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Lacunary Arithmetic Statistical Convergence For Double Sequences.

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***Abstract**: This paper extends the recently introduced summability concept of convergence namely; arithmetic statistical convergence and lacunary arithmeticstatisticalconvergence, to double sequences. We shall also investigate the relationship between these concepts and prove some inclusion theorems.*
***Keywords and Phrases:**Summability, Arithmetic statistical convergence, lacunary arithmetic statistical convergence and double sequences.*

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I. Introduction:

The concept of statistical convergence was introduced by Fast [4] and it was further investigated from the sequence space point of view and linked with summability theory by Fridy [2], Connor [3], Fridy and Orhan [1], Šalát [5] and many others.

The idea of arithmetic convergence was introduced by Ruckle [9]. Yaying and Hazarika [8] used this concept of arithmetic convergence introduced arithmetic statistical convergence and lacunary arithmetic statistic convergence of single sequence. We shall use the concept of statistical convergence of double sequences. [see Mursaleen (6)] to extend the results of Yaying and Hazarika [8] to double sequences.

II. Lacunary Arithmetic Statistical Convergence.

Definition 2.1:(Yaying and Hazarika [2017]) A sequence $x = (x_k)$ is called arithmetically convergent if for each $\varepsilon > 0$ there is an integer $l$such that for every integer k we have $|x_k - x_{\langle k,l\rangle}| < \varepsilon$, where the symbol $\langle k,l\rangle$ denotes the greatest common divisor of two integers k and l. We denote the sequence space of all arithmetic convergent sequence by AC.

Definition 2.2 :(Fridy and Orhan [1993]) Let $\theta = (k_r)$ be a lacunary sequence. A number sequence $x = (x_k)$ is said to be lacunary statistically convergent to $l$or S_θ-convergent to l, if, for each ε> 0,

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : |x_k - l| \geq \varepsilon\}| = 0$$

In this case, one writes $S_\theta - \lim x_k = l$ or $x_k \to (S_\theta)$.The set of all lacunary statistically convergence sequences is denoted by S_θ

Definition 2.3: (Yaying and Hazarika [2017]) A sequence $x = (x_k)$ is said to be arithmetic statistically convergent if for each ε> 0 , there is an integer l such that

$$\lim_{n\to\infty} \frac{1}{n} \left|\left\{k \in n: \left|x_k - x_{\langle k,l\rangle}\right| \geq \varepsilon\right\}\right| = 0$$

We shall use ASC to denote the set of all arithmetic statistical convergent sequences. Thus for $\varepsilon > 0$ and integer l

ASC $=\left\{(x_k): \lim_{n\to\infty} \frac{1}{n} \left|\left\{k \in n: \left|x_k - x_{\langle k,l\rangle}\right| \geq \varepsilon\right\}\right| = 0\right\}$.

We shall write $ASC - \lim x_k = x_{\langle k,l\rangle}$ to denote the sequence (x_k) is arithmetic statistically convergent to $x_{\langle k,l\rangle}$.

Definition 2.4: (Yaying and Hazarika [2017]) Let $\theta = (k_r)$ be a lacunary sequence. The number sequence $x = (x_k)$ is said to be lacunary arithmetic statistically convergent if for each $\varepsilon > 0$ there is an integer l such that

$$\lim_{r\to\infty} \frac{1}{h_r} \left|\left\{k \in I_r: \left|x_k - x_{\langle k,l\rangle}\right| \geq \varepsilon\right\}\right| = 0$$

We shall write
$ASC_\theta = \left\{x = (x_k): \lim_{r\to\infty} \frac{1}{h_r}\left|\left\{k \in I_r: \left|x_k - x_{\langle k,l\rangle}\right| \geq \varepsilon\right\}\right| = 0\right\}$.
We shall write $ASC_\theta - \lim x_k = x_{\langle k,l\rangle}$ to denote the sequence (x_k) is lacunary arithmetic statistically convergent to $x_{\langle k,l\rangle}$.

Definition 2.5: (Yaying and Hazarika [2017]) Let $\theta = (k_r)$ be a lacunary sequence. A lacunary refinement of θ is a lacunary sequence $\theta' = (k_r')$satisfying $(k_r) \subseteq (k_r')$. (See Freedman et al. [ ].

