac{\partial^2}{\partial X^2} - \frac{\partial}{\partial \tau^2} \right)\phi = \Theta \qquad (49)$$

Equation (36) becomes

$$C_T^2 \frac{\partial^2 \Theta}{\partial X^2} = \left(\delta + \tau_0 \frac{\partial^\alpha}{\partial \tau^\alpha} \right)\frac{\partial \Theta}{\partial \tau} + \varepsilon \left(\delta + \tau_0 \frac{\partial^\alpha}{\partial \tau^\alpha} \right)\frac{\partial^3 \phi}{\partial \tau \partial X^2} \quad (50)$$

VII. Solution In The Transformed Domain

We now define Laplace transform of a function $g(X,\tau)$ by

$$L[g(X,\tau)] = \overline{g}(X,s) = \int_0^\infty e^{-st} g(X,\tau)\, d\tau, \qquad (51)$$

where s is the transform parameter.
Taking Laplace transform of (35) and (36) and using the initial conditions, we obtain

$$\left(\frac{d^2}{dX^2} - s^2 \right)\overline{\phi} = \Theta, \qquad (52)$$

$$C_T^2 \frac{d^2 \overline{\Theta}}{dX^2} = s\left(\delta + \tau_0 s^\alpha\right)\overline{\Theta} + s\varepsilon\left(\delta + \tau_0 s^\alpha\right)\frac{d^2 \overline{\phi}}{dX^2}. \quad (53)$$

Further (37)-(40) in the Laplace transform domain become

$$\left(\frac{d^2}{dX^2} - \beta^2 s^2 \right)\overline{h}_z^0 = 0, \qquad (54)$$

$$\overline{\Sigma} = \frac{d\overline{U}}{dX} - \overline{\Theta} = \frac{d^2 \overline{\phi}}{dX^2} - \overline{\Theta}, \quad (55)$$

$$\overline{h}_z = -\frac{d\overline{U}}{dX} = -\frac{d^2 \overline{\phi}}{dX^2}. \qquad (56)$$

Elimination of $\overline{\Theta}$ from (47) and (48) yields

$$\left[\frac{d^4}{dX^4} - A \frac{d^2}{dX^2} + B \right]\overline{\phi} = 0, \qquad (57)$$

where

$$A = \left\{ s^2 + \frac{s\left(\delta + \tau_0 s^\alpha\right)(1+\varepsilon)}{C_T^2} \right\},\ B = \frac{s^3\left(\delta + \tau_0 s^\alpha\right)}{C_T^2}.$$

The solutions of equation (52) bounded at infinity can be written in the form:

$$\overline{\phi} = C_1 e^{-m_1 X} + C_2 e^{-m_2 X}, \quad (58)$$

where C_1 and C_2 are parameters depending on s to be determined from the boundary conditions and m_1 and m_2 are the roots with positive real parts of the characteristic equation

$$m^4 - Am^2 + B = 0, \qquad (59)$$

m_1 and m_2 are given by

$$m_{1,2} = \sqrt{\left(A \pm \sqrt{A^2 - 4B} \right)/2}. \qquad (60)$$

The expression for displacement and temperature can be written in the

$$\overline{u} = -m_1 C_1 e^{-m_1 X} - m_2 C_2 e^{-m_2 X}, \qquad (61)$$

$$\overline{\Theta} = \left(m_1^2 - s^2\right)C_1 e^{-m_1 X} + \left(m_2^2 - s^2\right)C_2 e^{-mX}. \qquad (62)$$

Further, (49) yields

$$\overline{h}_z^0(X,s) = C_3 e^{-s\beta X} \qquad (63)$$

The boundary conditions (40)-(42) in the Laplace transform domain reduce to the following:

$$\frac{d^2\overline{\phi}}{dX^2} - \overline{\Theta} + \beta_1 \overline{h}_z^0 = 0 \quad \text{on} \quad x = 0, \qquad (64)$$

$$\beta_2 \frac{d\overline{\phi}}{dX} - \frac{\partial \overline{h}_z^0}{\partial X} = 0 \quad \text{on} \quad x = 0, \qquad (65)$$

$$\overline{\Theta} = \frac{\omega}{s} = \frac{\theta_1}{T_0}\frac{1}{s} \quad \text{on} \quad x = 0. \quad (66)$$

