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Laplace Homotopy Analysis Method for Solving Fractional Order Partial Differential Equations

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Abstract: *In thispaper,we considered a non-linear systemsof fractional partial differential equations.They have been solved by a computational method which is so-called a modified Laplace Homotopy Analysis method .The fractional derivatives are described in the Caputo sense. The proposed technique is only a simple modification of the Homotopy Analysis Method .The method was applied for some illustrative examples to solve non-linear systems of fractional partial differential equations. From the result of the illustrative examples we conclude that the method is computationally efficient.*

Keywords: *Fractional calculus, system of fractional order partial differential equations, Laplace transform, Homotopy Analysis method.*

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I. Introduction

Fractional order partial differential equations are popularizations of classical partial differential equations. These have been of large attention in the recent literatures.

These topics have received a mighty deal of attention especially in the fields of viscoelasticity materials, electrochemical processes, dielectric polarization, colored noise, anomalous diffusion, signal processing, control theory and others.

Increasing extent, these models are used in applications such as fluid flow, finance and others. most nonlinear fractional differential equations do not have exact solutions, so approximation and numerical techniques must be applied .The Laplace Homotopy Analysis method (LHAM) is a combination of the Homotopy analysis method proposed by Liao in his Ph.D. Thesis [1] and the Laplace transform [2, 3] .

The Homotopy analysis method has been successfully employed to solve many types of nonlinear, homogeneous or nonhomogeneous equations and systems of equations as well as problems in science and engineering [4-5].

Various authors have proposed several schemes to solve system of fractional partial differential equations with Liouville-Caputo and Caputo-Fabrizio fractional operators.

Dehghan in [6] applied the HAM to solve linear partial differential equations, in this work, fractional derivatives are described in the Liouville-Caputo sense, Xu in [7] calculated analytically the time fractional wave-like differential equation with a changeable coefficients, the author reduced the governing equation to two fractional ordinary differential equations.

Jafari in [8] exercise the HAM to obtain the solution of multi-order fractional differential equation studied by Diethelmand Ford [9], Goufo et al. [10] developed a mathematical analysis of a model of rock fracture in the ecosystem and applied the Caputo-Fabrizio fractional derivative, where analytical and computational approaches are obtained. Other analytical approaches that could be of interest are presented in [12- 14].

This paper is organized as follows: in section 2 we recall the definitions of fractional derivatives and fractional integration simply, section 3 describes the formulation of Laplace Homotopy analysis method for solving system of fractional order P.D.Es.Some illustrative examples are given in section 4 finally a conclusion is given in section 5.

II. Fractional Order Derivative And Integral:

In this section, we review basic definitions of fractional order differentiation and fractional order integration such as:

Definition 2.1:The Riemann –Liouville fractional integral of order $\alpha > 0$ is defined as follows:

$I_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (t-\tau)^{\alpha-1} f(\tau) d\tau$, x> 0 ,$\alpha \in R^+$

Where $\Gamma(\alpha)$ is the Gamma function.

Definition 2.2: The Caputo fractional derivative of order $\alpha > 0$ is defined as follows:

$$ {}^{c}D_x^{\alpha} f(x) = \begin{cases} \frac{1}{\Gamma(x-\alpha)} \int_0^x \frac{f^{(m)}(\tau)}{(x-\tau)^{\alpha+1-m}} d\tau & m-1 < \alpha < m \\ \frac{d^m}{dx^m} f(x) & \alpha = m \end{cases} $$

For $\alpha > 0$, we have the following properties of the Caputo fractional derivative:

1- ${}^{c}D_x^{\alpha}(I_x^{\alpha} f(x)) = f(x)$.