Definition 2.6: (Yaying and Hazarika [2017])A function f defined on a subset $E\ of\ \mathbb{R}$ is said to be lacunary arithmetic statistical continuous if it preserves lacunary arithmetic statistical convergence i.e. if

$ASC_\theta - \lim x_k = x_{\langle k,l\rangle}$implies$ASC_\theta - \lim f(x_k) = f(x_{\langle k,l\rangle})$.

Theorem 2.1:(Yaying and Hazarika [2017]) Let $x = (x_k)$ and $y = (y_k)$ be two sequences.
(i) If $ASC - \lim x_k = x_{\langle k,l\rangle}$ and $a \in \mathbb{R}$, then $ASC - \lim ax_k = ax_{\langle k,l\rangle}$.
(ii) If $ASC - \lim x_k = x_{\langle k,l\rangle}$ and $ASC - \lim y_k = y_{\langle k,l\rangle}$, then $ASC - \lim(x_k + y_k) = (x_{\langle k,l\rangle} + y_{\langle k,l\rangle})$.

Theorem 2.2:(Yaying and Hazarika [2017]) Let $x = (x_k)$ and $y = (y_k)$ be two sequences.
(i) If$ASC_\theta - \lim x_k = x_{\langle k,l\rangle}$ and $a \in \mathbb{R}$, then $ASC_\theta - \lim cx_k = cx_{\langle k,l\rangle}$
(ii) If$ASC_\theta - \lim x_k = x_{\langle k,l\rangle}$ and $ASC_\theta - \lim y_k = y_{\langle k,l\rangle}$, then $ASC_\theta - \lim(x_k + y_k) = (x_{\langle k,l\rangle} + yk,l)$

Theorem 2.3: (Yaying and Hazarika [2017]) If $\theta' = (k_r')$ is a lacunary refinement of a lacunary sequence $\theta = (k_r)$ and $(x_k) \in ASC_{\theta'}$ then $(x_k) \in ASC_\theta$.

Theorem 2.4:(Yaying and Hazarika [2017]) Suppose $\beta = (l_r)$ is a lacunary refinement of a lacunary sequence $\theta = (k_r)$. Let $I_r = (k_{r-1}, k_r]$ and $J_r = (l_{r-1}, l_r]$, r = 1, 2, … If there exists $\delta > 0$ such that $\frac{|J_j|}{|I_i|} \geq \delta\ for\ every\ J_j \subseteq I_i$. Then $(x_k) \in ASC_\theta \Rightarrow (x_k) \in ASC_\beta$.

Theorem 2.5: (Yaying and Hazarika [2017]) Suppose $\beta = (l_r)$ and $\theta = (k_r)$ are two lacunary sequences. Let $I_r = (k_{r-1}, k_r]$, $J_r = (l_{r-1}, l_r]$, r = 1, 2, … $I_{ij} = I_i \cap J_j$, $i, j = 1, 2, 3 \ldots$ If there exists $\delta > 0$ such that
$\frac{|I_{ij}|}{|I_i|} \geq \delta\ for\ every\ i, j = 1, 2, 3, \ldots,\ I_{ij} \neq \emptyset$.
Then $(x_k) \in ASC_\theta \Rightarrow (x_k) \in ASC_\beta$.

Theorem 2.6: (Yaying and Hazarika [2017]) Let $\theta = (k_r)$, r = 1,2,3,. . . , be a lacunary sequence. If lim inf $q_r > 1$, then $ASC \subseteq ASC_\theta$.

Theorem 2.7: (Yaying and Hazarika [2017]) Forlim sup $q_r < \infty$, we have $ASC_\theta\ \ ASC$.

We shall now use analogy to extend the above concepts and results to double sequences;

III. Lacunary Arithmetic Statistical Convergence For Double Sequences.

Definition 3.1: A double sequence $x = (x_{k,m})$ is called arithmetically convergent if for each $\varepsilon > 0$ there is an integer *l*, *n* such that for every integer k, m we have $\left|x_{k,m} - x_{\langle k,l\rangle,\langle m,n\rangle}\right| < \varepsilon$, where the symbol $\langle k, l, m, n\rangle$ denotes the greatest common divisor of four integers *k*, *l*, *m* and *n*. We denote the double sequence space of all arithmetic convergent sequence by $(AC)_2$

Note that: $g = \langle(\langle k, l\rangle, \langle m, n\rangle)\rangle$where g denotes the greatest common divisor (gcd) for double sequences. Therefore we shall use g as the above equality throughout this paper.