From the boundary conditions (59)-(61), it follows that

$$\overline{\phi}(X,s) = \frac{\lambda_2}{M} e^{-m_1 X} - \frac{\lambda_1}{M} e^{-m_2 X}, \qquad (67)$$

$$\overline{U}(X,s) = -\frac{m_1\lambda_2}{M} e^{-m_1 X} + \frac{m_2\lambda_1}{M} e^{-m_2 X}, \qquad (68)$$

$$\overline{\Theta}(X,s) = \frac{\left(m_1^2 - s^2\right)\lambda_2}{M} e^{-m_1 X} - \frac{\left(m_2^2 - s^2\right)\lambda_1}{M} e^{-m_2 X}, \qquad (69)$$

$$\overline{h}_z(X,s) = -\frac{m_1^2\lambda_2}{M} e^{-m_1 X} + \frac{m_2^2\lambda_1}{M} e^{-m_2 X}, \qquad (70)$$

$$\overline{h}_z^0(X,s) = \frac{\omega s \beta_2 (m_1 - m_2)}{M} e^{-s\beta X}, \qquad (71)$$

$$\overline{\Sigma}(X,s) = \frac{s\lambda_2}{M} e^{-m_1 X} - \frac{s\lambda_1}{M} e^{-m_2 X}, \qquad (72)$$

where

$$\lambda_1 = \frac{\omega}{s}\left(s\beta + \beta_1\beta_2 m_1\right), \quad \lambda_2 = \frac{\omega}{s}\left(s\beta + \beta_1\beta_2 m_2\right),$$

$$M = (m_1 - m_2)\left[\beta_1\beta_2 s^2 + \beta s(m_1 + m_2) + \beta_1\beta_2 m_1 m_2\right]$$

This completes the solution of the problem in the Laplace transform domain.

VIII. Inversion Of The Laplace Transforms

It is difficult to find the analytical inverse Laplace transform of the complicated solutions for the displacement, temperature, stress and strain in Laplace transform domain. We will now outline the numerical inversion method to obtain the solution of the problem in the physical domain. Durbin [23] derived the approximation formula

$$f(t) = \frac{2e^{st}}{t_1}\left(-\frac{1}{2}Re[F(s)] + Re\sum_{n=0}^{N}\left[\begin{array}{l} Re\left(F\left(s + \frac{2in\pi}{t_1}\right)\right)\cos\left(\frac{2n\pi}{t_1}\right) \\ -Im\left(F\left(s + \frac{2in\pi}{t_1}\right)\right)\sin\left(\frac{2n\pi}{t_1}\right)\end{array}\right]\right)$$

where $st_1 = 5 - 10$ gives good results for N ranging from 50 to 1000.

It should be noted that a good choice of the free parameters N and st_1 is not only important for the accuracy of the results, but also for the application of the Korrecktur method and the methods for the acceleration of convergence. The values of all parameters in Eq. (47) are defined as $t_1 = 20$, $s = 0.25$ and $N = 1000$ in this paper.

In order to find the displacement distribution u, we use expression (20) with and replacing $f(t)$ and , $F(s)$ respectively. This procedure is repeated for the functions of the stress and temperature distribution.

IX. Numerical Results And Discussion

In order to illustrate the theoretical results obtained in preceding section and to compare these in the context of various theories of thermoelasticity, we now present some numerical results. In the calculation process, we consider the material medium as that of copper. The constants of the problem are taken as:

$$K = 368\,\mathrm{Wm^{-1}K^{-1}}, \quad \alpha_t = 1.78\times10^{-5}\,\mathrm{K^{-1}}, \quad C_E = 383.1\,\mathrm{JKg^{-1}K^{-1}},$$
$$\rho = 8954\,\mathrm{Kgm^{-3}}, \quad \lambda = 7.76\times10^{10}\,\mathrm{Nm^{-2}}, \quad \mu = 3.86\times10^{10}\,\mathrm{Nm^{-2}},$$
$$\eta = 8886.73\,\mathrm{sm^{-2}}, \quad T_0 = 293\,\mathrm{K}, \quad \varepsilon = 0.0168, \; g = 1.61,$$
$$\varepsilon_0 = 10^{-9}/36\pi\,\mathrm{Fm^{-1}}, \quad \mu_0 = 4\pi\times10^{-7}\,\mathrm{Hm^{-1}}, \quad H_0 = 10^7/4\pi\,\mathrm{Am^{-1}}.$$

Using this data, the values of the physical quantities are evaluated and plotted in Figures (1-10). The non-dimensional temperature Θ, displacement U, and total stress Σ distributions and perturbed magnetic fields both in the vacuum h_z^0 and in the half-space h_z were evaluated on the $X-$axis. All the figures show that the heat and elastic effects propagate with finite speeds. Initially, the inner surface is passive while the outer surface is affected by the thermal shock. As seen in the figures, the effect of the thermal shock propagates into the medium with a finite speed.

