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Some Generalized Difference Riesz Sequence Spaces and Related Matrix Transformations

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Abstract: *The generalized difference Riesz sequence space $r^q(\mathcal{M}, \Delta_n^m, u, p, s)$ of non absolute type was recently introduced and studied by some authors. This paper is devoted to characterize the classes $(r^q(\mathcal{M}, \Delta_n^m, u, p, s), \ell_\infty), (r^q(\mathcal{M}, \Delta_n^m, u, p, s), c)$ and $(r^q(\mathcal{M}, \Delta_n^m, u, p, s), c_0)$ of infinite matrices and characterize a basic theorem where ℓ_∞, c and c_0 denotes respectively the space of bounded sequences, space of all convergent sequences and space of all sequences converging to zero.*

Keywords: *Riesz sequence space, sequence space of non absolute type, paranormed sequence spaces and Matrix transformations.*

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I. Introduction

We denote the set of all sequences (real or complex) by ω. Any subspace of ω is called sequence space. Let $\mathbb{N}, \mathbb{R}, and\ \mathbb{C}$ denote the space of all non negative integers, the space of all real numbers and the space of all complex numbers respectively. Let $\ell_\infty, c\ and\ c_0$ respectively denotes the space of all bounded sequences, the space of all convergent sequences and the space of all sequences converging to zero. Also by $\ell_1, \ell(p), cs\ and\ bs$ we denote the space of all absolutely, p- absolutely convergent, convergent, and bounded series respectively. We use the convention that any term with negative subscript equal to zero.

Let X be a real or complex linear space h be a function from X to the set $\mathbb{R}$ of real numbers. Then the pair (X, h) called a paranormed space and h is a paranorm for X if the following axioms are satisfied:

1 $h(\theta) = 0$
2 $h(-x) = h(x)$
3 $h(x + y) \leq h(x) + h(y)$ and
4 scalar multiplication is continuous, that is $|\alpha_n - \alpha| \to 0 and\ h(x_n - x) \to 0, imply\ h(\alpha_n x_n - \alpha x) \to 0,$ for all $\alpha\ in \mathbb{R}$ and x in X. where θ is the zero vector in the space X. Assuming here and after (p_k) be a bounded sequence of strictly positive real numbers with $\sup_k p_k = H\ and\ M = \max[1, H]$. Then, the linear space $\ell(p)$ was defined by Maddox [2] as follows:

$$\ell(p) = \left\{x = (x_k) \in \omega: \sum_k |x_k|^{p_k} < \infty\right\}$$

which is complete space paranormed by

$$h_1(x) = \left[\sum_k |x_k|^{p_k}\right]^{1/M}$$

We shall assume throughout the paper that $\frac{1}{p_k} + \frac{1}{p'_k} = 1\ provided\ 1 < \inf p_k \leq H < \infty$.

Let X and Y be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers . Then the matrix A defines the A-transformation from X into Y, if for every sequence $x = (x_k) \in X$ the sequence $Ax = [(Ax)_n]$, the A-transform of x exists and is in Y; where

$$(Ax)_n = \sum_k a_{nk} x_k.$$

For simplicity in notation, here and in what follows, the summation without limits runs from $0\ to\ \infty$. By $A \in (X, Y)$ we mean the characterizations of matrices from X to Y i.e $A: X \to Y$. A sequence x is said to be A-summable to l if Ax converges to l, which is called the A-limit of x. For a sequence space X, the matrix domain of an infinite matrix A is define as

$X_A = \{x = (x_k) \in \omega: Ax \in X\}$
(1.1)

The theory of matrix transformations is a wide field in summability; it deals with the characterization of classes of matrix mappings between sequence spaces by giving necessary and sufficient conditions on the entries of the infinite matrices.

The classical summability theory deals with a generalization of convergence of sequences and series. One original idea was to assign a limit to divergent sequence or series . Toeplitz [1] was the first to study summability method as a class of transformations of complex sequences by infinite matrices.