Definition 3.2 : Let$\theta = (k_{r,s})$ be a lacunary double sequence. A double sequence $x = (x_{k,m})$ is said to be lacunary statistically convergent to $l$or $S_{\theta_{r,s}}$-convergent to l, if, for each ε> 0,

$$\lim_{r,s\to\infty} \frac{1}{h_{r,s}}\left|\left\{k, m \in I_{r,s}: \left|x_{k,m} - l\right| \geq \varepsilon\right\}\right| = 0$$

In this case, one writes $S_{\theta_{r,s}} - \lim x_{k,m} = l$ or $x_{k,m} \to (S_{\theta_{r,s}})$.The set of all lacunary statistically convergence double sequences is denoted by $S_{\theta_{r,s}}$

Definition 3.3 : A double sequence $x = (x_{k,m})$ is said to be arithmetic statistically convergent if for each ε> 0 , there is an integer *l*, *n* such that

$$\lim_{n\to\infty} \frac{1}{n}\left|\left\{k, m \in n: \left|x_{k,m} - x_g\right| \geq \varepsilon\right\}\right| = 0$$

We shall use $(ASC)_2$ to denote the set of all arithmetic statistical convergent double sequences. Thus for $\varepsilon > 0$ and integer *l*, *n*

$(ASC)_2 = \left\{(x_{k,m}) : \lim_{n\to\infty} \frac{1}{n}\left|\{k, m \in n : |x_{k,m} - x_g| \geq \varepsilon\}\right| = 0\right\}$.

We shall write $(ASC)_2 - \lim x_{k,m} = x_g$ to denote the double sequence $(x_{k,m})$ is arithmetic statistically convergent to x_g

Definition 3.4 : Let $\theta = (k_{r,s})$ be a lacunary double sequence. The double sequence $x = (x_{k,m})$ is said to be lacunary arithmetic statistically convergent for double sequences if for each $\varepsilon > 0$ there is an integer l, n such that for every integer $k, m \geq l, n$

$$\lim_{r,s\to\infty} \frac{1}{h_{r,s}}\left|\{k, m \in I_{r,s} : |x_{k,m} - x_g| \geq \varepsilon\}\right| = 0$$

We shall write

$ASC_{\theta_{r,s}} = \left\{x = (x_{k,m}) : \lim_{r,s\to\infty} \frac{1}{h_{r,s}}\left|\{k, m \in I_{r,s} : |x_{k,m} - x_g| \geq \varepsilon\}\right| = 0\right\}$.

We shall write $ASC_{\theta_{r,s}} - \lim x_{k,m} = x_g$ to denote the double sequence $(x_{k,m})$ is lacunary arithmetic statistically convergent to x_g

Definition 3.5 : Let $\theta = (k_{r,s})$ be a lacunary double sequence. A lacunary refinement of θ is a lacunary double sequence $\theta' = (k'_{r,s})$satisfying $(k_{r,s}) \subseteq (k'_{r,s})$. (See Freedman et al. [7].**)**

Theorem 3.1 : Let $x = (x_{k,m})$ and $y = (y_{k,m})$ be two double sequences.

(i) If $(ASC)_2 - \lim x_{k,m} = x_{\langle k,l\rangle,\langle m,n\rangle}$ and $a \in \mathbb{R}$, then $(ASC)_2 - \lim a x_{k,m} = a x_{\langle k,l\rangle\langle m,n\rangle}$.

(ii) If $(ASC)_2 - \lim x_{k,m} = x_{\langle k,l\rangle,\langle m,n\rangle}$and $(ASC)_2 - \lim y_{k,m} = y_{\langle k,l\rangle,\langle m,n\rangle}$, then $(ASC)_2 - \lim(x_{k,m} + y_{k,m}) = (x_{\langle k,l\rangle,\langle m,n\rangle} + y_{\langle k,l\rangle,\langle m,n\rangle})$.

