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The M-step iterative procedure for finite family of uniformly continuous asymptotically Nonexpansive ln the intermediate sense Maps.

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Abstract: *Let K be a closed convex nonempty subset of a uniformly convex Banach space E and let* $\{T_i\}_{i=1}^{m}$ *be a finite family of self maps on K such that* T_1 *is uniformly continuous asymptotically Nonexpansive ln the intermediate sense Maps. with* $F=\cap_{i=1}^{m}F(T_i)\neq\emptyset$*, an m-step iteration process was used and sufficient conditions for the strong convergence of the process to a common fixed point of the family are proved.*
MSC(2010):- 47H10, 47J25.
Key words/phrases:-*uniformly continuous; asymptotically Nonexpansive ln the intermediate sense Maps; finite family, common fixed point; M-step iteration process; strong convergence.*

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I. Introduction

The Mann iteration scheme [**Error! Reference source not found.**], introduced in 1953, was used to prove the convergence of the sequence to the fixed points of mappings of which the Banach principle is not applicable. In 1974, Ishikawa [**Error! Reference source not found.**] devised a new iteration scheme and established it's convergence to a fixed point of Lipschitzian pseudocontractive map when Mann iteration process failed to converge. Noor [**Error! Reference source not found.**] introduced the three-step iteration process for solving nonlinear operator equations in real Banach spaces as follows.
Let E be a real Banach space, K a nonempty convex subset of E and $T:K\to K$, a mapping. For an arbitrary $x_o\in K$, the sequence $\{x_n\}\subset K$ defined by

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n+\alpha_n Ty_n, \\ y_n &= (1-\beta_n)x_n+\beta_n Tz_n \\ z_n &= (1-\gamma_n)x_n+\gamma_n Tx_n \end{aligned} \tag{1}$$

Where $\{\alpha_n\},\{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in [0,1] is called the three-step iteration (or the Noor iteration). When $\gamma_n=0$, then the three-step iteration reduces to the Ishikawa iterative sequence. If $\beta_n=\gamma_n=0$, then the three-step iteration reduces to the Mann iteration.

Glowinski and Le Tallec (1989) used the three-step iterative schemes to solve elastoviscoplasticity, liquid crystal and eigen-value problems. They have shown that the three-step approximation scheme performs better than the two-step and one-step iterative methods. Haubruge et al (1998) have studied the convergence analysis of three-step iterative schemes and applied these three-step iteration to obtain new splitting type algorithms for solving variational inequalities, separable convex programming and minimization of a sum of convex functions. They also proved that three-step iterations also lead to highly parallelized algorithms under certain conditions. Thus, it is clear that three-step schemes play an important part in solving various problems, which arise in pure and applied sciences.
In 2013, Chika Moore et al [**Error! Reference source not found.**] introduced the M-step process and proved strong convergence of the process to the common fixed point of finite family of hemicontractive maps. Let K be

a nonempty convex subset of a normed linear space E and let $T:K \longrightarrow K$ be a map. For any given $x_o \in K$, The m-step iterative process is defined by

$$\begin{aligned} y_{n,0} &= x_n \\ y_{n,i} &= (1-\alpha_n)x_n + \alpha_n T y_{n,i-1}; \quad i=1,\dots,m \qquad (2) \\ y_{n,m} &= x_{n+1} = y_{n+1,0} \quad \text{where} \quad n+1 \equiv im; \quad n \geq 0 \end{aligned}$$

For a finite family $\{T_i\}_{i=1}^{m}$ of m-maps,the m-step iterative process becomes

$$\begin{aligned} y_{n,0} &= x_n \\ y_{n,i} &= (1-\alpha_n)x_n + \alpha_n T_{m+1-i} y_{n,i-1}; \quad i=1,\dots,m \\ &(3) \\ y_{n,m} &= x_{n+1} = y_{n+1,0} \\ \text{where} \quad & n+1 \equiv im \quad \left(\text{or } i(n) = Res\left[\frac{n+1}{m}\right] = m\left(\frac{n+1}{m}\right) - \left[\frac{n+1}{m}\right]\right), n \geq 0 \end{aligned}$$

In the case where at least one of the maps in the finite family has some asymptotic behaviour (satisfies an asymptotic condition) then the iterative process becomes:

$$\begin{aligned} y_{n,0} &= x_n \\ y_{n,i} &= (1-\alpha_n)x_n + \alpha_n T^r_{m+1-i} y_{n,i-1}; \quad i=1,\dots,m \\ y_{n,m} &= x_{n+1} = y_{n+1,0}; \end{aligned} \qquad (4)$$

with n and m as in equation(3) and $r=1+\left[\frac{n}{m}\right]$. Our purpose in this paper is to prove strong convergence of the *m*-step iteration process to a common fixed point of a family of uniformly continuous asymptotically nonexpansive in the intermediate sense Maps when such point are known to exists.