Figure 1 indicates variation of temperature versus distance X. It can be found from Figure 1 that for fixed time $\tau = .02$, the temperature has a non-zero value only in a bounded region of space at a given instant. Outside this region the value vanishes and this means that the region has not felt thermal disturbance yet. At different instants, the non-zero region moves forward correspondingly with the passage of time. This indicates that heat propagates as a wave with finite velocity in medium. It is completely different from the case for the classical theories of thermoelasticity where an infinite speed of propagation is inherent and hence all the considered functions have a non-zero (although may be very small) value for any point in the medium.

From Figure 2, it can be observed that the medium adjacent to the half space surface $X = 0$ undergoes expansion deformation because of thermal shock while the others compressive deformation. The deformation is a dynamic process. With the passage of time, the expansion region moves insides gradually and becomes larger and larger. Thus the displacement distribution becomes larger and larger. At a given instant, the non-zero region of displacement is finite, which is due to the wave effect of heat. It indicates that heat transfers into the deep of the medium with a finite velocity with the time passing. The more the considered instant, the more the thermal disturbed region and the radial displacement correspondingly.

In Figure 3, shows that the total stress with distance X for various values of the order of fractional derivative α. The medium close to the surface $X = 0$ suffers from tensile stress. This is corresponding to the expansion deformation of the medium shown in Figure 2. It also can be find that the tensile stress region becomes larger while the compressed becomes smaller with the time passing, which is corresponding to the dynamic expansion effect described above. It can also be found from Figure 3, at some instant, the non-zero region of stress is finite, which indirectly proves the wave effect of heat.

Figure 4 shows that the perturbed field gradually increases with distance. Further for time $\tau = 0.02$, the value of perturbed field first gradually increases with distance $X(0 \le X \le .4)$ and then again it gradually decreases. The electromagnetic medium is placed in an initial magnetic field, and it deforms because of thermal shock. This makes the magnetic flux traversing the cross section of the medium changed. Thus there result an induced magnetic filed in vacuum and an induced magnetic field in the medium. It also can be found the induced magnetic fields vary with heat wave transferring into the deep of the half space, and again proves the wave effect of heat.

In Figs. (1-5) the temperature, the stress and the displacement distributions and perturbed magnetic fields both in the vacuum and in the half-space are plotted against X respectively for various values of the order of fractional derivative α of Caputo derivative. The different values of the parameter α with wide range $(0 < \alpha \le 1)$ cover the two cases of the conductivity; $(0 < \alpha < 1)$ for weak conductivity and $\alpha = 1$ for normal conductivity (ordinary heat conduction equation). It should be pointed that, the increasing of the value of the parameter α causes increasing in the speed of the waves propagation of the stresses, the displacement and perturbed magnetic field in the medium distributions, whereas the distribution of the temperature and perturbed magnetic field in the vacuum decreasing. We have also noticed that, the value of α has a significant effect on all distributions.

The distributions of the non-dimensional temperature, displacement, stress and induced magnetic fields both in the vacuum and in the half-space at $\tau = 0.02$ and $\tau = 0.03$ are shown in Figures (6-10) respectively. Figure 6 depict the distributions of temperature Θ versus X at the boundary for different values of time (e.g. $\tau = 0.2$, $\tau = 0.03$). Also we can see that the temperature decreases monotonically with X as increasing the time τ. Figure 6 shows that the values of temperature Θ for $\tau = 003$ are large compared with those for $\tau = 0.02$. From Figure 7, we notice that the magnitude of displacement U gradually decreases for fixed value of space variable X as time τ increases. Values of total stress Σ for $\tau = 003$ are large compared to those for $\tau = 003$, as shown in Figure 8. From figures 9 and 10, it is clear that the absolute value of magnetic fields gradually decreases with greater wave length as τ increases and finally vanishes. We can also see that, the changes in the time values make a significant change in all the fields and it is very obvious at the peak points of the curves.

