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A Proof of Goldbach's Conjecture via Surjective Mapping.

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Abstract: *The aim of this research is to prove the elusive Goldbach's Conjecture through the medium of mapping of sets in Set Theory. Some previous research attempts at proving the conjecture had been based on the belief that it is a problem for Number Theory which this researcher believes is not strictly so.*

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I. Introduction/Brief History

For over 270 years Goldbach's Conjecture has so far remained one of the unsolved problems in mathematics. It is said that it started with a letter written by a German mathematician by the name of Christian Goldbach to a great and prolific mathematician by the name of Leonhard Euler in which he stated that:

Every even integer greater than 2 can be expressed as the sum of two primes.

Euler in his reply of 30th June, 1742 agreed with the statement which he described as a 'theorem' but added that he could not prove it. Encouraged by Euler's description of the statement as a theorem, this researcher decided to see it the same way and search for a proof.

THE SEARCH FOR A PROOF

Euler's description of Goldbach's conjecture as a theorem encourages one to believe that there must be a proof. The idea that the problem is a subject for Number Theory appears to have led many researchers into concentrating on the even integers in the hope that an example could be found to contradict the conjecture. The very idea of looking for a contradiction gave this researcher the idea of looking elsewhere in the wide subject of mathematics. Hence a closer study of the statement of the conjecture which led him to look in on sets. Mapping under Set Theory is a powerful tool for comparing and matching sets.

Mapping

Definition: Let X and Y be non-empty sets.

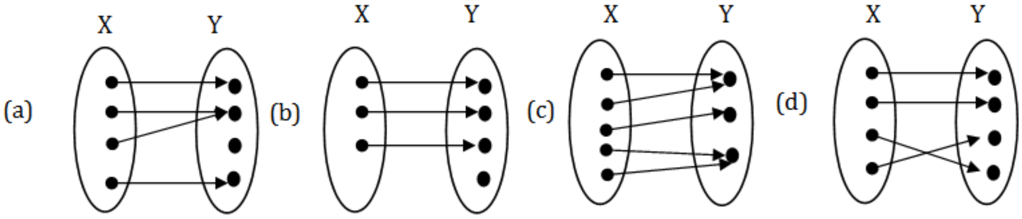
A mapping f of X into Y is a subset of X x Y such that for every $x \in X$ there is a unique
$y \in Y$ in the elements of the subset. y is said to be the image of x in the mapping.
For any mapping of X into Y, the set X is called the domain and the set Y is called the codomain of f.
Three immediately related mappings are the injective, surjective and bijective mappings.

Definition: A mapping $f : X \rightarrow Y$ is injective or one – to – one mapping of X into Y if the image of distinct elements of X are distinct elements of Y.

Definition: A mapping $f : X \rightarrow Y$ is surjective if for every element y in the co-domain Y of f there is at least one element x in the domain X of f such that f(x) = y. The element x need not be unique because f may map one or more elements of X to the same element of Y.

Definition: A mapping $f : X \rightarrow Y$ is bijective if every element of the codomain is mapped to exactly one element of the domain. This means that the mapping is both injective and surjective.

Below we give elementary picture of these mappings and then proceed to dwell on surjective mappings which is the relevant mapping for the purpose of our proof of the Goldbach conjecture.



Fig(i)

Fig (i) shows four mappings. (a) is neither injective nor surjective. (b) is injective but not surjective. (c) is surjective but not injective. (d) is bijective.

Surjective Mapping
f : X $\longrightarrow$ Y is surjective (onto) mapping if $\forall y \in Y \; \exists \; x \in$ X such that f (x) = y.

Examples. In the examples that follow we show how to prove that a mapping is surjective (onto).

1. Prove that f : R $\longrightarrow$ R defined by
   f (x) = mx + c, m ≠o, is surjective .

Proof: Let us write X for the domain R and Y for the codomain R. So we are to prove that
f :X $\longrightarrow$ Y is surjective.

From the definition of surjective mapping above, it is clear that for f to be surjective, we must have, for any y∈Y,

$$f(x) = y.$$

Now f(x) = y $\Rightarrow$ mx + c = y

$$\Rightarrow x = \frac{y-c}{m}$$

Then for an arbitrary y∈R, take

$$x = \frac{y-c}{m}$$

Therefore we have

$$f(x) = f\left(\frac{y-c}{m}\right) = m\left(\frac{y-c}{m}\right) + c$$
$$= y - c + c$$
$$\Rightarrow f(x) = y$$

Hence f is surjective.