The following definitions is vital in this work. Let E be a normed linear space and let $J:E \to 2^{E^*}$ be the normalized duality mapping defined by

$$J(x) = \{f \in E^* : \langle x,f \rangle = |x|^2 = |f|^2\}; \ \forall \ x \in E$$

where E^* denotes the dual space of E and $\langle .,. \rangle$ denotes the generalised duality pairing between E and E^*. The single-valued normalised duality mapping is denoted by j. Let K be a nonempty closed convex subset of E and $T:K \to K$ be a map. The mapping T is said to be *uniformly L-Lipschitzian* if there exists a constant $L>0$ such that for any $x,y \in K$ and $\forall n \geq 1$

$$|T^n x - T^n y| \leq L|x-y|$$

Let K be a nonempty subset of a real Hilbert space H. A mapping $T:K \longrightarrow K$ is called *nonexpansive* if and only if for all $x,y \in K$, we have that

$$|Tx - Ty| \leq |x-y|. \qquad (5)$$

The mapping T is called *asymptotically nonexpansive mapping* if and only if there exists a sequence $\{\mu_n\}_{n \geq 1} \subset [0,+\infty)$, with $\lim_{n \to \infty} \mu_n = 0$ such that for all $x.y \in K$,

$$|T^n x - T^n y| \leq (1+\mu_n)|x-y| \qquad \forall \ n \in N \qquad (6)$$

T is called *asymptotically nonexpansive in the intermidiate sense* (ANIS) if and only if there exist two sequences $\{\mu_n\}_{n \geq 1}, \{\eta_n\}_{n \geq 1} \subset [0,+\infty)$, with $\lim_{n \to \infty} \mu_n = 0 = \lim_{n \to \infty} \eta_n$ such that for all $x,y \in K$,

$$|T^n x - T^n y| \le (1+\mu_n)|x-y| + \eta_n \qquad n \ge 1 \qquad (7)$$

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [] as a generalisation of nonexpansive mappings. As further generalisation of class of nonexpansive mappings, Alber, Chidume and Zegeye [**Error! Reference source not found.**] introduced the class of total asymptotically nonexpansive mappings, where a mapping $T:K \longrightarrow K$ is called *total asymptotically nonexpansive* (TAN) if and only if there exist two sequences $\{\mu_n\}_{n\ge1}, \{\eta_n\}_{n\ge1} \subset [0,+\infty)$, with $\lim_{n\to\infty} \mu_n = 0 = \lim_{n\to\infty} \eta_n$ and nondecreasing continous function $\phi:[0,+\infty) \longrightarrow [0,+\infty)$ with $\phi(0)=0$ such that for all $x,y \in K$,

$$|T^n x - T^n y| \le |x-y| + \mu_n \phi(|x-y|) + \eta_n \qquad n \ge 1 \qquad (8)$$

Ofoedu and Nnubia [**Error! Reference source not found.**] gave an example to show that the class of asymptotically nonexpansive mappings is properly contained in the class of total asymptotically nonexpansive mappings. The class of asymptotically nonexpansive type mappings includes the class of mappings which are asymptotically nonexpansive in the intermediate sense. These classes of mappings had been studied extensively by several authors (see e.g.[], [**Error! Reference source not found.**], [**Error! Reference source not found.**]). A map T is said to satisfies *condition B* if there exists $f:[0,\infty)\to[0,\infty)$ strictly increasing, continuous, $f(0)=0, f(r)>0 \ \forall \ r>0$ such that for all $x \in D(T), |x-Tx| \ge f(d(x,F))$ where $F=F(T)=\{x \in D(T): x=Tx\}$ and $d(x,F)=\inf\{|x-y|: y \in F\}$.

II. Preliminaries

We shall make use of the following lemmas. We need the following lemma in this work.