X. Conclusion

A new model of thermoelasticity theory was investigated in the context of a new consideration of heat conduction with fractional derivatives. This model based on the heat conduction equation with the Caputo fractional derivative of order α. The solution is obtained by applying the Laplace integral transform. The numerical results for temperature, displacement, and stresses are computed and illustrated graphically. The results are graphically described for the medium of copper.

The analysis of the results can be summarized as follows:

- The dependence of the fractional parameter has a significant effect on the thermal and mechanical interactions, and plays a significant role in all the physical quantities.
- At any point the distributions of displacement, stress, and perturbed magnetic field in the medium are increased with an increase in α but the effect of fractional parameters is to decrease the values of the temperature field and perturbed magnetic field in the vacuum with a wide range $(0 < \alpha \le 1)$.
- It is clear from Figures (6-10) that the different times play a significant role in all the physical quantities.
- All the physical quantities satisfy the boundary conditions and initial conditions.
- It is also observed that the theories of coupled thermoelasticity and generalized thermoelasticity with one relaxation time can be obtained as limited cases.
- The values of the distributions of all the physical quantities converge to zero with increasing the distance X.
- The phenomenon of finite speeds of propagation is manifested in all these figures.
- As a final remarks, the results presented in this paper should prove useful for researchers in material science, designers of new materials, low temperature physicists as well as for those working on the development of a theory of hyperbolic thermoelasticity with fractional order.

References

[1] J. Nowinski, *Theory of thermoelasticity with applications* (Sijtho and Noordho International Publishers-Alphen Aan Den Rijn, 1978).

[2] P. Chadwick, R. Hill and I.N. Sneddon, *Progress in solid mechanics* (North Holland- Amsterdam, 1960).

[3] H. Hany, H. Sherief, A.E.M. Elmisiery, and M.A. Elhagary, Generalized thermoelastic problem for an infinitely long hollow cylinder for short times, *J. Therm. Stress, 27*, 2004 , 885-902.

[4] R.S. Dhaliwal, A. Singh, Dynamic Coupled Thermoelasticity, Hindustan Publishing Corporation, *Delhi*,1980, 631-636.

[5] N. Noda, T. Furukawa and F. Ashida, Generalized thermoelasticity in an infinite solid with a hole, *J. Therm. Stress. 12*,1989, 385-402.

[6] R.S. Dhaliwal, H.S. Saxena, Generalized magneto-thermo-elastic waves in an infinite elastic solid with a cylindrical cavity, *J. Therm. Stress*, 14,1991, 353-369.

[7] H. Sherief, M.A. Ezzat, Thermal-shock problem in magneto-thermo-elasticity with thermal relaxation, *Int. J. Solids Struct, 33*, 1996, 4449-4459.

[8] M. Solodyak, A. Gachkevich, Thermoelasticity of ferromagnetic solids by deep induction heating, in: Mathematical Methods in Electromagnetic Theory, Conference Proceedings,1996, 219-222.

[9] L. Liang, F. Herlach, Deformation analysis of pulsed magnets with internal and external reinforcement, *Measur. Sci. Technol, 6*,1995, 1035-1042.

[10] K.B. Oldham, J. Spanier, *The Fractional Calculus* (Theory and Application of Differentiation and Integration to Arbitrary Order, Academic Press, New York, 1974).

[11] S.G. Samko, A.A. Kilbas and O.I. Marichev, *Fractional Integrals and Derivatives* (Theory and Applications, Gordon and Breach Science Publishers, London, 1993).

[12] K.S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations* (Wiley, New York, 1993).

[13] I. Podlubny, *Fractional Differential Equations* (Academic Press, New York 1999).

[14] YZ. Povstenko, Fractional heat conduction equation and associated thermal stress. *J Therm Stresses, 28*, 2005, 83-102.

[15] M. Caputo, Linear model of dissipation whose Q is almost frequency independent-II. *Geophys J R Astron Soc ,13*,1967, 529-539.

[16] H. Youssef, Theory of fractional order generalized thermoelasticity. *J Heat Trans,132*, 2010, 1-7.

[17] H.H. Sherief, El-Sayed A, El-Latief A Fractional order theory of thermoelasticity. *Int J Solids Struct, 47*, 2010, 269-275.