2-$I_x^{\alpha}\left({}^{c}D_x^{\alpha} f(x)\right) = f(x) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{x^k}{k!}$

3- ${}^{c}D_x^{\alpha}(c)=0$, $c \in R$

4- ${}^{c}D_x^{\alpha}(x^{\gamma}) = \begin{cases} \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} x^{\gamma-\alpha} & \gamma \in \{0,1,2,3,\dots\}, \gamma \geq [\alpha] \\ 0 & \gamma \in \{0,1,2,3,\dots\}, \gamma < [\alpha] \end{cases}$

Where $[\alpha]$ is the floor function of α

III. The Approach

Let us consider the following system of non-linear fractional partial differential equations.

$$ {}^{c}D_t^{\alpha_i} u_i(x,t) + R_i(u_1, u_2, \dots, u_m) + N_i(u_1, u_2, \dots, u_m) = g_i(x,t), \; n-1 < \alpha_i \leq n, i = 1,2,..m \quad \dots (1) $$

With initial data

$$ u_i(x,0) = f_i(x) \quad \dots (2) $$

Where ${}^{c}D_t^{\alpha_i}$ are the Caputo fractional derivatives of order α_i, R_i and N_i , $i = 1,2,\dots,m$ are non-linear operators, respectively, and g_i are source terms.

In order to solve this system by using Laplace Homotopy Analysis method, first we employing the Laplace transform to the both sides of (1) yields:

$$ \mathcal{L}[u_i(x,t)] = \frac{1}{s^{\alpha_i}} \sum_{k=0}^{m-1} s^{\alpha_i-1-k} u_i(x,0) - \frac{1}{s^{\alpha_i}} \mathcal{L}[R_i(u_1, u_2, \dots, u_m) + N_i(u_1, u_2, \dots, u_m)] + \frac{1}{s^{\alpha_i}} \mathcal{L}[g_i(x,t)] \quad \dots (3) $$

The so-called zero-order deformation equation of the Laplace equation (3) has

$$ (1-q)[\mathcal{L}\emptyset_i(x,t;q) - \mathcal{L}u_{i0}(x,t)] = qh \begin{bmatrix} \mathcal{L}\emptyset_i(x,t;q) - \frac{1}{s^{\alpha_i}} \sum_{k=0}^{m-1} s^{\alpha_i-1-k} f_i(x) + \frac{1}{s^{\alpha_i}} \mathcal{L}_t[g_i(x,t)] + \\ \frac{1}{s^{\alpha_i}} \mathcal{L}\begin{bmatrix} R_i(\emptyset_1(x,t,q), \emptyset_2(x,t,q), \dots, \emptyset_m(x,t,q) + N_i((\emptyset_1(x,t,q), \emptyset_2(x,t,q), \dots, \\ \emptyset_m(x,t,q) \end{bmatrix} \end{bmatrix} \quad \dots (4) $$

Subject to the initial conditions:

$$ \emptyset_i(x,0,q) = f_i(x), \quad i = 1,2,\dots m \quad \dots (5) $$

Where $q \in [0,1]$ is an embedding parameter when q=0 we have $\mathcal{L}\emptyset_i(x,t;0) = \mathcal{L}u_{i0}(x,t)$

And when $q = 1$, $h = -1$ the zero-order deformation eq (4) and (5) equivalent to (3) and (2), respectively, provides

$\mathcal{L}\emptyset_i(x,t;1) = \mathcal{L}u_i(x,t)$.

Thus as q increasing from 0 to 1, $\mathcal{L}\emptyset_i(x,t,q)$ varies from $\mathcal{L}u_0(x,t)$ to $\mathcal{L}u(x,t)$.