Proof 3.1 :

(i) The result is obvious when *a* = 0 . Suppose *a* ≠ 0, then for integer *l* , *n*

$$\frac{1}{uv}\left|\{k \leq u, m \leq v : |a x_{k,m} - a x_g| \geq \varepsilon\}\right|$$

$$= \frac{1}{uv}\left|\left\{k \leq u, m \leq v : |x_{k,m} - x_g| \geq \frac{\varepsilon}{|a|}\right\}\right|$$

Which gives the result

The result of (ii) follows from

$$\frac{1}{uv}\left|\{k \leq u, m \leq v : |(x_{k,m} + y_{k,m}) - (x_{\langle k,l\rangle,\langle m,n\rangle} + y_{\langle k,l\rangle,\langle m,n\rangle})| \geq \varepsilon\}\right|$$

$\leq \frac{1}{uv}\left|\left\{k \leq u, m \leq v : |x_{k,m} - x_{\langle k,l\rangle,\langle m,n\rangle}| \geq \frac{\varepsilon}{2}\right\}\right| + \frac{1}{uv}\left|\left\{k \leq u, m \leq v : |y_{k,m} - y_{\langle k,l\rangle,\langle m,n\rangle}| \geq \frac{\varepsilon}{2}\right\}\right|$

□

Thus we defined a related concept of convergence in which the set $\{k, m : k, m \leq uv\}$is replaced by the set $\{k, m : k_{r-1,s-1} \leq k, m \leq k_{r,s}\}$, for some lacunary double sequence $(k_{r,s})$. (see definition 3.4)

Theorem 3.2 : Let$x = (x_k)$ and $y = (y_k)$ be two sequences.

(iii) If$ASC_\theta - \lim x_k = x_{\langle k,l\rangle}$ and $a \in \mathbb{R}$, then $ASC_\theta - \lim c x_k = c x_{\langle k,l\rangle}$

(iv) If$ASC_\theta - \lim x_k = x_{\langle k,l\rangle}$ and $ASC_\theta - \lim y_k = y_{\langle k,l\rangle}$, then $ASC_\theta - \lim(x_k + y_k) = (x_{\langle k,l\rangle} + yk,l)$

Proof3.2 :

(i) The result is obvious when *a* = 0 . Suppose *a* ≠ 0, then for integer *l* , *n*

$$\frac{1}{h_{r,s}}\left|\{k, m \in I_{r,s} : |a x_{k,m} - a x_g| \geq \varepsilon\}\right|$$

$$= \frac{1}{h_{r,s}}\left|\left\{k, m \in I_{r,s} : |x_{k,m} - x_g| \geq \frac{\varepsilon}{|a|}\right\}\right|$$

Which gives the result

The result of (ii) follows from

$$\frac{1}{h_{r,s}}\left|\left\{k,m \in I_{r,s} : \left|\left(x_{k,m} + y_{k,m}\right) - (x_g + y_g)\right| \geq \varepsilon\right\}\right|$$

$$\leq \frac{1}{uv}\left|\left\{k \leq u, m \leq v : \left|x_{k,m} - x_g\right| \geq \frac{\varepsilon}{2}\right\}\right| + \frac{1}{uv}\left|\left\{k \leq u, m \leq v : \left|y_{k,m} - y_g\right| \geq \frac{\varepsilon}{2}\right\}\right|$$

$$\leq \frac{1}{h_{r,s}}\left|\left\{k,m \in I_{r,s}: \left|x_{k,m} - x_g\right| \geq \frac{\varepsilon}{2}\right\}\right| + \frac{1}{h_{r,s}}\left|\left\{k,m \in I_{r,s} : \left|y_{k,m} - y_g\right| \geq \frac{\varepsilon}{2}\right\}\right|$$

□

Theorem 3.3 : If$\theta' = (k'_{r,s})$ is a lacunary refinement of a lacunary double sequence $\theta = (k_{r,s})$ and $\left(x_{k,m}\right) \in ASC_{\theta'_{r,s}}$ then $\left(x_{k,m}\right) \in ASC_{\theta_{r,s}}$.