Let $A = (a_{nk})$ be any matrix. Then a sequence x is said to be summable to l, written $x_k \to l$ if and only if $A_n(x) = \sum_k a_{nk}\, x_k$ exists for each n and $A_n(x) \to l(n \to \infty)$. For example, if I is a unit matrix, then $x_k \to l(I)$ means precisely that $x_k \to l(k \to \infty)$ in ordinary sense of convergence.

We denote by (A) the set of all sequences which are summable A. The set (A) is called summability field of the matrix A. Thus if $Ax = (a_{nk}(x))$ then $(A) = \{x \in \omega: Ax \in c\}$, where c is the set of all convergent sequences. For example $(I) = c$

An infinite matrix $A = (a_{nk})$ is said to be regular [2] if and only if the following conditions (Toeplitz conditions) hold.

1 $\lim_{n\to\infty} \sum_k a_{nk} = 1$

2 $\lim_{n\to\infty} a_{nk} = 0, \quad (k = 0,1, \dots\}$

3 $\sum_k |a_{nk}| < M, \quad (M > 0,\ n = o, 1,2 \dots)$

Let (q_k) be a sequence of positive numbers and let us write $Q_k = \sum_{k=0}^{n} q_k$ for all positive integers. Then the matrix $R^q = (r_{nk}^q)$ of Riesz mean (R, q_n) is given by

$$r_{nk}^q = \begin{cases} \dfrac{q_k}{Q_k} & if\ 0 \le k \le n \\ 0 & if\ k > n \end{cases}$$

The Riesz mean (R, q_n) is regular if $Q_n \to \infty, as\ n \to \infty$ (see Petersen [3]).

The sequence space $r^q(u,p)$ introduced by Sheikh and Ganie [4] as

$$r^q(u,p) = \left\{ x = (x_k) \in \omega: \sum_k \left| \frac{1}{Q_k} \sum_{j=0}^{k} u_j q_j x_j \right|^{p_k} < \infty \right\}$$

Where $0 \le p_k \le H < \infty$.

Also FazlurRahman and Rezaulkarim [10] introduced and studied the sequence space

$$r^q(u,p,s) = \left\{ x = (x_k) \in \omega: \sum_k \left| \frac{1}{Q_n^{s+1}} \sum_{k=1}^{n} u_k q_k x_k \right|^{p_k} < \infty \right\}$$

The notion of difference sequence spaces was introduced by Kizmaz [5], who studied the difference sequence spaces $l_\infty(\Delta),\ c(\Delta)$ and $c_0(\Delta)$. This notion was further generalized by Et and Colak [6] defined the sequence spaces $l_\infty(\Delta^m),\ c(\Delta^m)$ and $c_0(\Delta^m)$. Let n, m be non negative integers then for Z a given sequence space we have the following spaces:

$$Z(\Delta_n^m) = \{x = (x_k) \in \omega: (\Delta_n^m x_k) \in Z\}$$

For $Z = c, c_0, and\ l_\infty$ where

$$\Delta_n^m x = (\Delta_n^m x_k) = (\Delta_n^{m-1} x_k - \Delta_n^{m-1} x_{k+1}), \text{ and } \Delta_n^0 x_k = x_k$$

For all $k \in N$, which is equivalent to binomial representation

$$\Delta_n^m x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} x_{+ni}$$

It was proved that the generalized sequence space $Z(\Delta_n^m)$, where $Z = \ell_\infty, c$ or c_0, is a Banach space with norm defined by

$$\|x\|_{\Delta_n^m} = \sum_{i=1}^{m} |x_i| + sup|\Delta_n^m x_k|.$$

Altay and Basar [7] defined the sequence spaces of p- bounded variation $bv_p = \left(l_p\right)_\Delta, \quad 1 \le p < \infty$

Where Δ denotes the matrix $\Delta = (\Delta_{nk})$ and is defined as

$$\Delta_{nk} = \begin{cases} (-1)^{n-k} & , if\ n-1 \le k \le n, \\ 0 & if\ k < n-1 or\ k > n. \end{cases}$$

Neyaz and Hamid [8] introduced the space $r^q\left(\Delta_u^p\right)$ as

$$r^q\left(\Delta_u^p\right) = \left\{ x = (x_k) \in \omega: \sum_k \left| \frac{1}{Q_k} \sum\nolimits_{j=0}^{k} u_j q_j \Delta x_j \right|^{p_k} < \infty \right\},$$

Where $0 \le p_k \le H < \infty$.