Lemma 2.1 *[**Error! Reference source not found.**] Let $\{\mu_n\},\{\beta_n\},\{\gamma_n\}$ be sequences of nonnegative numbers satisfying the conditons: $\sum_{n\ge0}^{\infty} \beta_n = \infty$, $\beta_n \longrightarrow 0$ as $n \longrightarrow \infty$ and $\gamma_n = 0(\beta_n)$. Suppose that*

$$\mu_{n+1}^2 \le \mu_n^2 - \beta_n \psi(\mu_{n+1}) + \gamma_n; \quad n=1,2,...$$

where $\psi:[0,1)\to[0,1)$ is a strictly increasing function with $\psi(0)=0$. Then $\mu_n \to 0$ as $n \to \infty$

3 Main Result

Proposition 3.1 *Let H be a real Hilbert space, let K be a nonempty closed convex subset of H and let $T_i:K \longrightarrow K$ where $i \in I=\{1,2,...,m\}$. be m uniformly continuous asymptotically nonexpansive in the intermediate sense mapping from K into itself, then there exist sequences $\{\mu_n\}_{n\ge1},\{\eta_n\}_{n\ge1} \subset [0,+\infty)$, with $\lim_{n\to\infty} \mu_n = 0 = \lim_{n\to\infty} \eta_n$ such that $\forall x,y \in K$,*

$$|T_i^n x - T_i^n y| \le (1+\mu_n)|x-y| + \eta_n \qquad n \ge 1, \forall i \in I \qquad (9)$$

Proof Since $T_i:K \longrightarrow K$ where $i \in I$ are asymptotically nonexpansive in the intermediate sense maps, there exist sequences $\{\mu_{in}\}_{n\ge1},\{\eta_{in}\}_{n\ge1} \subset [0,+\infty)$ such that $\lim_{n\to\infty} \mu_{in} = 0 = \lim_{n\to\infty} \eta_{in}$ and $\forall \ x,y \in K$

$$|T_i^n x - T_i^n y| \le (1+\mu_{in})|x-y| + \eta_{in} \qquad n \ge 1, \forall \ i \in I. \qquad (10)$$

Now, setting $\mu_n := \max_{i\in I}\{\mu_{in}\}, \eta_n := \max_{i \in I}\{\eta_{in}\}$. we have that $\{\mu_n\}_{n\ge1},\{\eta_n\}_{n\ge1} \subset [0,+\infty)$, $\lim_{n\to\infty} \mu_n = 0 = \lim_{n\to\infty} \eta_n$ and $\forall x,y \in K$,

$$|T_i^n x - T_i^n y| \le (1+\mu_n)|x-y| + \eta_n \qquad n \ge 1, \forall \ i \in I \qquad (11)$$

Thus, completing the proof.

Theorem 3.1 *Let H be a normed linear space, let K be a closed convex nonempty subset of H and let $T_i : K \longrightarrow K$ where $i \in I = \{1,2,...,m\}$. be m uniformly continuous asymptotically nonexpansive in the intermediate sense mappings from K into itself with sequences $\{\mu_{in}\}_{n\geq 1}, \{\eta_{in}\}_{n\geq 1} \subset [0,+\infty)$ such that $\lim_{n\to\infty} \mu_{in} = 0 = \lim_{n\to\infty} \eta_{in}$ and Let $\mu_n = \max_{i\in I}\{\mu_{in}\}$ and $\eta_n = \max_{i\in I}\{\eta_{in}\}$. Suppose that $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ and let $\{x_n\}_{n\geq 1}$ be a sequence generated iteratively by (4, where $\{\alpha_n\}_{n\geq 1}$ is a sequence in (0,1) satisfying the following conditions:*

$\sum_{n=1}^{\infty} \alpha_n < \infty,\ 0 < \zeta < \alpha_n < \varepsilon < 1\ \forall n \geq 1$. *Let $x^* \in F$ then $\lim_{n\to\infty} |x_n - x^*|$ exists for $x^* \in F$*

Proof; Let $x^* \in F$.

Now, from (4) we have that

$$
\begin{aligned}
|y_{n,j} - x^*| &\leq (1-\alpha_n)|x_n - x^*| + \alpha_n |T_j^n y_{n,j-1} - x^*| \\
&\leq (1-\alpha_n)|x_n - x^*| + \alpha_n(\mu_{n,j}|y_{n,j-1} - x^*| + \eta_{n,j}) \\
&\leq (1-\alpha_n)|x_n - x^*| + \alpha_n \mu_{n,j}|y_{n,j-1} - x^*| + \alpha_n \eta_{n,j}
\end{aligned}
$$