Proof3.3 :

Suppose for each $I_{r,s}$ $of\ \theta$ contains the point $(k'_{r,s,t})_{t=1}^{\mu(r,s)}$ $of\ \theta'$ such that

$$k_{r-1,s-1} < k'_{r,s,1} < k'_{r,s,2} < \cdots < k'_{\mu,\mu(r,s)} = k_{r,s}$$

Where $I'_{r,s} = (k'_{r,s-1} , k'_{r,s}]$

Since $\left(k_{r,s}\right) \subseteq (k'_{r,s}) , so\ \forall\ r , s\ \mu(r,s) \geq 1$

Let $(I^*)_{r,s=1}^{\infty}$ be the double sequence of interval $(I^*_{r,s})$ ordered by increasing right end points. Since $\left(x_{k,m}\right) \in ASC_{\theta'_{r,s}}$ then for each $\varepsilon > 0$ and an integer *l*,*n*

$$lim \sum_{I^*_{r,s} \subset I_{r,s}} \sum_{I^*_{r,s} \subset I_{r,s}} \frac{1}{h^*_{r,s}}\left|\left\{k,m \in I^*_{r,s} : \left|x_{k,m} - x_g\right| \geq \varepsilon\right\}\right| = 0$$

Also since $h_{r,s} = k_{r,s} - k_{r-1,s-1}$ $, so\ h'_{r,s} = k'_{r,s} - k'_{r-1,s-1}$

For each $\varepsilon > 0$ and integer *l*,*n*

$$\frac{1}{h_{r,s}}\left|\left\{k,m \in I_{r,s}: \left|x_{k,m} - x_g\right| \geq \varepsilon\right\}\right| = \frac{1}{h_{r,s}}\sum_{I^*_{r,s} \subset I_{r,s}} h^*_{r,s} \frac{1}{h^*_{r,s}}\left|\left\{k,m \in I^*_{r,s} : \left|x_{k,m} - x_g\right| \geq \varepsilon\right\}\right|$$

$$\to 0\ as\ r,s \to \infty$$

This implies $\left(x_{k,m}\right) \in ASC_{\theta_{r,s}}$ ■

Theorem 3.4 : Suppose $\gamma = (l_{r,s})$ is a lacunary refinement of a lacunary double sequences$\theta = (k_{r,s})$. Let $I_{r,s} = (k_{r-1,s-1} , k_{r,s}]$ and $J_{r,s} = (l_{r-1,s-1} , l_{r,s}]$, r = 1, 2, … If there exists $\delta > 0$ such that $\frac{|J_{g,h}|}{|I_{i,j}|} \geq \delta\ for\ every\ J_{g,h} \subseteq I_{i,j}$. Then $\left(x_{k,m}\right) \in ASC_{\theta_{r,s}} \Rightarrow \left(x_{k,m}\right) \in ASC_{\gamma_{r,s}}$.

Proof 3.4 :

For any $\varepsilon > 0$and integer *l*,*n*every $J_{g,h}$ we can find $I_{i,j}$ such that $J_{g,h} \subseteq I_{i,j}$, then we have

$$\frac{1}{|J_{g,h}|}\left|\left\{k,m \in J_{g,h} : \left|x_{k,m} - x_g\right| \geq \varepsilon\right\}\right| = \left(\frac{|I_{i,j}|}{|J_{g,h}|}\right)\left(\frac{1}{|I_{i,j}|}\right)\left|\left\{k,m \in J_{g,h} : \left|x_{k,m} - x_g\right| \geq \varepsilon\right\}\right|$$

$$\leq \left(\frac{|I_{i,j}|}{|J_{g,h}|}\right)\left(\frac{1}{|I_{i,j}|}\right)\left|\left\{k,m \in I_{i,j} : \left|x_{k,m} - x_g\right| \geq \varepsilon\right\}\right|$$

$$\leq \left(\frac{1}{\delta}\right)\left(\frac{1}{|I_{i,j}|}\right)\left|\left\{k,m \in I_{i,j} : \left|x_{k,m} - x_g\right| \geq \varepsilon\right\}\right| ■$$

Which gives the result.