[18] A.E. Abouelregal, Fractional Order generalized thermo-piezoelectric semi-infinite medium with temperature-dependent properties subjected to a ramp-type heating, *Journal of Thermal Stresses, 34(11)*, 20111139-1155.

[19] H.W.Lord, Y.Shulman, A Generalized Dynamical Theory of Thermoelasticity, *Journal Mech. Phys. Solids, 15*, 1967, 299-309.

[20] A.E.Green, K.A. Lindsay, Thermoelasticity, *J. Elast, 2*, 1972, 1-7.

[21] A. E. Green, P. M. Naghdi, Thermoelasticity Without Energy Dissipation, *J. Elasticity, vol. 31*,1993, pp. 189-209.

[22] J. Ignaczak, M. Ostoja-Starzewski, *Thermoelasticity with Finite Wave Speeds*, (Oxford University Press, New York), 2010,413.

[23] F. Durbin, Numerical inversion of Laplace transforms: an efficient improvement to Dubner and Abate's method. *Computer,17*, 1974, 371-376.

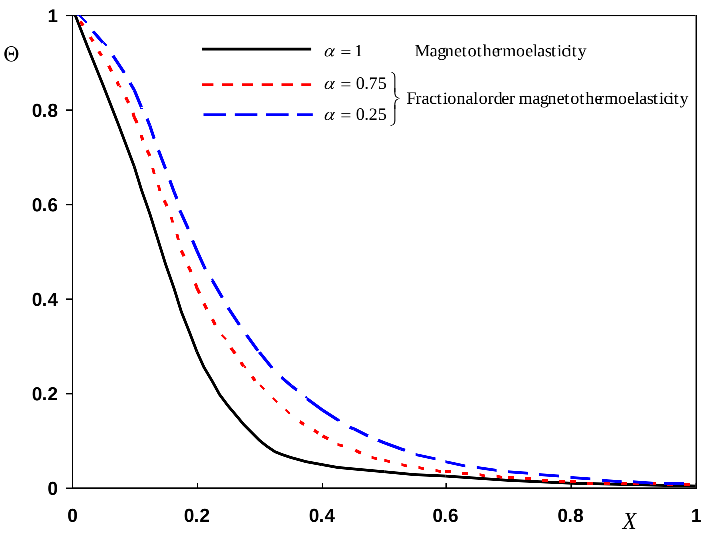

Figure 1: The effect of the fractional order parameter α on temperature distribution Θ