Expanding $\mathcal{L}\emptyset_i(x,t;q)$ in Taylor series with respect to q, one has

$$ \mathcal{L}\emptyset_i(x,t;q) = \mathcal{L}u_{i0}(x,t) + \sum_{m=1}^{\infty} \mathcal{L}u_{im}(x,t) q^m, i = 1,2,\dots,m \quad \dots (6) $$

Where

$$ \mathcal{L}u_{im}(x,t) = \frac{1}{m!} \frac{\partial^m}{\partial x^m} \mathcal{L}\emptyset_i(x,t;q) \Bigg|_{q=0} \quad \dots (7) $$

Define the vectors

$$ \overrightarrow{\mathcal{L}u_{\iota m}(x,t)} = \{\mathcal{L}u_{i0}(x,t), \mathcal{L}u_{i1}(x,t), \mathcal{L}u_{i2}(x,t), \dots, \mathcal{L}u_{im}(x,t)\}, \; i = 1,2,\dots,m \quad \dots (8) $$

Differentiating equation (4) m times with respect to the embedding parameter q, and setting q=0, h= -1 and finally dividing them by m!, we have the so-called m^{th} order deformation equation for $i = 1,2,\dots n$.

$$ \mathcal{L}u_{im}(x,t) = x_m \mathcal{L}u_{im-1}(x,t) - R_{im}(\overrightarrow{\mathcal{L}u_{\iota m-1}(x,t)}) \quad \dots (9) $$

Where

$$R_{im}(\overrightarrow{\mathcal{L}u_{im-1}(x,t)}) = \mathcal{L}u_{m-1}(x,t) + \frac{1}{s^{\alpha_i}}\left(\frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}[\mathcal{L}[R_i(\emptyset_1(x,t,q),\emptyset_2(x,t,q),\dots,\emptyset_m(x,t,q)) + N_i((\emptyset_1(x,t,q),\emptyset_2(x,t,q),\dots,\emptyset_m(x,t,q)]\right)\Bigg|_{q=0} - (1-x_m)\left(\frac{1}{s^{\alpha_i}}\sum_{k=0}^{m-1} s^{\alpha_i-1-k}f_i(x) + \frac{1}{s^{\alpha_i}}\mathcal{L}\big[g_i(x,t)\big]\right)$$

… (10)

And $x_m = \begin{cases} 0 & , \quad m \le 1 \\ 1 & , \quad m > 1 \end{cases}$ … (11)

Applying the inverse Laplace transform of both sides of (9), then we have a power series solution of (1) which can be expressed as:

$u_i(x,t) = \sum_{n=0}^{\infty} u_{in}(x,t) \quad , i = 1,2,\dots m$ ... (12)

IV. Illustrative Examples

In this section we will apply the LHAM to systems of non-linear fractional partial differential equations (FPDEs).

Example 1: Consider the following system of linear FPDEs:

$${}^{C}D_t^{\alpha}u(x,t) - v_x(x,t) + v(x,t) + u(x,t) = 0$$
$${}^{C}D_t^{\beta}v(x,t) - u_x(x,t) + v(x,t) + u(x,t) = 0 \quad \dots (13)$$

With initial conditions as

$u(x,0) = \sinh(x), v(x,0) = cosh(x) \quad , (0 < \alpha, \beta < 1)$ … (14)

The exact solution is given in [13] as $u(x,t) = \sinh(x-t), v(x,t) = \cosh(x-t)$

Taking the Laplace transform to the both sides of eq (13) and using (14), we have

$$\left.\begin{aligned} \mathcal{L}u(x,t) - \frac{\sinh(x)}{s} - \frac{1}{s^{\alpha}}\mathcal{L}[v_x(x,t)] + \frac{1}{s^{\alpha}}\mathcal{L}[v(x,t) + u(x,t)] = 0 \\ \mathcal{L}v(x,t) - \frac{\cosh(x)}{s} - \frac{1}{s^{\beta}}\mathcal{L}[u_x(x,t)] + \frac{1}{s^{\beta}}\mathcal{L}[v(x,t) - u(x,t)] = 0 \end{aligned}\right\} \quad \dots (15)$$