2. The mapping f : N $\rightarrow$ N (from the set of natural numbers to the set of natural numbers) given by f(x) = 2x is not surjective because nothing in N can be mapped, for example, to 5 by the function.
   However the mapping f(x) = 2x from the set of natural numbers N to the set of non-negative even integers is a surjective mapping.
   It should be noted that the definition of surjective (onto) mapping implies that **every element** of the codomain of the function is mapped **to** by **at least** one element of the domain. This means that the image and the codomain of the function are equal.

3. Prove f : Z x Z $\rightarrow$ Z given by
   f (m , n) = 2m – n
   is surjective. (Z is set of integers).
   Let y be an arbitrary element of Z.
   Then f(m , n) = y ⇒ 2m – n = y
   Take m = o.
   Then 2m – n =o – n = y
   ⇒ n = - y.

So that $(o, -y) \in Z \, x \, Z$ because o$\in Z \; and \; n = -y \in Z.$

Hence $f(m, n) = f(o, -y)$
$= 2(o) - (-y) = y$
$\Rightarrow f(o, -y) = y$.

Therefore f is surjective(onto).
From the examples (1) and (3) we see that to prove
$f : X \longrightarrow Y$ is surjective, we,
(1). Pick any $y \in Y$ (that is y is an arbitrary element of Y).
$y \in Y$ being arbitrary ensures that the proof is general for all elements of Y
(2). Find an $x \in X$ such that $f(x) = y$.

This procedure allows us to deal with infinite sets like N and Z above without the need for ascertaining the composition of every element of Y.

GOLDBACH'S CONJECTURE STATES:

Every even integer greater than 2 can be expressed as the sum of two primes.

A careful study of the statement clearly shows that there are two sets:
(1) A set G of all even integers greater than 2, that is, the set (4,6,8,10……) = [2n], n=2,3…

And,

(2) A set X of **sum-pairs** of prime integers, where a **sum-pair** means that if p_1 and p_2 are prime numbers, then $p_1 + p_2$ is a **sum-pair.**

The Goldbach's set is an infinite set of even integers greater than 2 and the set of prime integers is also an infinite set.

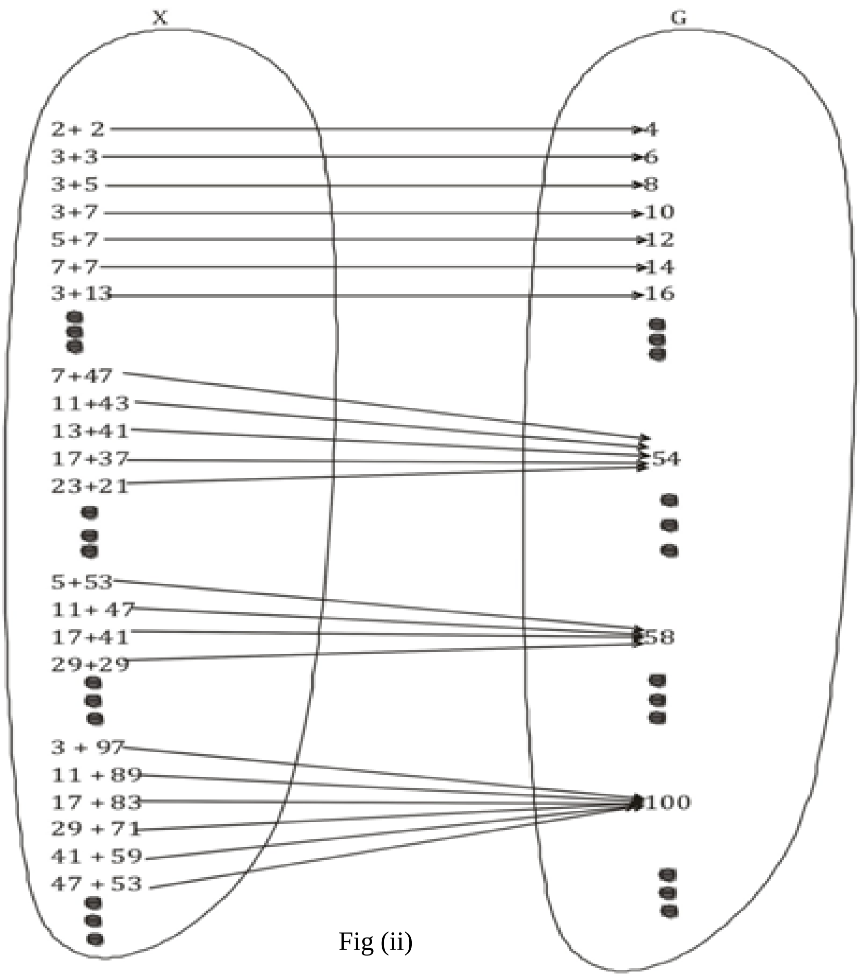


Fig (ii)

Fig (ii) shows an elementary picture of the relation between the two sets.