So,

$$
|y_{n,1} - x^*| \leq (1+\alpha_n(\mu_{n,1}-1))|x_n - x^*| + \alpha_n \eta_{n,1} \tag{12}
$$

$$
\begin{aligned}
|y_{n,2} - x^*| &\leq (1-\alpha_n)|x_n - x^*| + \alpha_n \mu_{n,2}[(1+\alpha_n(\mu_{n,1}-1))|x_n - x^*| + \alpha_n \eta_{n,1}] \\
&\leq +\alpha_n \eta_{n,2}[1+\alpha_n(\mu_{n,2}-1) + \alpha_n^2 \mu_{n,2}(\mu_{n,1}-1)]|x_n - x^*| \\
&\quad + \alpha_n^2 \mu_{n,2}\eta_{n,1} + \alpha_n \eta_{n,2}
\end{aligned} \tag{13}
$$

$$
\begin{aligned}
|y_{n,3} - x^*| &\leq [1+\alpha_n(\mu_{n,3}-1) + \alpha_n^2 \mu_{n,3}(\mu_{n,2}-1) + \alpha_n^3 \mu_{n,3}(\mu_{n,1}-1)]|x_n - x^*| \\
&\quad + \alpha_n \eta_{n,3} + \alpha_n^2 \mu_{n,3}\eta_{n,2} + \alpha_n^3 \mu_{n,3}\mu_{n,2}\eta_{n,1}
\end{aligned} \tag{14}
$$

Hence

$$
\begin{aligned}
|y_{n,j} - x^*| &\leq (1 + \sum_{t=0}^{j-1} \alpha_n^{t+1} \Pi_{s=0}^{t-1} \mu_{n,j-s}(\mu_{n,j-t}-1))|x_n - x^*| \\
&\quad + \sum_{t=0}^{j-1} \alpha_n^{t+1} \Pi_{s=0}^{t-1} \mu_{n,j-s}\eta_{n,j-t}
\end{aligned} \tag{15}
$$

$$
|x_{n+1} - x^*| \leq (1 + \sum_{t=0}^{m-1} \alpha_n^{t+1} \Pi_{s=0}^{t-1} \mu_{n,m-s}(\mu_{n,m-t}-1))|x_n - x^*|
$$

$$+\sum_{t=0}^{m-1}\alpha_n^{t+1}\Pi_{s=0}^{t-1}\mu_{n,m-s}\eta_{n,m-t}$$

$$\leq (1+2\,\tfrac{m}{2}b\sum_{j=1}^{m}(\mu_{n,j}-1))|x_n-x^*|+2\,\tfrac{m}{2}b\sum_{j=1}^{m}\eta_{n,j} \tag{16}$$

Thus by our hypothesis and lemma **Error! Reference source not found.**, we have that $\lim_{n\to\infty}|x_n-x^*|$ exist, this completes the proof.

Theorem 3.2 *Let E be a uniformly convex Banach space and K a nonempty closed convex subset of E , let* $\{T_i\}_{i=1}^{m}$ *be a finite family of uniformly continuous asymptotically nonexpansive ln the intermediate sense maps form K to itself with sequences* $\{\mu_{in}\}_{n\geq 1},\{\eta_{in}\}_{n\geq 1}\subset[0,+\infty)$ *such that* $\lim_{n\to\infty}\mu_{in}=0=\lim_{n\to\infty}\eta_{in}$ *and Let* $\mu_n=\max_{i\in I}\{\mu_{in}\}$ *and* $\eta_n=\max_{i\in I}\{\eta_{in}\}$ *such that* $\lim_{n\to\infty}\mu_n=1,\ \lim_{n\to\infty}\eta_n=0,\ \sum_{n=0}^{\infty}(\mu_n-1)<\infty\ \ \sum_{n=0}^{\infty}\eta_n<\infty$

Suppose that $F=\bigcap_{i=1}^{N}F(T_i)$ *is not empty and let* $\{x_n\}_{n\geq 1}$ *be a sequence generated iteratively by (4) where* $\{\alpha_n\}_{n\geq 1}$ *is sequences in (0,1) satisfying the following conditions:*

$\sum_{n=1}^{\infty}\alpha_n<\infty,\ 0<\zeta<\alpha_n<\varepsilon<1\ \ \forall n\geq 1$, *then*

$\forall j\in\{1,2,\ldots,m\},\ \lim_{n\to\infty}|x_n-T_jx_n|=0$

Proof. Let $x^*\in F$

$$\begin{aligned}|y_{n,j}-x^*|^2 &\leq (1-\alpha_n)|x_n-x^*|^2+\alpha_n|T_j^n y_{n,j-1}-x^*|-\alpha_n(1-\alpha_n)g(|x_n-T_j^n y_{n,j-1}|)\\ &\leq (1-\alpha_n)|x_n-x^*|^2+\alpha_n(\mu_{n,j}|y_{n,j-1}-x^*|+\eta_{n,j})^2\\ &\quad -\alpha_n(1-\alpha_n)g(|x_n-T_j^n y_{n,j-1}|)\\ &\leq (1-\alpha_n)|x_n-x^*|^2+\alpha_n\mu_{n,j}^2|y_{n,j-1}-x^*|^2\\ &\quad +\alpha_n(2\mu_{n,j}|y_{n,j-1}-x^*|+\eta_{n,j})\eta_{n,j}-(1-\alpha_n)g(|x_n-T_j^n y_{n,j-1}|)\end{aligned} \tag{17}$$