Furthermore, we can construct the Homotopy as follows

$$R_{1m}(\overrightarrow{u_{m-1}}, \overrightarrow{v_{m-1}}) = \mathcal{L}u_{m-1}(x,t) - \frac{1}{s^{\alpha}}\mathcal{L}\left[\left(v_{m-1}(x,t)\right)_x\right] + \frac{1}{s^{\alpha}}\mathcal{L}[v_{m-1}(x,t) + u_{m-1}(x,t)] - (1-x_m)\frac{\sinh(x)}{s} \quad \dots (16)$$

$$R_{2m}(\overrightarrow{u_{m-1}}, \overrightarrow{v_{m-1}}) = \mathcal{L}v_{m-1}(x,t) - \frac{1}{s^{\beta}}\mathcal{L}\left[\left(u_{m-1}(x,t)\right)_x\right] + \frac{1}{s^{\beta}}\mathcal{L}[v_{m-1}(x,t) - u_{m-1}(x,t)] - (1-x_m)\frac{\sinh(x)}{s} \quad \dots (17)$$

and the m^{th} order deformation equations for $m \ge 1$ become

$\mathcal{L}u_m(x,t) = x_m\mathcal{L}u_{m-1}(x,t) - R_{1m}(\overrightarrow{u_{m-1}}, \overrightarrow{v_{m-1}})$ … (18)

$\mathcal{L}v_m(x,t) = x_m\mathcal{L}v_{m-1}(x,t) - R_{2m}(\overrightarrow{u_{m-1}}, \overrightarrow{v_{m-1}})$ … (19)

From Eqs (14), (18) and (19) and subject to initial condition

$u_{m-1}(x,0) = 0, v_{m-1}(x,0) = 0, \quad m \ge 1$

We successively obtain

$$\begin{cases} \mathcal{L}u_0(x,t) = \dfrac{\sinh(x)}{s} \\ \mathcal{L}v_0(x,t) = \dfrac{\cosh(x)}{s} \end{cases}, \begin{cases} \mathcal{L}u_1(x,t) = \dfrac{-1}{s^{\alpha+1}}\cosh(x) \\ \mathcal{L}v_1(x,t) = \dfrac{-1}{s^{\beta+1}}\sinh(x) \end{cases}$$

$$\begin{cases} \mathcal{L}u_2(x,t) = \dfrac{-1}{s^{\alpha+\beta+2}}\cosh(x) + \dfrac{1}{s^{\alpha+\beta+2}}\sinh(x) + \dfrac{1}{s^{2\alpha+2}}\cosh(x) \\ \mathcal{L}v_2(x,t) = \dfrac{-1}{s^{\alpha+\beta+2}}\sinh(x) + \dfrac{1}{s^{2\beta+2}}\sinh(x) + \dfrac{1}{s^{\alpha+\beta+2}}\cosh(x) \end{cases}$$

$\vdots$

And so on.