The figure displays a few **sum-pairs** of prime integers and their corresponding even integers. It is clear from it that a mapping of the set of **sum-pairs** of prime integers into the corresponding even integers of Goldbach's set is a many-to-one mapping. Therefore the mapping is not injective and, by the same token, it is not bijective.

We are therefore left with the possibility that it is a surjective (onto) mapping. Obviously what Goldbach's conjecture seeks to ascertain is whether every member of the Goldbach's set G is mapped onto by at least a member of set X because the relationship between two sets is achieved by mapping one to the other.
Hence the conjecture will be proved if we can produce a surjective (onto) mapping from X to G.
Now it is a known fact that the sum of two prime integers, both greater than 2, is always an even integer.
This is a very important fact which follows from the fact that prime integers are odd integers.
We are now set to prove the Goldbach conjecture:

Every even integer greater than 2 can be expressed as the sum of two primes.

The Proof

Definition: $\varphi : X \rightarrow Y$ *is surjective (onto) mapping*
If $\forall\, y \in Y\ \exists\ x \in X$ *such that* $\varphi(x) = y$.

Let {P} be the infinite set of prime integers and let G = {2n} be the infinite set of even integers greater than 2.

Let φ : P x P $\rightarrow$ G be given by:
φ (r, s) = r + s for $r \geq 2$, $s \geq 2$ and $r, s \in P$
but r + s $\notin$ P because r + s is even.

Take r = 7 ∈ P.

Then for an arbitrary y ∈ G, we have,

$$\varphi\ (r\,, s\,) =\ \varphi\,(7\,, s\,) = y\ \Rightarrow 7 + s = y$$

So that s = y - 7 ∈ P.
∴(7 , y – 7) ∈ P x P because 7 is a prime and s = y – 7 is also a prime.

$$\begin{aligned} \text{Hence}\ \ \varphi(r\,, s\,) &=\ \varphi\,(7, y - 7) \\ &=\ 7 + y\ - 7 \\ &=\ 7 - 7 + y \\ \Rightarrow\ \varphi\,(r\,, s\,) &=\ y \end{aligned}$$

Therefore φ is a surjection and hence the conjecture is proved.

Q. E. D

II. Discussion

The seemingly simple proof above has been made possible by appeal to mapping under Set Theory, thanks to the German mathematician, George Ferdinand Cantor (1845-1918) who invented set theory and the 20thcentury group of mathematicians known as Nicholas Bourbaki whose works brought to light the ideas of injective, bijective and surjective mappings. In order to make progress in tackling the problem the researcher first removed the word 'conjecture' by agreeing with the great Euler that it is a theorem even though he could not prove it. That done, the researcher shifted the problem away from Number Theory to Algebra, shoved it into Set Theory under mapping and proved it via surjective mapping.

The great mathematician Euler and his friend Goldbach cannot be blamed for not being able to prove the conjecture because they lived in the 18th century when set theory had not been invented.

The beauty of surjective mapping is that it makes it unnecessary to consider the composition of the even integers because $y \in Y$ of the surjective mapping is an arbitrary (even integer) element of Y. This means that it is immaterial whether an even integer in the Goldbach's set is a one digit, two digits or even a string of so many digits from Lagos to London for the proof of the conjecture. What matters is that it is an even integer.
This obviously misled some researchers who had attempted to prove or disprove the conjecture by the use of computers or by appeal to sophisticated mathematics.

III. Conclusion And Recommendations

This research has shown that the elusive Goldbach's conjecture can be proved without resort to sophisticated mathematics or even the use of brute force of computers. The proof confirms Euler's description of the conjecture as a theorem. It should therefore be taken to be so. The power of surjective mapping in dealing with finite and infinite sets made the proof simple. This should be explored when such cases arise that involve sets.

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