Theorem 3.5 : Suppose$\gamma = (l_{r,s})$ and $\theta = (k_{r,s})$ are two lacunary double sequences. Let $I_{r,s} = (k_{r-1,s-1} , k_{r,s}]$,$J_{r,s} = (l_{r-1,s-1} , l_{r,s}]$,r,s= 1, 2, … and$I_{a,b} = I_{wx} \cap J_{yz}$, $a,b = 1,2,3 \ldots\ and\ where\ a = wx\ and\ b\ yz$ If there exists $\delta > 0$ such that $\frac{|I_{a,b}|}{|I_{wx}|} \geq \delta\ for\ every\ y,z = 1,2,3,\ldots,\ I_{y,z} \neq \emptyset.$

Then $\left(x_{k,m}\right) \in ASC_{\theta_{r,s}} \Rightarrow \left(x_{k,m}\right) \in ASC_{\gamma_{r,s}}$

Proof 3.5 :

Let $= \gamma \cup \theta$. Then μ is a lacunary refinement of θ. The interval sequence of $\mu$is $\{I_{a,b} = I_{wx} \cap I_{yz} : I_{a,b} \neq \emptyset.\ where\ a = wx\ and\ b\ yz.\}$. Using theorem 3.4 and the condition $\frac{|I_{a,b}|}{|I_{wx}|} \geq \delta$ gives$\left(x_{k,m}\right) \in ASC_{\theta_{r,s}} \Rightarrow$

$(x_{k,m}) \in ASC_{\gamma_{r,s}}$. Since μ is a lacunary refinement of the lacunary double sequences , from theorem 3.3, we have $(x_{k,m}) \in ASC_{\mu_{r,s}} \Rightarrow (x_{k,m}) \in ASC_{\gamma_{r,s}}$ ■

Theorem 3.6: Let $\theta = (k_{r,s})$, r,s = 1,2,3,. . . , be a lacunary double sequences. If liminf$q_{r,s} > 1$, then $(ASC)_2 \subseteq ASC_{\theta_{r,s}}$.

Proof 3.6 :

Let $(x_{k,m}) \in (ASC)_2$ and liminf$q_{r,s} > 1$. Then there exist $\alpha > 1$ such that $q_{r,s} = \frac{k_{r,s}}{k_{r-1,s-1}} \geq 1 + \alpha$ for sufficiently larger r,s which implies that $\frac{h_{r,s}}{k_{r,s}} \geq \frac{\alpha}{1+\alpha}$

Then, for sufficiently large r,s and integer k,m

$$\frac{1}{k_{r,s}}\left|\left\{k,m \in k_{r,s}: \left|x_{k,m} - x_g\right| \geq \varepsilon\right\}\right| \geq \frac{1}{k_{r,s}}\left|\left\{k,m \in I_{r,s}: \left|x_{k,m} - x_g\right| \geq \varepsilon\right\}\right|$$
$$\geq \frac{\alpha}{1+\alpha}\frac{1}{h_{r,s}}\left|\left\{k,m \in I_{r,s}: \left|x_{k,m} - x_g\right| \geq \varepsilon\right\}\right|$$

Thus $x = (x_{k,m}) \in (ASC)_2 \Rightarrow (x_{k,m}) \in ASC_{\theta_{r,s}}$ ■

Theorem 3.7 : Forlim sup $q_{r,s} < \infty$, we have $ASC_{\theta_{r,s}} \subseteq (ASC)_2$.

Proof 3.7 :

Let lim sup$q_{r,s} < \infty$ then there exist $\omega > 0$ such that $q_{r,s} < \omega$ for every r,s. Let $\tau_{r,s} = |\{k,m \in I_{r,s}: |x_{k,m} - xg \geq \varepsilon$ where *l,n* is an integer. Now for $\varepsilon > 0$ and $xk,m \in ASC\theta r,s$ there exists N such that

$$\frac{\tau_{r,s}}{h_{r,s}} < \varepsilon \text{for every } r,s \geq N$$

Let $M = Max\{\tau_{r,s} : 1 \leq r,s \leq N\}$and let p be any integer with $k_{r,s} \geq p \geq k_{r-1,s-1}$. Then for an integer *l,n*