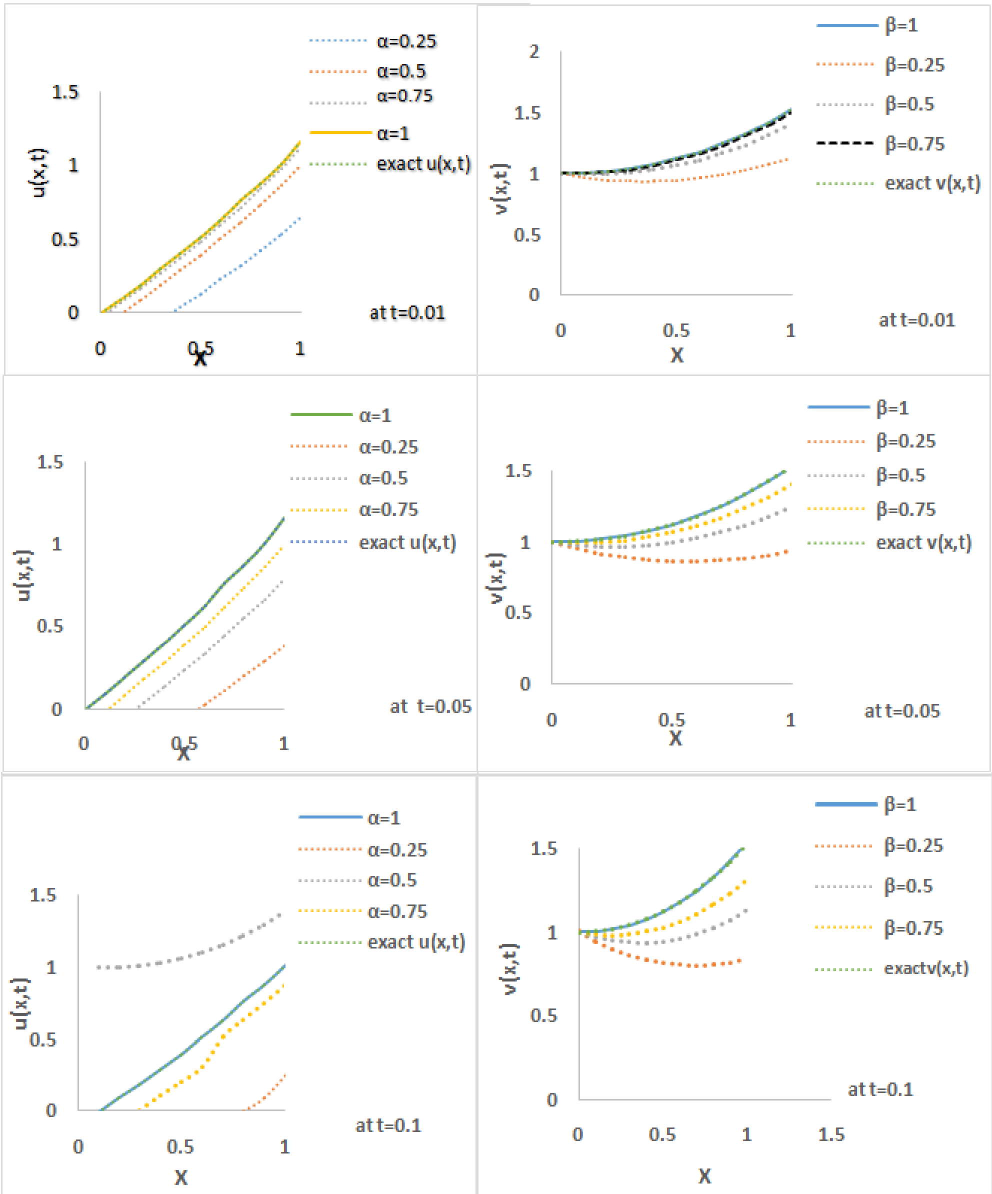
upon applying the inverse Laplace transform to the above equations and by using (12), one can get:

$$u(x,t) = \left(1 - \frac{t^{\alpha+\beta+1}}{\Gamma(\alpha+\beta+2)} + \dots\right)\sinh(x) + \left(\frac{-t^{\alpha}}{\Gamma(\alpha+1)} - \frac{t^{\alpha+\beta+1}}{\Gamma(\alpha+\beta+2)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + \cdots\right)\cosh(x)$$

$$v(x,t) = \left(1 + \frac{t^{\alpha+\beta+1}}{\Gamma(\alpha+\beta+2)} + \cdots\right)\cosh(x) + \left(\frac{-t^{\beta}}{\Gamma(\beta+1)} - \frac{t^{\alpha+\beta+1}}{\Gamma(\alpha+\beta+2)} + \frac{t^{2\beta+1}}{\Gamma(2b+2)} + \cdots\right)\sinh(x)$$

… (20)

Following figure (1) represent the approximate solution of problem (13) for different values of α and β compared with the exact solution when $\alpha = \beta = 1$ at t=0.01, 0.05, 0.1



Fig(1) the approximate solution of problem (13) for different values of α and β compared with the exact solution when $\alpha = \beta = 1$