Raj and Anand [9] also introduced and studied the space $r^q(\mathcal{M}, \Delta_n^m, u, p)$ as

$$r^q(\mathcal{M},\Delta_n^m,u,p)=\left\{x=(x_k)\in\omega:\ \sum_k\left|\frac{1}{Q_k}\sum_{j=0}^{k}M_j\left(\left|u_j q_j\Delta_n^m x_j\right|\right)\right|^{p_k}\right\}$$

Where $0<p_k\leq H<\infty$.

The main purpose of this paper is to characterize the matrix classes

$(r^q(\mathcal{M},\Delta_n^m,u,p,s),\ell_\infty),(r^q(\mathcal{M},\Delta_n^m,u,p,s),c)\ and\ (r^q(\mathcal{M},\Delta_n^m,u,p,s),c_0)$ which filled up the gap in the existing literature.

II. Matrix transformation on the space $(r^q(\mathcal{M},\Delta_n^m,u,p,s)$

In this section, we characterize the matrix mappings from the space $(r^q(\mathcal{M},\Delta_n^m,u,p,s)$ to the space $\ell_\infty,c,\ c_0$.

Theorem 2.1 Let $\mathcal{M}=(M_i)$ be Musielak-Orlicz function, $u=(u_i)$ be a sequence of strictly positive real numbers and $p=(p_k)$ be a bounded sequence of positive real numbers.

(i) Let $0\leq p_k\leq H<\infty$ for $k\in\mathbb{N}$. Then $A\in(r^q(\mathcal{M},\Delta_n^m,u,p,s),\ell_\infty)$ if and only if there exists an integer $B>1$ such that

$$C(B,S)=\sup_n\sum_k\left|\left[\left(\frac{\alpha_{nk}}{M_k(u_kq_k)}+\left(\frac{1}{M_k(u_kq_k)}-\frac{1}{M_{k+1}(u_{k+1}q_{k+1})}\right)\sum_{i=k+1}^{n}\alpha_{ni}\right)Q_k^{s+1}\right]B^{-1}\right|^{p_k'}<\infty$$

And $\{\alpha_{nk}\}_{k\in\mathbb{N}}\in cs. for\ each\ n\in\mathbb{N}$ (2.1)

(ii) Let $0\leq p_k\leq 1$, for every $k\in\mathbb{N}$. Then $A\in(r^q(\mathcal{M},\Delta_n^m,u,p,s),\ell_\infty)$ if and only if

$$\sup_{n,k}\left|\left[\left(\frac{\alpha_{nk}}{M_k(u_kq_k)}+\left(\frac{1}{M_k(u_kq_k)}-\frac{1}{M_{k+1}(u_{k+1}q_{k+1})}\right)\sum_{i=k+1}^{n}\alpha_{ni}\right)Q_k^{s+1}\right]\right|^{p_k}<\infty$$

And $\{\alpha_{nk}\}_{k\in\mathbb{N}}\in cs. for\ each\ n\in\mathbb{N}$
(2.2)

Proof. We shall prove only (i) and the proof of (ii) will follow on applying similar argument.