So

$$\begin{aligned}|y_{n,1}-x^*|^2 &\leq (1+\alpha_n(\mu_{n,1}^2-1))|x_n-x^*|^2+\alpha_n(2\mu_{n,1}|x_n-x^*|+\eta_{n,1})\eta_{n,1}\\ &\quad -\alpha_n(1-\alpha_n)g(|x_n-T_1^n x_n|)\end{aligned} \tag{18}$$

$$\begin{aligned}|y_{n,2}-x^*|^2 &\leq (1+\alpha_n(\mu_{n,2}^2-1)+\alpha_n^2\mu_{n,2}^2(\mu_{n,1}^2-1))|x_n-x^*|^2\\ &+\alpha_n(2\mu_{n,2}|y_{n,1}-x^*|+\eta_{n,2})\eta_{n,2}+\alpha_n^2\mu_{n,2}^2(2\mu_{n,1}|x_n-x^*|+\eta_{n,1})\eta_{n,1}\\ &\quad -\alpha_n(1-\alpha_n)g(|x_n-T_2^n x_n|)-\alpha_n^2\mu_{n,2}^2(1-\alpha_n)g(|x_n-T_1^n x_n|)\end{aligned} \tag{19}$$

So,

$$|y_{n,j}-x^*|^2 \leq (1+\sum_{t=0}^{j-1}\alpha_n^{t+1}\Pi_{s=0}^{t-1}\mu_{n,j-s}^2(\mu_{n,j-t}^2-1))|x_n-x^*| + \sum_{t=0}^{j-1}\alpha_n^{t+1}(2\mu_{n,j-s}|y_{n,j-t-1}-x^*|+\eta_{n,j-t})\eta_{n,j-t}\Pi_{s=0}^{t-1}\mu_{n,j-s}^2 -\alpha_n(1-\alpha_n)\sum_{t=0}^{j-1}\alpha_n^t g(|x_n-T_{j-t}^n y_{n,j-t-1}|)\Pi_{s=0}^{t-1}\mu_{n,j-s}^2 \quad (20)$$

Hence,

$$|x_{n+1}-x^*|^2 \leq (1+\sum_{t=0}^{m-1}\alpha_n^{t+1}\Pi_{s=0}^{t-1}\mu_{n,m-s}^2(\mu_{n,m-t}^2-1))|x_n-x^*| + \sum_{t=0}^{m-1}\alpha_n^{t+1}\Pi_{s=0}^{t-1}\mu_{n,m-s}^2(2\mu_{n,m-s}|y_{n,m-t-1}-x^*|+\eta_{n,m-t})\eta_{n,m-t} -\alpha_n(1-\alpha_n)\sum_{t=0}^{m-1}\alpha_n^t\Pi_{s=0}^{t-1}\mu_{n,m-s}^2 g(|x_n-T_{m-t}^n y_{n,m-t-1}|)$$

$$\leq (1+2^m b\sum_{j=1}^{m}(\mu_{n,j}^2-1))|x_n-x^*|+2^m b\sum_{j=1}^{m}(2\mu_{n,j}|y_{n,j-1}-x^*|+\eta_{n,j})\eta_{n,j}| -a^{m+1}(1-b)\sum_{j=1}^{m}g(|x_n-T_j^n y_{n,j-1}|) \quad (21)$$

So, $\lim_{n\to\infty} g(|x_n-T_j^n y_{n,j-1}|)=0$, thus $\lim_{n\to\infty}|x_n-T_j^n y_{n,j-1}|=0\ \forall\ j=1,\ldots,m$.

$$\begin{aligned} |x_n-T_j^n x_n| &\leq |x_n-T_j^n y_{n,j-1}|+|T_j^n y_{n,j-1}-T_j^n x_n| \\ &\leq |x_n-T_j^n y_{n,j-1}|+\mu_{n,j}|y_{n,j-1}-x_n|+\eta_{n,j} \\ &= |x_n-T_j^n y_{n,j-1}|+\mu_{n,j}\alpha_n|x_n-T_{j-1}^n y_{n,j-2}|+\eta_{n,j} \end{aligned} \quad (22)$$

