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Trinomial Model on Employee Stock Option Valuation

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Abstract: *This paper discusses the pricing of employee stock options (ESO) using the Boyle and Kamrad-Ritchken trinomial models as a different alternative to the binomial model used by Hull and White. From the numerical results, it was found that Kamrad-Ritchken's trinomial model with λ = 1.11803 gave the best results in the ESO price estimation. Also obtained the relationship between ESO price with some specific parameter to the ESO's values i.e: exit rate and vesting period. The interesting thing happens to the multiplier of the implementation price (M) as a trigger to execute an option that is, there is an M value at which the OSK price will be maximum.*

Keywords: *Early Exercise, Employee Stock Options, Exit Rate,Hull-White Model, Vesting Period, Trinomial Model*

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I. Introduction

Employee stock option (ESO) is a call option (gives the holder right to buy) given by the company to a particular group of employees for the shares of the company, which will be exercised at a strike price after a certain period in the future (vesting period) and have a certain time period (maturity time) usually 10 years. This option can be viewed as a part of the remuneration package for the employees that encourages employees not to leave the company and work harder to improve the performance of the company. It will impact the company's future stock price that will be increased and the increase in the income of the employees who own the ESO.

ESO has some special features that accommodate the concerns of companies and employees. The ESO special features that make it different from the options traded in the regular market (e.g: see Rubinstein [8]) are as follows:

a. There is a vesting period of during which the options cannot be exercised.
b. When employees leave the company (either voluntarily or not) within the vesting period then the option will be forfeited.
c. When employees leave the company (either voluntarily or not) after the vesting period, the option becomes forfeit (if that are out of the money) or the option can be immediately exercised (if that are in the money).
d. Employees are not allowed to sell their options. The possibility that can be done is to take early exercise after the vesting period until the maturity time.

The lattice (binomial and trinomial) method is a simple and intuitive method for valuation the option. In terms of accuracy, it can be seen that the standard option pricing obtained by using the binomial model will converge to the exact Black-Scholes solution when many steps are taken quite large. But unfortunately this binomial model has only two possible stock price movements i.e: stock prices rise or stock prices down at every time step. The trinomial model is an extension of the binomial model, where at each time step the stock price is assumed to move up, fixed or down with a certain probability, making it more realistic in estimating stock price movements.

So far there have been several of ESO pricing models introduced both numerically and analytically such as the utility-maximizing model proposed by Kulatilaka and Markus [7], Huddart [4] and Rubinstein [8] that developed a binomial model with an exercise strategy that maximizes utility which are expected by the option holder when they cannot sell or protect the option, the Hull and White model [5] that modifies the binomial model CRR [2] to accommodate the ESO features; the most recent proposed by Cvitanić, Wiener and Zapatero [3] in the form of analytical solutions for ESO prices without incorporating reload, reset feature and dilution effects.

In this paper, we will determine the ESO price using the trinomial model proposed by Boyle and Kamrad-Ritchken using the Hull-White pricing model. Because in both trinomial models there are stretch parameters λ that are very influential in building stock prices then first will be seen the effect of the selection of values λ on the determination of ESO prices. Furthermore, the option price obtained with this trinomial model will be compared with the binomial model to see which model gives the best option pricing estimates with the Cvitanić, Wiener and Zapatero (CWZ) model as its benchmark. To see the effectiveness of stretch parameters λon each trinomial model then we calculated RMS (root mean square) error formulated as follows:

$$RMS = \sqrt{\frac{\sum_{j=1}^{n}(\hat{P} - P)}{n}}$$

where $\hat{P}$is the estimated price of the option and P is the actual price provided by CWZ.

II. Trinomial Model For Option Pricing Valuation

The expansion of the popular binomial model is the trinomial model. In this model, between two nodes i and $i+1$, stock price ratio $\frac{S_{i+1}}{S_i}$ take the value $\{d, m, u\}$ where $d < m < u$, with probability p_u, p_m and p_d.As in the binomial model, to obtain parameters in the trinomial model is done by equating the first and second moments continuous and discrete models of stock price movements are:

$$p_u u + p_m m + p_d d = \exp(r\delta t)$$
$$p_u u^2 + p_m m^2 + p_d d^2 = \exp(2r + \sigma^2)\,\delta t$$

with the relation of $\sum_{j=1}^{3} p_j = 1$ and $p_u, p_m, p_d > 0$, will be obtained three equations with six unknown variables.