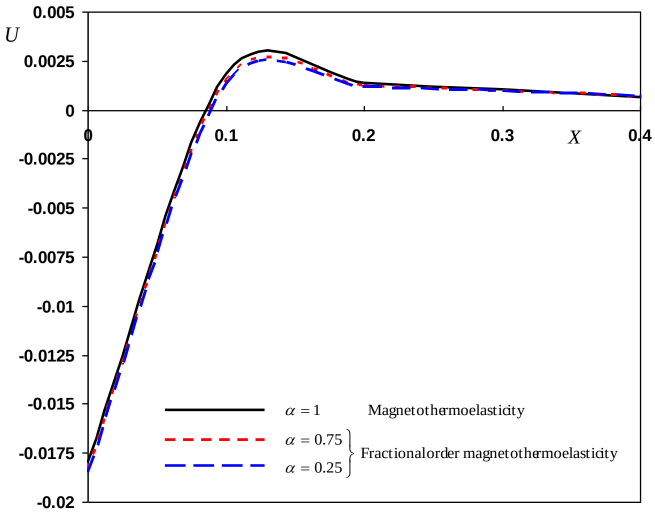

Figure 2: The effect of the fractional order parameter α on displacement U

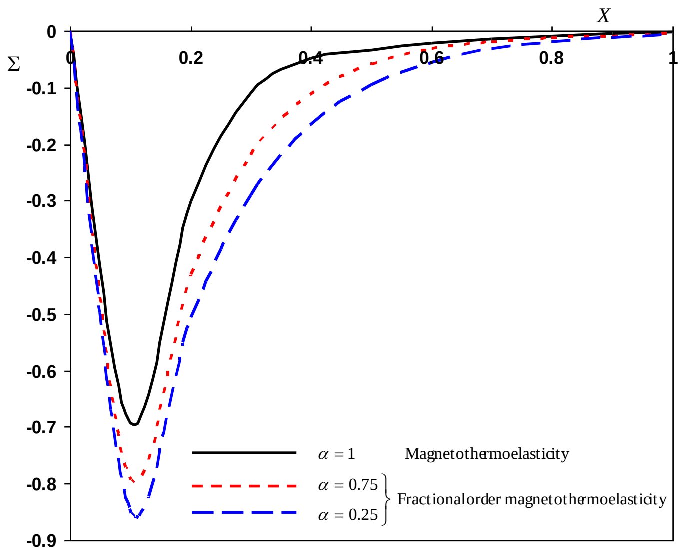

Figure 3: The effect of the fractional parameter α on the total stress Σ

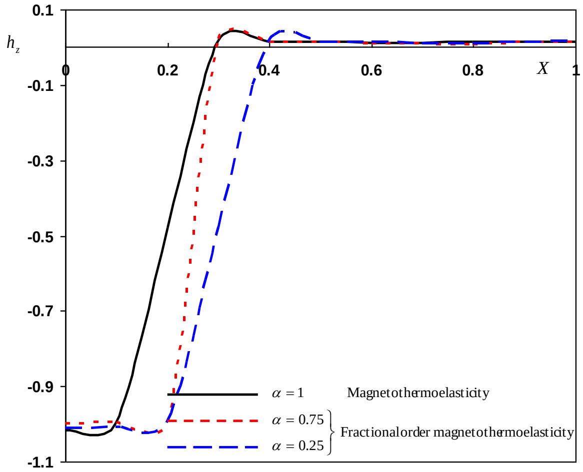

Figure 5: The effect of the fractional parameter α on the induced magnetic field h_z

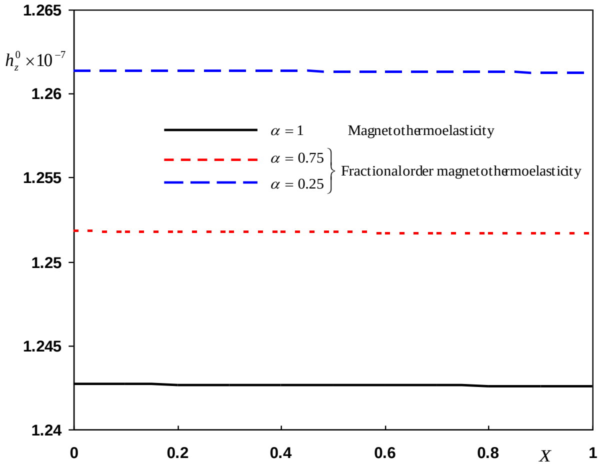

Figure 5: The effect of the fractional parameter α on the induced magnetic field in vacuum

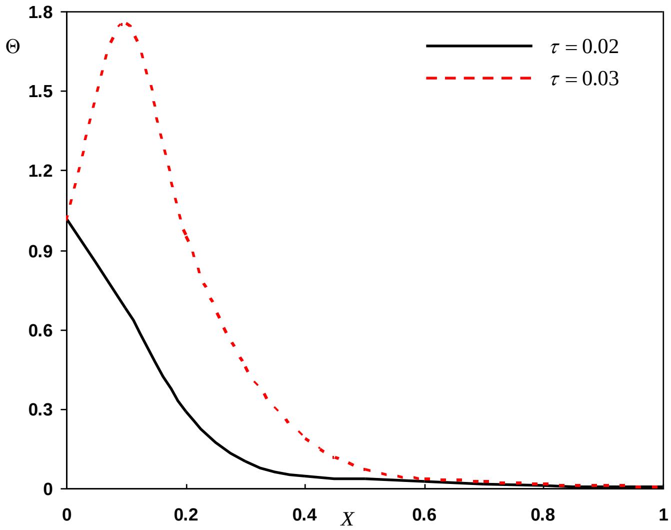

Figure 6: Temperature distribution for two different of time t .

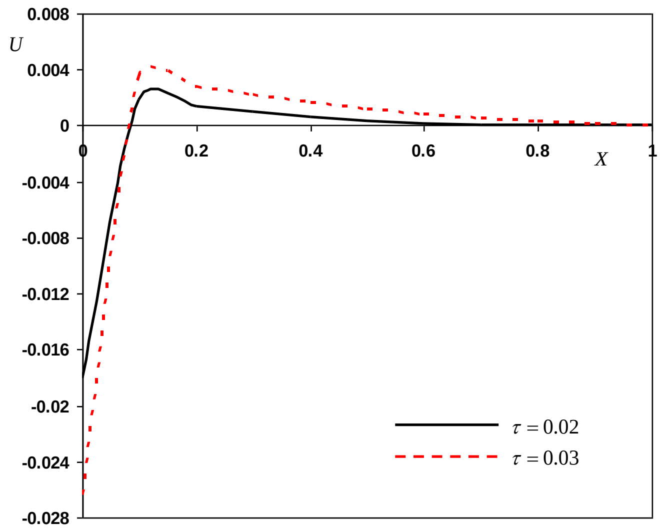

Figure 7: Displacement distribution for two different of time t .