Example 2: Consider the nonlinear system of FPDEs:

$${}^{C}D_t^{\alpha}u(x,t) + v(x,t)u_x(x,t) + u(x,t) = 1$$

$${}^{C}D_t^{\alpha}v(x,t) - u(x,t)v_x(x,t) - v(x,t) = 1 \qquad , 0 < \alpha \le 1 \qquad \dots (21)$$

With initial conditions

u(x, 0)=e^{x} ,v(x,0)=e^{-x} … (22)

Taking the Laplace transform to the both sides of eq. (21) and using (22), we have

$$\left.\begin{aligned} \mathcal{L}u(x,t) - \frac{e^{x}}{s} + \frac{1}{s^{\alpha}}\mathcal{L}[v(x,t)u_x(x,t)] + \frac{1}{s^{\alpha}}\mathcal{L}[u(x,t)] - \frac{1}{s^{\alpha+1}} = 0 \\ \mathcal{L}v(x,t) - \frac{e^{-x}}{s} - \frac{1}{s^{\alpha}}\mathcal{L}[u(x,t)v_x(x,t)] - \frac{1}{s^{\alpha}}\mathcal{L}[v(x,t)] - \frac{1}{s^{\alpha+1}} = 0 \end{aligned}\right\} \qquad \dots (23)$$

Furthermore, we can construct the Homotopy as follows

$$R_{1m}(\overrightarrow{u_{m-1}}, \overrightarrow{v_{m-1}}) = \mathcal{L}u_{m-1}(x,t) + \frac{1}{s^{\alpha}}\mathcal{L}\left[v_{m-1}(x,t)\left(u_{m-1}(x,t)\right)_x + u_{m-1}(x,t)\right] - (1 - x_m)\left(\frac{e^{x}}{s} + \frac{1}{s^{\alpha+1}}\right)$$

$$R_{2m}(\overrightarrow{u_{m-1}},\overrightarrow{v_{m-1}}) = \mathcal{L}v_{m-1}(x,t) - \frac{1}{s^{\alpha}}\mathcal{L}\left[u_{m-1}(x,t)\big(v_{m-1}(x,t)\big)_x + v_{m-1}(x,t)\right] - (1-x_m)\left(\frac{e^{-x}}{s} + \frac{1}{s^{\alpha+1}}\right)$$

and the m^{th} order deformation equations for $m \geq 1$ become

$$\mathcal{L}u_m(x,t) = x_m\mathcal{L}u_{m-1}(x,t) - R_{1m}(\overrightarrow{u_{m-1}},\overrightarrow{v_{m-1}}) \quad \ldots (24)$$
$$\mathcal{L}v_m(x,t) = x_m\mathcal{L}v_{m-1}(x,t) - R_{2m}(\overrightarrow{u_{m-1}},\overrightarrow{v_{m-1}}) \quad \ldots (25)$$

From Eqs (22), (24) and (25) and subject to initial condition
$u_{m-1}(x,0) = 0, v_{m-1}(x,0) = 0, \ \ m \geq 1$

We successively obtain

$$\begin{cases}\mathcal{L}u_0(x,t) = \dfrac{e^{x}}{s}\\ \mathcal{L}v_0(x,t) = \dfrac{e^{-x}}{s}\end{cases}, \quad \begin{cases}\mathcal{L}u_1(x,t) = \dfrac{-e^{x}}{s^{\alpha+1}}\\ \mathcal{L}v_2(x,t) = \dfrac{e^{-x}}{s^{\alpha+1}}\end{cases}$$

$$\begin{cases}\mathcal{L}u_2(x,t) = \dfrac{-\Gamma(2\alpha+1)t^{3\alpha}}{\Gamma(\alpha+1)\Gamma(\alpha+1)s^{3\alpha+1}} + \dfrac{e^{x}}{s^{2\alpha+2}}\\ \mathcal{L}v_2(x,t) = \dfrac{\Gamma(2\alpha+1)t^{3\alpha}}{\Gamma(\alpha+1)\Gamma(\alpha+1)s^{3\alpha+1}} + \dfrac{e^{-x}}{s^{2\alpha+2}}\end{cases}$$