2.1 Boyle Model

This model was developed by Phelim Boyle in 1986 [1]. In this model, three additional assumptions are given $m = 1, ud = 1,$ dan $u = \exp\left(\lambda\sigma\sqrt{\delta t}\right)$ where $\lambda > 1$called stretch parameters, so it is obtained

$$p_u = \frac{(\exp((2r+\sigma^2)\delta t) - \exp(r\delta t)) - (\exp(r\delta t) - 1)}{(\exp\left(\lambda\sigma\sqrt{\delta t}\right) - 1)(\exp\left(2\lambda\sigma\sqrt{\delta t}\right) - 1)}$$

$$p_d = \frac{(\exp((2r+\sigma^2)\delta t) - \exp(r\delta t))\exp(2\lambda\sigma\sqrt{\delta t}) - (\exp(r\delta t) - 1)\exp(3\lambda\sigma\sqrt{\delta t})}{(\exp(\lambda\sigma\delta t) - 1)(\exp(2\lambda\sigma\delta t) - 1)}$$

$$p_m = 1 - p_u - p_d$$

2.2 Kamrad-Ritchken Model

This model was developed by BardiaKamrad and Peter Ritchken in 1991 [6]. Suppose random variable: $\ln\left(\frac{S(t+\delta t)}{S(t)}\right) \sim N\left(\left(r - \frac{\sigma^2}{2}\right)\delta t, \sigma^2\delta t\right)$ then $\ln\left(S(t+\delta t)\right) = \ln\left(S(t)\right) + \xi(t)$ with $\xi \sim N\Big(\Big(r -$ *σ22δt,σ2δt*. In this model *ξ(t)*approximated by a discrete random variable defined as follows:

$$\xi^a(t) = \begin{cases} s & \text{, with probability} p_u \\ 0 & \text{, with probability} p_m \\ -s & \text{, with probability } p_d \end{cases}$$

where $s = \lambda\sigma\sqrt{\delta t}$ dan $\lambda \geq 1$. Note that $u = \exp(s)$, $m = 1$ and $d = \exp(-s)$. The mean and variance of the approximating distribution are chosen to equal the mean and variance of $\xi(t)$.

$$s(p_u - p_d) = \left(r - \frac{\sigma^2}{2}\right)\delta t$$
$$s^2(p_u + p_d) - s^2(p_u - p_d)^2 = \sigma^2\delta t$$

Substituting $s = \lambda\sigma\sqrt{\delta t}$ and for sufficiently small δt , while recognizing $\sum_{j=1}^{3} p_j = 1$ yields

$$p_u = \frac{1}{2\lambda^2} + \frac{(r - \frac{\sigma^2}{2})\sqrt{\delta t}}{2\lambda\sigma}, p_d = \frac{1}{2\lambda^2} - \frac{(r - \frac{\sigma^2}{2})\sqrt{\delta t}}{2\lambda\sigma}$$

$$p_m = 1 - \frac{1}{\lambda^2}$$

Next will be built trinomial tree stock price as Figure 1:

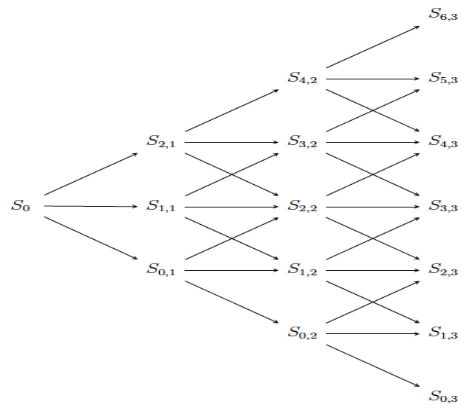

***Figure 1**. Trinomialtreestockprice*

Let $\delta t = \frac{T}{n}$ denote the spacing between successive time points, where T is the maturity date. Then we divided time interval $[0, T]$ into n sub-interval where $0 = t_0 < t_1 < t_2 < \cdots < t_n = T$ with $t_j = j\delta t(j = 0,1,2,\ldots,n$.