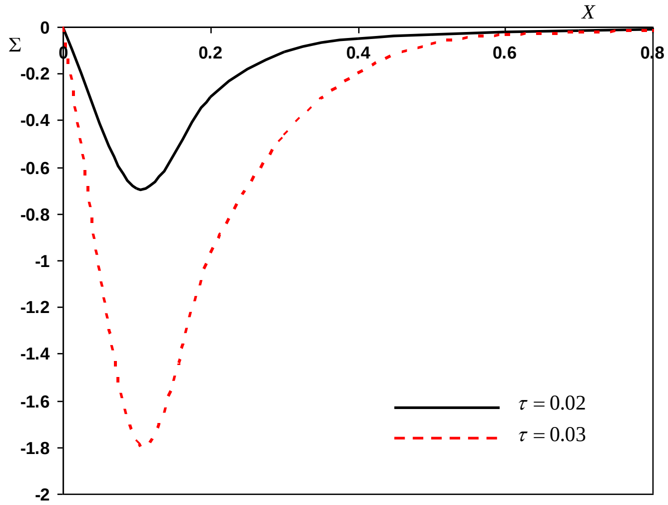

Figure 8: The total stress distribution for two different of time t .

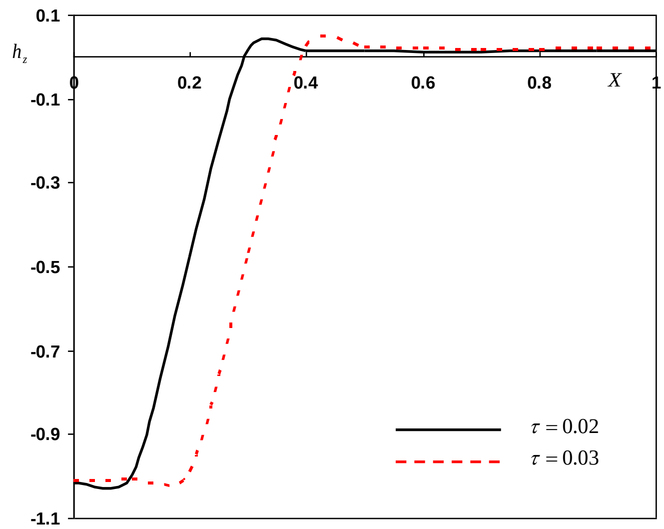

Figure 9: The induced magnetic field distribution for two different of time t .

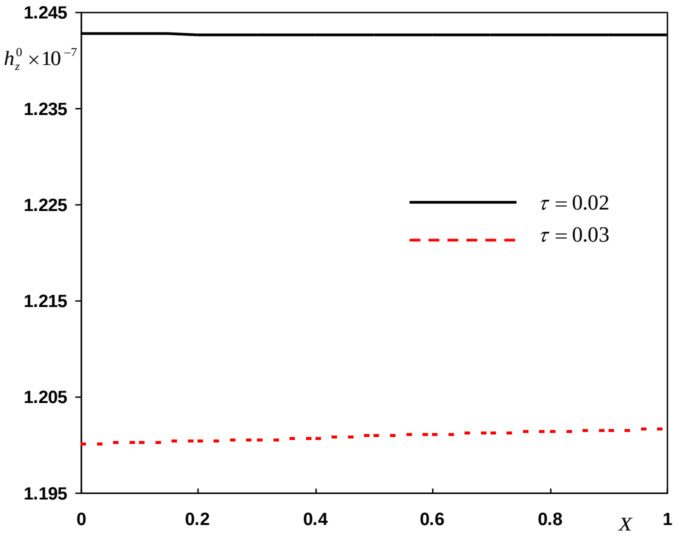

Figure 10: The induced magnetic field in vacuum distribution for two different of time t .

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