Let $A\in(r^q(\mathcal{M},\Delta_n^m,u,p,s),\ell_\infty)$ and $0\leq p_k\leq D<\infty$, for every $k\in\mathbb{N}$. Then Ax exists for $x\in(r^q(\mathcal{M},\Delta_n^m,u,p,s)$ and implies that $\{\alpha_{nk}\}_{k\in\mathbb{N}}\in(r^q(\mathcal{M},\Delta_n^m,u,p,s)$ for each $n\in\mathbb{N}$. Hence the necessity of (2.1) holds. Conversely, suppose that (2.1) holds and $x\in(r^q(\mathcal{M},\Delta_n^m,u,p,s)$ since $\{\alpha_{nk}\}_{k\in\mathbb{N}}\in\{(r^q(\mathcal{M},\Delta_n^m,u,p,s)\}^\beta$ for every fixed $n\in\mathbb{N}$, so the A-transform of x exists.

Consider the following equality obtained by using $\{(r^q(\mathcal{M},\Delta_n^m,u,p,s)\}^\beta$ that

$$\sum_{k=0}^{t}\alpha_{nk}x_k=\sum_{k=0}^{t}\left[\left(\frac{\alpha_{nk}}{M_k(u_kq_k)}+\left(\frac{1}{M_k(u_kq_k)}-\frac{1}{M_{k+1}(u_{k+1}q_{k+1})}\right)\sum_{i=k+1}^{t}\alpha_{ni}\right)Q_k^{s+1}\right]y_k\qquad(2.3)$$

Taking into account the assumption, we derive from (3.3) as $t\to\infty$ that

$$\sum_{k=0}^{\infty}\alpha_{nk}x_k=\sum_{k=0}^{\infty}\left[\left(\frac{\alpha_{nk}}{M_k(u_kq_k)}+\left(\frac{1}{M_k(u_kq_k)}-\frac{1}{M_{k+1}(u_{k+1}q_{k+1})}\right)\sum_{i=k+1}^{\infty}\alpha_{ni}\right)Q_k^{s+1}\right]y_k\qquad(2.4)$$

Now by combining (2.4) and the inequality which hold for any $B>0$ and any complex numbers a,b $|ab|\leq B\left(|aB^{-1}|^{p'}+|b|^p\right)\ with\ p^{-1}+\{p'\}^{-1}=1$ (see [2]) (2.5)

We can see that

$$\sup_{n\in\mathbb{N}}\left|\sum_k\alpha_{nk}x_k\right|\leq\sup_{n\in\mathbb{N}}\sum_k\left|\left[\left(\frac{\alpha_{nk}}{M_k(u_kq_k)}+\left(\frac{1}{M_k(u_kq_k)}-\frac{1}{M_{k+1}(u_{k+1}q_{k+1})}\right)\sum_{i=k+1}^{\infty}\alpha_{ni}\right)Q_k^{s+1}\right]\right||y_k|$$

$$\leq B[c(B,S)+h_1^B(y)]$$
$$<\infty$$

This shows that $Ax\in\ell_\infty\ whenever\ x\in(r^q(\mathcal{M},\Delta_n^m,u,p,s)$ Hence the proof.

Theorem 2.2 Let $\mathcal{M}=(M_i)$ be Musielak-Orlicz function, $u=(u_i)$ be a sequence of strictly positive real numbers and $p=(p_k)$ be a bounded sequence of positive real numbers.

(i) Let $0\leq p_k\leq D<\infty$ for $k\in\mathbb{N}$. Then $A\in(r^q(\mathcal{M},\Delta_n^m,u,p,s),c)$ if and only if there exists an integer $B>1$ such that

$\alpha_{nk}\to\beta_n\ \ (n\to\infty,k\ fixed)$ (2.6)

(ii) $C(B,S)=\sup_n\sum_k\left|\left[\left(\frac{\alpha_{nk}}{M_k(u_kq_k)}+\left(\frac{1}{M_k(u_kq_k)}-\frac{1}{M_{k+1}(u_{k+1}q_{k+1})}\right)\sum_{i=k+1}^{n}\alpha_{ni}\right)Q_k^{s+1}\right]B^{-1}\right|^{p_k'}<\infty$

And $\{\alpha_{nk}\}_{k\in\mathbb{N}}\in cs. for\ each\ n\in\mathbb{N}$ (2.7)