Suppose when $t_0 = 0$, the stock price is S_0, then when $t_1 = 1\delta t$ the stock price will be given by $S_0 u, S_0,\ or\ S_0 d$. Next when t_2, the stock price will take one of $S_0 u^2, S_0 u, S_0, S_0 d, or\ S_0 d^2$. By continuing this step, when t_j there will be $2j + 1$ possible stock price, given by

$$S_{i,j} = S_0 u^i d^j, i = 0,1,2, \ldots, 2j$$

with $S_{i,j}$ denotes the stock price when time t_j. In maturity time, $t_n = n\delta t = T$there are $2n + 1$ stock prices i.e: $\{S_{i,n}\}_{n=0,1,2,\ldots,2n}$. If $\{C_{i,n}\}_{n=0,1,2,\ldots,2n}$denotes payoff values at maturity for a European call option then

$$C_{i,n} = max\{S_{i,n} - K, 0\}$$

Next trinomial model working backward (in time) to obtain the option price at the time $t_0 = 0$. Option price at the time t_j is $C_{\mathrm{i,j}}$ calculated as the present value of the expected options values at the time t_{j+1}:

$$C_{i,j} = e^{-r\delta t}\left[p_u C_{i+2,j+1} + p_m C_{i+1,j+1} + p_d C_{i,j+1}\right]$$

For options that allow early exercise facilities, the above recursive formulation should be modified by adding a comparison test of value$C_{i,j}$ above with the payoff value obtained if the early exercise is done at the timet_j i.e.:

$$C_{i,j} = \max\{max(S_{i,j} - K, 0), e^{-r\delta t}\left[p_u C_{i+2,j+1} + p_m C_{i+1,j+1} + p_d C_{i,j+1}\right]\}$$

III. Hull-White Model

This model modified binomial model CRR [2] by explicitly inserting the possibility of employees leaving the company (voluntarily or not) before or after the vesting period and also including an employee strategy during the early exercise that is assumed to occur when the market price of the stock reaches a certain multiple, *M*, of the strike price. Based on that, the rule for pricing is established as follows:

a. Options only exercised after the vesting period ends.
b. After the vesting period, the option will be exercised until maturity if the stock price is at least *M* times the strike price.
c. There is a probability $\omega\delta t$ that the option will be forfeited in each short time interval δt during the vesting period, where ω is an employee exit rate.
d. There is a probability $\omega\delta t$ that the option will be terminated in each short time interval δt after the vesting period. When this happens then the option will be forfeited if the condition is out of the money and will be exercise immediately when the condition is in the money.

Suppose $f_{i,j}$ is the value of the option when the stock price $S_{i,j}$.Define v as the time when the vesting period ends, r as risk-free interest rate and K as the strike price of the option. At the maturity time ($t_n = n\delta t = T$) for each node at the end of the binomial tree, option price given by the values of the payoff at the maturity time i.e.

$$f_{i,n} = max\{S_{i,n} - K, 0\}.$$

For other nodes in the binomial model with $0 \leq j \leq n-1$ given the following rules:

1. During the vesting period i.e. $j\delta t < v$, option price can be calculated as:
$$f_{i,j} = e^{-\omega\delta t}\, e^{-r\delta t}\left[p_u.f_{i,j+1} + p_d f_{i+1,j+1}\right]$$
2. After the vesting period i.e. $j\delta t \geq v$:
If $S_{i,j} \geq MK$, then the option will be exercise i.e.:
$$f_{i,j} = S_{i,j} - K$$
if $S_{i,j} < MK$, then the option given by:
$$f_{i,j} = \left(1 - e^{-\omega\delta t}\right)\cdot max\{S_{i,j} - K, 0\} + e^{-\omega\delta t}\, e^{-r\delta t}\left[p_u.f_{i,j+1} + p_d f_{i+1,j+1}\right]$$

At the end of the process, we get the ESO price given by the value $f_{0,0}$.

IV. Numerical Result

To see the effect of selecting parameter λ to the ESO price, the following calculation shows the numeric price of ESO by using the trinomial model with the number of different time step for the values$S = 100, K = 100, r = 0.06, T = 10, \sigma = 0.2, M = 1.5, \omega = 0.04$ and $v = 2$ .As a benchmark, CWZ analytical solutions are used for the ESO prices with the same inputs as the trinomial models. This analytical solution is then called the 'CWZ' price. Both of these models, Hull-White and CWZ together do not include reloading, resetting and dilution effects in the option pricing. This is because these features are not features that are generally present in the ESO and the existence of these features in the model will complicate the computational process.