$$\frac{1}{p}\left|\left\{k,m \in p : \left|x_{k,m} - x_g\right| \geq \varepsilon\right\}\right|$$
$$\leq \frac{1}{k_{r-1,s-1}}\left|\left\{k,m \in k_{r,s}: \left|x_{k,m} - x_g\right| \geq \varepsilon\right\}\right|$$
$$= \frac{1}{k_{r-1,s-1}}\{\tau_1 + \tau_2 + \cdots + \tau_N + \tau_{N+1} + \cdots + \tau_{r,s}\}$$
$$\leq \frac{MN}{k_{r-1,s-1}} + \frac{1}{k_{r-1,s-1}}\left\{h_{N+1}\frac{\tau_{R+1}}{h_{N+1}} + \cdots + h_{r,s}\frac{\tau_{r,s}}{h_{r,s}}\right\}$$
$$\leq \frac{MN}{k_{r-1,s-1}} + \frac{1}{k_{r-1,s-1}}\left(\sup_{r,s>N}\frac{\tau_{r,s}}{h_{r,s}}\right)\{h_{N+1} + \cdots + h_{r,s}\}$$
$$\leq \frac{MN}{k_{r-1,s-1}} + \varepsilon\frac{k_{r,s} - k_N}{k_{r-1,s-1}}$$
$$\leq \frac{MN}{k_{r-1,s-1}}T + \varepsilon\, q_{r,s}$$
$$\leq \frac{MN}{k_{r-1,s-1}}T + \varepsilon\, K \blacksquare$$

Which gives$(x_{k,m}) \in (ASC)_2$

Corollary 3.1.

From there 2.6 and 2.7 , if $\theta = (k_r)$ be a lacunary double sequences and if

$$1 < \lim inf q_r \leq \lim\sup q_r < \infty$$

Then $(ASC)_2 = ASC_\theta$

In (2016) Yaying and Hazarika introduced lacunary arithmetic convergent sequence AC_θ as follow:

$$AC_\theta = \left\{(x_k) : \lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\left|x_k - x_{\langle k,l\rangle}\right| = 0 \; for\ integer\ l\right\}$$

Analogously, we define double lacunary arithmetic convergence

From theorem 3.6 and 3.7 , if $\theta = (k_{r,s})$ be a lacunary double sequences and if

$$1 < \lim inf q_{r,s} \leq \lim\sup q_{r,s} < \infty$$

Then $(ASC)_2 = ASC_{\theta_{r,s}}$

Now we introduce lacunary arithmetic convergent sequence $AC_{\theta_{r,s}}$ as follow:

$$AC_{\theta_{r,s}} = \left\{ (x_{k,m}) : \lim_{r,s\to\infty} \frac{1}{h_{r,s}} \sum_{k,l\in I_r} \sum_{k,l\in I_r} |x_{k,m} - x_g| = 0 \; some\ integers\ l,n \right\}$$

In relation to this we shall introduce for double sequences space and give some relation between the double spaces $AC_{\theta_{r,s}}$ and $ASC_{\theta_{r,s}}$

Theorem 3.8 : Let$\theta = (k_{r,s})$ be a lacunary double sequence; then if $(x_{k,m}) \in (AC_\theta)_2$ then $(x_{k,m}) \in (ASC_\theta)_2$

Proof 3.8 : Let$(x_{k,m}) \in (AC_\theta)_2$ and $\varepsilon > 0$. We can write, for an integer *l*,*n*

$$\sum_{k,m\in I_{r,s}} |x_{k,m} - x_g|$$

$$\geq \sum_{\substack{k,m\in I_{r,s} \\ |x_{k,m}-x_g|\geq\varepsilon}} |x_{k,m} - x_g| + \sum_{\substack{k,m\in I_{r,s} \\ |x_{k,m}-x_g|<\varepsilon}} |x_{k,m} - x_g|$$

$$\geq \sum_{\substack{k,m\in I_{r,s} \\ |x_{k,m}-x_g|\geq\varepsilon}} |x_{k,m} - x_g|$$

$$\geq \varepsilon \left| \{k,m \in I_{r,s} : |x_{k,m} - x_g| \geq \varepsilon\} \right|$$

■

Which gives the result.

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