Proof. Necessity, suppose $x\in(r^q(\mathcal{M},\Delta_n^m,u,p,s),c)$. Then $A_n(x)$ exists for each $n\geq 1\ and\ \lim_{n\to\infty}A_n(x)$ exists for every $x\in(r^q(\mathcal{M},\Delta_n^m,u,p,s)$. Therefore by an argument similar to that in theorem (2.1). We have condition (2.7), condition (2.6) is obtained by taking $x = e^k \in (r^q(\mathcal{M}, \Delta_n^m, u, p, s)$ where e^k is a sequence with 1 at the k^{th}place and zero elsewhere.

Sufficiency. The conditions of the theorem imply that

$$C(B,S) = \sup_n \sum_k^\infty \left|\left[\left(\frac{\alpha_{nk}}{M_k(u_k q_k)} + \left(\frac{1}{M_k(u_k q_k)} - \frac{1}{M_{k+1}(u_{k+1}q_{k+1})}\right)\sum_{i=k+1}^n \alpha_{ni}\right) Q_k^{s+1}\right] B^{-1}\right|^{p_k'} \leq c(B,S) < \infty$$

(2.8)

By the above condition (2.8), it is clear that $\sum_k \alpha_{nk} x_k$ is absolutely convergent for each $x \in (r^q(\mathcal{M}, \Delta_n^m, u, p, s)$.

For each

$x \in (r^q(\mathcal{M}, \Delta_n^m, u, p, s)$ and $\epsilon > 0$, we can choose an integer $m_0 > 1$, such that

$$g_{m_0}(x) = \sum_{k=m_0} \left|\frac{1}{Q_k^{s+1}} \sum_{j=0}^{k-1} \left(M_j(u_j q_j) - M_{j+1}(u_{j+1}q_{j+1})\right)x_j + \frac{M_k(u_k q_k)x_k}{Q_k^{s+1}}\right|^{p_k} < \epsilon^M$$

Then by equality (2.5) we have

$$\sum_{k=m_0}^\infty |\alpha_{nk} - \alpha_k||x_k|$$

$$\leq B\left(\sum_{k=m_0}^\infty \left|\left(\frac{\alpha_{nk}}{M_k(u_k q_k)} + \left(\frac{1}{M_k(u_k q_k)} - \frac{1}{M_{k+1}(u_{k+1}q_{k+1})}\right)\sum_{i=k+1}^n a_{ni}\right) Q_k^{s+1} B^{-1}\right|^{p_k'} + 1\right)\left(g_{m_0}(x)\right)^{1/M}$$

$$< B(2(c(B,S) + 1)\epsilon$$

And

$$\left(\sum_{k=m_0}^\infty \left|\left(\frac{\alpha_{nk}}{M_k(u_k q_k)} + \left(\frac{1}{M_k(u_k q_k)} - \frac{1}{M_{k+1}(u_{k+1}q_{k+1})}\right)\sum_{i=k+1}^n a_{ni}\right) Q_k^{s+1} B^{-1}\right|^{p_k'}\right) \leq 2c(B,S) < \infty$$

It follows immediately that

$$\lim_{n\to\infty} \sum_k \alpha_{nk} x_k = \sum_k \alpha_k x_k$$

This shows that $A \in (r^q(\mathcal{M}, \Delta_n^m, u, p, s), c)$ which proved the theorem.

Corollary 2.3 Let $1 < p_k \leq H < \infty$. Then $A \in (r^q(\mathcal{M}, \Delta_n^m, u, p, s), c_0)$ if and only if

(i) Condition of theorem 2.2 holds

(ii) $a_{nk} \to 0, (n \to \infty, k\ fixed)$

i.e placing $\beta = 0$ in the theorem 2.2 (i) i.e (2.6)

III. Conclusion

The results obtained filled the gap in the existing literature and give room for further generalization of our characterization as well as extension to double sequence space.

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