Parameter λin each trinomial model is chosen so that on the Boyle model $\lambda = \{1.1, 1.2, 1.3\}$ denoted as:{B1, B2, B3} while on the Kamrad-Ritchken model $\lambda = \{1.291, 1.22474, 1.11803\}$ denoted as:{KR1, KR2, KR3}.

Table 1. *ESO price with trinomial model*

Analytic price 'CWZ': 27.8551

n	Boyle with λ			KR with λ		
	1.1 (B1)	1.2 (B2)	1.3 (B3)	1.29100 (KR1)	1.22474 (KR2)	1.11803 (KR3)
50	30.3556	28.6532	29.6146	29.3989	28.7543	30.4123
100	28.2708	29.3382	28.0703	27.9248	29.5287	28.3919
250	28.8889	28.6553	28.1856	28.0763	28.8845	29.0655
500	28.7529	27.9211	28.9129	28.8127	28.1563	28.0209
750	27.8934	28.1665	28.3016	28.2077	28.4099	28.0816
1000	28.2339	27.9365	28.1731	28.0817	28.1794	28.4283
1250	28.0914	27.9311	28.2489	28.1583	28.1755	28.2859
1500	28.0883	28.0392	28.4247	28.3342	28.2860	28.2838
1750	28.1652	28.2095	28.0746	27.9861	27.9111	27.8609
2000	28.2908	27.9126	28.3733	28.2840	28.1589	28.0226
2250	28.0132	28.1712	28.1708	28.0827	27.9377	28.2097
2500	28.2134	27.9877	28.0250	27.9378	28.2358	27.9937
3000	27.8869	28.1616	28.2979	28.2099	27.9937	28.0830

Table 1 shows that the price of the option not uniformly convergence towards the 'CWZ' price for all parameter selections, for both Boyle and Kamrad-Ritchken models. In addition, both trinomial models always overestimate the ESO price. This can also be seen in Figure 2, both for n small and n large.

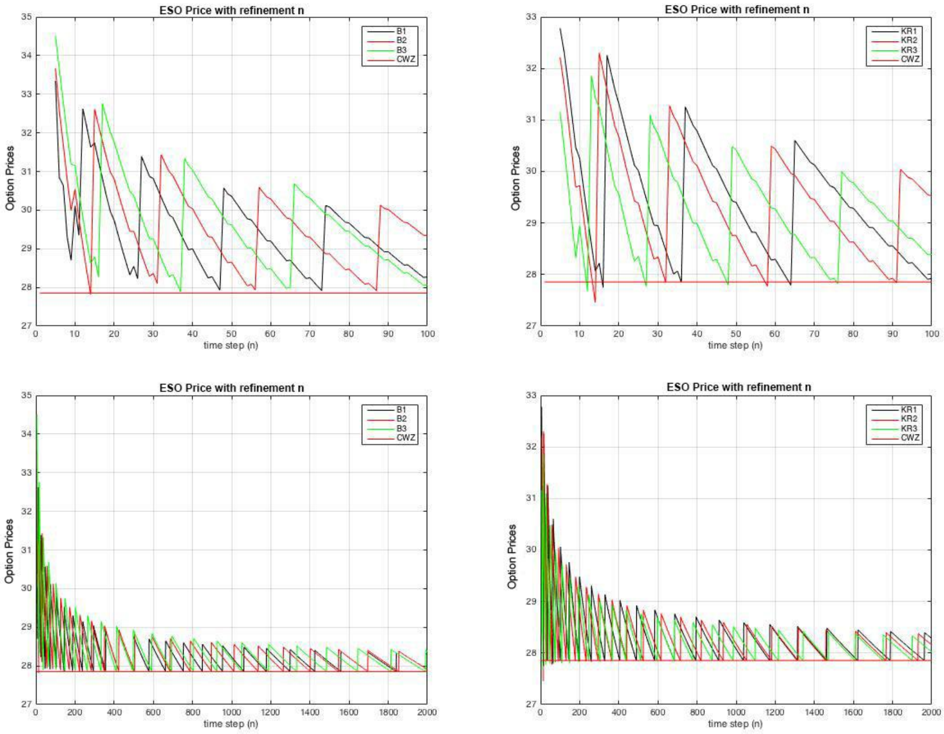

Figure 2. *Convergence of the ESO price with Boyle and Kamrad-Ritchken model with refinement n for n = 100 (up) and n = 2000 (down)*

The values used for RMS error calculation to see the effectiveness of stretch parameters λ is still the same as before.