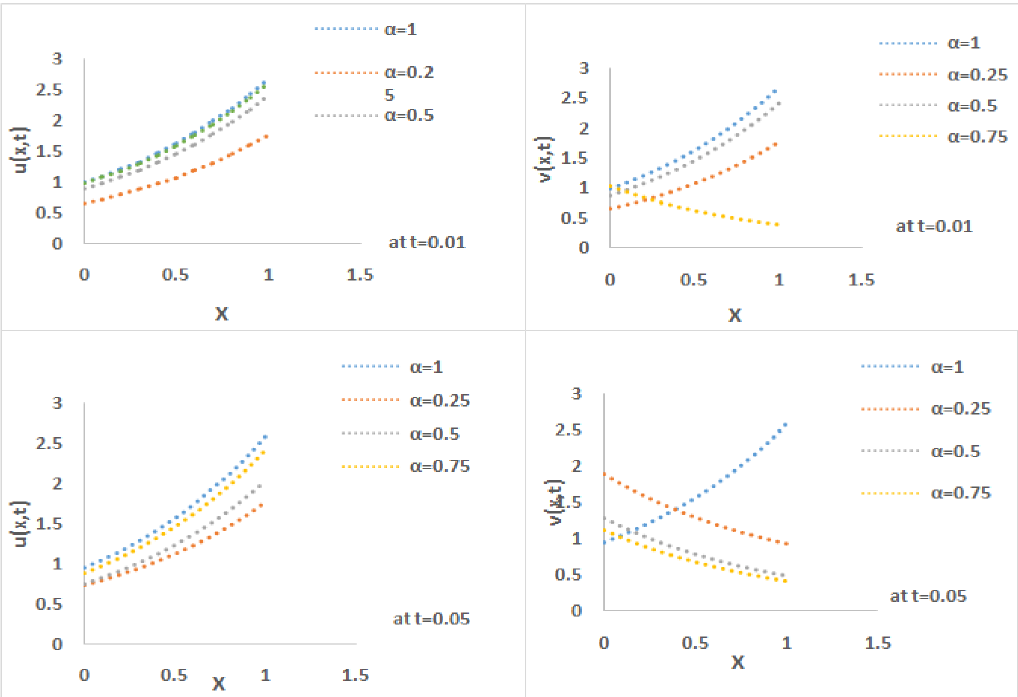
$\vdots$

Upon applying the inverse Laplace transform to the above equations and by using (12), one can get:

$$u(x,t) = \left(1 - \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + \ldots\right)e^{x} + \left(\frac{-\Gamma(2\alpha+1)t^{3\alpha}}{\Gamma(\alpha+1)\Gamma(\alpha+1)\Gamma(3\alpha+1)} + \cdots\right)$$

$$v(x,t) = \left(1 - \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + \ldots\right)e^{-x} + \left(\frac{\Gamma(2\alpha+1)t^{3\alpha}}{\Gamma(\alpha+1)\Gamma(\alpha+1)\Gamma(3\alpha+1)} + \cdots\right)$$

Following fig (2) represent the approximate solution of problem (21) for different of α