So, $\lim_{n\to\infty}|x_n-T_j^n x_n|=0$

$|x_{n+1}-x_n|=\alpha_n|x_n-T_m^n y_{n,m-1}|$ so $\lim_{n\to\infty}|x_{n+1}-x_n|=0$

$|x_n-T_j x_n|\leq|x_n-T_j^n x_n|+|T_j^n x_n-T_j x_n|$

$$\begin{aligned} |x_n-T_j^{n-1}x_n| &\leq |x_n-x_{n-1}|+|x_{n-1}-T_j^{n-1}x_{n-1}|+|T_j^{n-1}x_{n-1}-T_j^{n-1}x_n| \\ &\leq (1+\mu_{n-1,j})|x_n-x_{n-1}|+|x_{n-1}-T_j^{n-1}x_{n-1}|+\eta_{n-1,j} \end{aligned} \quad (23)$$

hence, $\lim_{n\to\infty}\|x_n - T_j^{n-1}x_n\|=0$ so that by uniform continuity of $T_j \forall j$ $\lim_{n\to\infty}\|T_j^n x_n - T_j x_n\|=0$ and hence $\lim_{n\to\infty}\|x_n - T_j x_n\|=0 \;\forall\, j=1,\ldots,m.$ Thus, completes the proof.

Theorem 3.3 *Let $E,K,T,F(T),\{x_n\}$ be as in Theorem Then, $\{x_n\}$ converges strongly to a fixed point of T if and only if $\liminf_{n\to\infty} d(x_n,F)=0$.*

(where F= F(T)) **Proof** Let $\tau_n=\sqrt{2}^m b \sum_{j=1}^{m}(\mu_{n,j}-1), \nu_n=\sqrt{2}^m b \sum_{j=1}^{m}\eta_{n,j}$

Hence,

$$\|x_{n+1}-x^*\|\le(1+\tau_n)\|x_n-x^*\|+\nu_n \tag{24}$$

So, $d(x_{n+1},F)\le(1+\tau_n)d(x_n,F)+\nu_n$,

hence, by $\lim_{n\to\infty} d(x_n,F)$ exists, thus, $\liminf_{n\to\infty} d(x_n,F)=0$ implies $\lim_{n\to\infty} d(x_n,F)=0$.

Now,

$$\begin{aligned}\|x_{n+k+1}-x^*\| &\le \Pi_{j=0}^{k}(1+\tau_{n+j})\|x_n-x^*\|+\sum_{j=0}^{k}\nu_{n+k-j}\Pi_{r=0}^{j-1}(1+\tau_{n+k-r})\\ &\le \Pi_{j=0}^{k}(1+\tau_{n+j})(\|x_n-x^*\|+\sum_{j=0}^{k}\nu_{n+j})\\ &\le Q(\|x_n-x^*\|+\sum_{j=0}^{k}\nu_{n+j})\end{aligned}$$

So that given any $\varepsilon>0\ \exists n_0>0$, integer, such that $\forall n\ge n_0$, $d(x_n,F)<\frac{\varepsilon}{4(Q+1)}$ and $\nu_{n_0+j}<\frac{\varepsilon}{4m(Q+1)}\forall\, j=1,2,\ldots,m.$. So $\exists\, x^*\in F$ such that $d(x_{n_0},x^*)<\frac{\varepsilon}{4(Q+1)}$ thus, $\|x_{n_0}-x^*\|<\frac{\varepsilon}{4(Q+1)}$

$$\begin{aligned}\|x_{n+k}-x_n\|\le\|x_{n+k}-x^*\|+\|x_n-x^*\| &\le 2Q(\|x_{n_0}-x^*\|+\sum_{j=0}^{k}\nu_{n_0+j})\\ &< 2Q(\frac{\varepsilon}{4(Q+1)}+m\frac{\varepsilon}{4m(Q+1)})\\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon\end{aligned}$$

So, $\{x_n\}$ is Cauchy and so converges to some $u^*\in K$. But, $x_n-T_i x_n\to 0$ as $n\to\infty\ \forall\ i$ and T_i is continous $\forall\ i$. Hence, $\lim_{n\to\infty}(x_n-T_i x_n)=\lim_{n\to\infty}x_n-T_i(\lim_{n\to\infty}x_n)=u^*-T_i u^*$, so that $u^*\in F$. i.e $u^*=x^*\in F$. Hence, $x_n\to x^*$ as $n\to\infty$. This completes the proof.