Table 2. *RMS error trinomial model for each stretch parameter λ*

Sub-interval	n	Boyle			Kamrad-Ritchken		
		B1	B2	B3	KR1	KR2	KR3
1	5-100	1.9362	2.0900	2.2518	1.9659	1.9096	1.6585
2	5-200	1.5397	1.6432	1.7779	1.5970	1.5344	1.3422
3	5-300	1.3218	1.4179	1.5541	1.4083	1.3226	1.1693
4	5-400	1.1905	1.2725	1.3858	1.2608	1.2049	1.0699
5	5-500	1.0892	1.1646	1.2715	1.1675	1.1079	0.9837

Table 2 shows that the smallest RMS error given by KR3 ($\lambda = 1.11803$). These results are different from those presented in the Kamrad-Ritchken [6] paper, which is the best value that gives the smallest error in the standard option pricing.

Next, we will see the ESO price sensitivity to ESO features using the Kamrad-Ritchken trinomial model with$\lambda = 1.11803$.

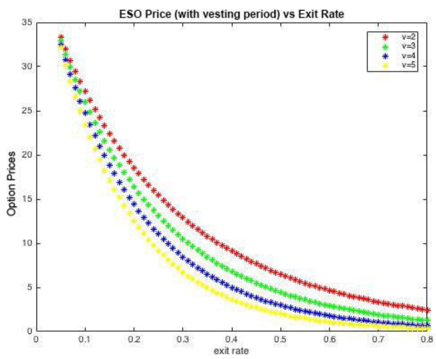

Figure 3. *The effect of parameterω (exit rate) to the ESO price together with a vesting period*

Figure 3 shows the effect of ω values to the ESO price with different vesting period for the values $S = 100, K = 100, r = 0.06, T = 10, \sigma = 0.2, M = 2$ and $n = 500$. From these results can be seen that the price of ESO decreases as the exit rate increases.

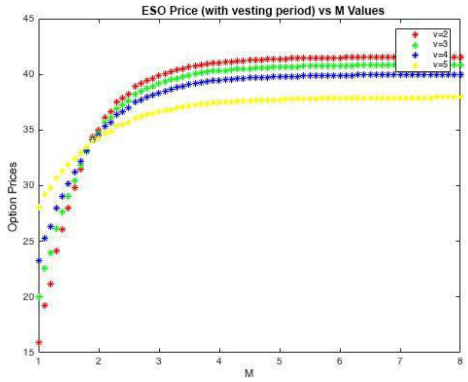

Figure 4. *The effect of* M *value to the ESO price together with vesting period*

Figure 4 shows the relationship of M values that affect the early exercise strategy to the ESO prices with different vesting periods for values$S = 100, K = 100, r = 0.06, T = 10, \sigma = 0.2, \omega = 0.04$ dan$n = 500$. From these results indicate that there is an M value saying M_{max} where the option has the maximum value. This means that there is a trusted in employees that the option will not wait until it is greater than M_{max} to immediately exercise. There is also an interesting phenomenon where for a certain M, say m,di mana $m \subset M$, when $m < M$ had a negative correlation between the vesting period with the ESO price but when$m > M$a positive correlation exists between the vesting period and the ESO price.

V. Conclusion

From the results that have been obtained before, it can be concluded things as follows:

1. The selection of parameters for the Boyle and Kamrad-Ritchken trinomial models will influence the ESO price. It can be seen that for each model, the smallest λwill give the best results in estimating the ESO price. In this study selected trinomial model Kamrad-Ritchken with $\lambda = 1.11803$.
2. ESO prices decrease when exit rate increases.
3. There is an M value where the option is maximum.

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