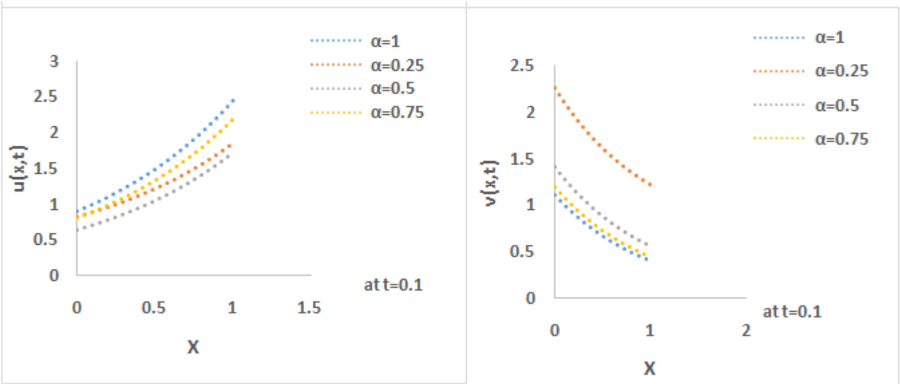


Figure (2) the approximate solution of problem (21) for different values of α

V. Conclusions

We present in this paper a computational method for solving a system of fractional order partial differential equations which is known as Laplace Homotopy analysis method. The method considered proved that it is a powerful tool which enables us to handle a wide class of non-linear fractional partial differential equations in a simple way and in order to reach the desired accuracy, all what we have to do is to increase the number of iterations.If the non-linear problems has an exact solution ,then after a certain stages ,every iteration leads to the same exact solution .Therefore Laplace Homotopy analysis method is adequate for both linear and non-linear problems

References

[1] SJ .Liao, The proposed homotopy analysis technique for the solution of nonlinear problems. PhD thesis, Shanghai Jiao Tong University; 1992.
[2] Gepree.M, Gepreel.MS,Al-Malki. KA,and Al-Humyani. FA. M: Approximate solutions of the generalized Abel's integralequations using the extension Khan's homotopy analysis transformation method. J. Appl. Math. 2015, 357861 (2015)
[3] Gupta. VG,Kumar. P: Approximate solutions of fractional linear and nonlinear differential equations using Laplace Homotopy analysis method. Int. J. Nonlinear Sci. 19(2), 113-120 (2015)
[4] Jafari. H, Seifi S: Homotopy analysis method for solving linear and nonlinear fractional diffusion-wave equation. Commun Nonlinear Sci Numer Simulat 2009; 14:2006–12.
[5] Liao .S, Homotopy analysis method: a new analytical technique for nonlinear problems. Commun Nonlinear Sci Numer Simulat 1997;2(2):95–100
[6] Dehghan .M, Manafian. J and Saadatmandi. A: The solution of the linear fractional partial differential equations using thehomotopy analysis method. Z. Naturforsch. A 65(11), 935-949 (2010)
[7] Xu. H, Cang. J: Analysis of a time fractional wave-like equation with the homotopy analysis method. Phys. Lett. A 372(8), 1250-1255 (2008)
[8] Jafari. H, Das. S, Tajadodi. H: Solving a multi-order fractional differential equation using homotopy analysis method. J. King Saud Univ., Sci. 23(2), 151-155 (2011)
[9] Diethelm. K, Ford, NJ: Multi-order fractional differential equations and their numerical solution. Appl. Math. Comput. 154, 621-640 (2004)
[10] Goufo. EF, Doungmo, MKP, Mwambakana, JN: Duplication in a model of rock fracture with fractional derivative without singular kernel. Open Math. 13(1), 839-846 (2015)
[11] Yousefi. SA, Dehghan. M, Lotfi. A: Generalized Euler-Lagrange equations for fractional variational problems with free boundary conditions. Comput. Math. Appl. 62(3), 987-995 (2011).
[12] Mohebbi. A, Abbaszadeh. M and Dehghan. M: High-order difference scheme for the solution of linear time fractional Klein-Gordon equations. Numer. Methods Partial Differ. Equ. 30(4), 1234-1253 (2014)
[13] Atangana. A: On the new fractional derivative and application to nonlinear Fisher's reaction-diffusion equation. Appl .Math. Comput. 273, 948-956 (2016)
[14] Atangana. A, Alkahtani. BST: Analysis of the Keller-Segel model with a fractional derivative without singular kernel. Entropy 17(6), 4439-4453 (2015)

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