Theorem 3.4 *Let $E,K,T,F(T),\{\alpha_n\},\{x_n\}$ be as in Theorem , then $\{x_n\}$ converges strongly to some point of F if T_j for some $j\in\{1,2,\ldots,m\}$ satisfy condition B.*

Proof Let T_{j_0} satisfy condition B. $\exists f:[0,+\infty)$ \o $(\longrightarrow)$ $[0,+\infty)$ such that $f(0)=0$ and $f(r)>0\forall\ r>0$ such that

$$f(d(x_n,F))\leq|x_n-T_{j_0}x_n|$$

Hence

$$\lim_{n\to\infty} f(d(x_n,F))\leq \lim_{n\to\infty}|x_n-T_{j_0}x_n|=0$$

and hence,

$$\lim_{n\to\infty} d(x_n,F)=0$$

Thus $\{x_n\}$ converges strongly to some point $x^*\in F$.

Remark 3.1 *Observe that if T is total asmptotically nonexpansive mapping and we assume that* $\exists\ c>0,k>0$ *constants such that* $\phi(t)\leq ct\ \forall \geq k$*, T collapse to asymptotically nonexpasive in the itermediate sense.*

Hence we have the following Corrollaries;

Corollary 3.1 *Let E be a normed linear space and K be a nonempty closed convex subset of E. Let* $T_i:K\longrightarrow K$ *where* $i\in I=\{1,2,...,m\}$*. be m uniformly continuous total asymptotically nonexpansive mapping from K into itself with sequences* $\{\mu_{in}\}_{n\geq 1},\{\eta_{in}\}_{n\geq 1}\subset[0,+\infty)$ *such that* $\lim_{n\to\infty}\mu_{in}=0=\lim_{n\to\infty}\eta_{in}$ *and with function* $\phi_i:[0,+\infty)\longrightarrow[0,+\infty)$ *satisfying* $\phi_i(t)\leq M_0 t\ \ \forall\ t>M_1$ *for some constants* $M_0,M_1>0$*, Let* $\mu_n=\max_{i\in I}\{\mu_{in}\}$ *and* $\eta_n=\max_{i\in I}\{\eta_{in}\}$ *and,* $\phi(t)=\max_{i\in I}\{\phi_i(t)\}\ \forall\, t\in[0,\infty)$ *Suppose that* $F=\cap_{i=1}^{N}F(T_i)\neq\emptyset$ *and let* $\{x_n\}_{n\geq 1}$ *be a sequence generated iteratively by (4) where* $\{\alpha_n\}_{n\geq 1}$ *is sequences in (0,1) satisfying the following conditions:*

$\sum_{n=1}^{\infty}\alpha_n<\infty,\ 0<\zeta<\alpha_n<\varepsilon<1\ \ \forall\, n\geq 1$ *. Let* $x^*\in F$ *then* $\lim_{n\to\infty}|x_n-x^*|$ *exists.*

Corollary 3.2 *Let E be a uniformly convex Banach space and K a nonempty closed convex subset of E. Let* $T_i:K\longrightarrow K$ *where* $i\in I=\{1,2,...,m\}$*. be m uniformly continuous total asymptotically nonexpansive mapping from K into itself with sequences* $\{\mu_{in}\}_{n\geq 1},\{\eta_{in}\}_{n\geq 1}\subset[0,+\infty)$ *such that* $\lim_{n\to\infty}\mu_{in}=0=\lim_{n\to\infty}\eta_{in}$ *and with function* $\phi_i:[0,+\infty)\longrightarrow[0,+\infty)$ *satisfying* $\phi_i(t)\leq M_0 t\ \ \forall\ t>M_1$ *for some constants* $M_0,M_1>0$*, Let* $\mu_n=\max_{i\in I}\{\mu_{in}\}$ *and* $\eta_n=\max_{i\in I}\{\eta_{in}\}$ *and,* $\phi(t)=\max_{i\in I}\{\phi_i(t)\}\ \forall\, t\in[0,\infty)$ *Suppose that* $F=\cap_{i=1}^{N}F(T_i)\neq\emptyset$ *and let* $\{x_n\}_{n\geq 1}$ *be a sequence generated iteratively by (4) where* $\{\alpha_n\}_{n\geq 1}$ *is sequences in (0,1) satisfying the following conditions:*

$\sum_{n=1}^{\infty}\alpha_n<\infty,\ 0<\zeta<\alpha_n<\varepsilon<1\ \ \forall\, n\geq 1$ *. Let* $x^*\in F$ *then* $\forall j\in\{1,2,...,m\},\ \lim_{n\to\infty}|x_n-T_jx_n|=0$ *and* $\{x_n\}_{n\geq 1}$ *converges strongly to some point of F if and only if* $\liminf_{n\to\infty} d(x_n,F)=0$*.*

Corollary 3.3 *Suppose* $E,K,T_i,F,\{x_n\}_{n\geq 1}$ *are as in corollary above, then,* $\{x_n\}_{n\geq 1}$ *converges strongly to some point of F if any of the following conditions is satisfied;*

1. $\{x_n\}$ *has a convergent subsequence*
2. *If* T_i *for some* $i\in\{1,...,m\}$ *satisfies condition B.*

References

[1] Ya. Alber, C. E. Chidume and H. Zegeye, Approximating fixed points of total asymptotically nonexpansive mappings.*Fixed point theory and applications*, 2006 (2006), article ID 10673.

[2] K. Goebel and Kirk, A fixed point theorem for asymptotically nonexpansive mappings *Proc. Amer. Math. Soc*. 35(1972),171-174.

[3] Zegeye, H. and Shahzad (2013), Approximation of the common fixed point of finite family of asymptotically nonexpansive mappings, Fixed point theory and applications. 11186/1687-1812.

[4] L.J. Ciric, J.S. Ume, *Ishikawa Iterative process for strongly pseudo-contractive operators in arbitrary Banach spaces,* Math. Commun.8(2003),43-48.

[5] R. Glowinski, P. LeTallee, *Augmented Lagrangian and operator-splitting methods in nonlinear mechanics*,SIAM publishing Co, Philadephia,1989.

[6] S. Haubruge,V.H. Nguyen, J.J. Strodiot,*Convergence analysis and applications of the Glowinski-LeTallee splitting method for finding a zero of the sum of two maximal monotone operators,* J.Optim.Theory Appl.97(1998)645-673.

[7] Z. Huang, *Approximating fixed points of ϕ -hemicontractive mappings by the Ishikawa iteration process with errors in uniformly smooth Banach spaces,* Comput.Math. Appl.39(2)(1998), 13-21.

[8] Z. Huang, *Equivalence theorems of the convergence between Ishikawa and Mann iterations with errors for generalized strongly successively ϕ -pseudocontractive mappings without Lipschitz assumptions* J. Math. Anal. Appl. 329(2007), 935-947.

[9] I.S. Ishikawa, *Fixed points by New Iteration Methods*, Proc. Amer. Math. Soc., 44 (1974), 147-150.

[10] T.C.Lim and H.K. Xu Fixed point theorems for asymtotically nonexpansive mappings, *Nonlinear Anal.*, 22(1994), 1345-1355.

[11] W.R. Mann, *Mean Value methods in Iteration*, Proc. Amer. Math. Soc.,4(1953).

[12] Chika Moore, A.C Nnubia AND Mogbademu Adesanmi;*Iterative procedure for uniformly continuous mappings.* Mathematical Theory and Modeling. 3(2013) no.8 53-59.

[13] C. Moore and B.V.C. Nnoli, *Iterative solution of nonlinear equations involving set-valued uniformly accretive operators*, Comput. Math. Anal. Appl. 42(2001), 131-140.

[14] Chika Moore and B.V.C. Nnoli, Strong convergence of averaged approximants for Lipschitz Pseudocontrative maps, *J.Math.Anal.Appl.*260(2006) (1) 169-178.

[15] Chika Moore and E. U. Ofoedu, Convergence of iteration process to random fixed points of uniformly L- Lipschitzian asymptotically hemi-contractive random maps in Banach spaces, *J. Nigerian Math. Soc.*, 25(2006) 87-94.

[16] M.A Noor, T. M Kassias, Z. Huang, *Three-step iterations for nonlinear accretive operator equations*, J. Math. Anal. Appl.,274(2002), 59-68.

[17] M.A. Noor, *New approximation schemes for general variational inequalities*, J. Math. Anal. Appl. 251(2000), 217-229.

[18] E.U. Ofoedu and Agatha C. Nnubia, Strong Convergence Theorem for Minimum-norm Fixed Point of Total asymptotically Nonexpansive Mapping. *Springer Verlag Journal-Afrika Matematika*(2014) Doi: 1007/513370-014-0240-4.

[19] A. Rafiq, *On Modified Noor iteration for nonlinear equations in Banach spaces,* Appl. Math. Comput. 182(2006), 589-595.

[20] J. Schu, *Iterative construction of fixed points of asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc. 113(1999), 727-731.

[21] Z. Xue and R. Fan, *Some comments on Noors iterations in Banach spaces,* Appl. Math. Comput. 206(2008),12-15.

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