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Contribution of Fixed point theorem for Compact Metric Spaces.

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Abstract. *In the present study, the aim was made at studying the compact space and fixed point theorem in the spaces. The present paper may help to understand the fixed point with the help of Compact metric spaces.*
Keywords*: Euclidean Space, complete metric Space, Compact Space, Compact metric Space.*

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I. Introduction

In mathematics and more specifically in general topology, **Compactness** is a property that generalizes the notion of a subset of Euclidean space being closed and bounded, i.e. every sequence of points must have an infinite subsequence that converges to some point of the space is called compact space. The Heine-Borel theorem states that a subset of Euclidean space is compact in the sequential sense if and only if it is closed and bounded.

The term compact was introduced in mathematics by **Maurice Frechet** in 1904.Compactness is the more general situation plays an important role in mathematical analysis, because many classical and important theorem of 19^{th} century analysis such as the extreme value theorem are easily generalized to this situation. The term compact set is sometimes a synonym for compact space, but usually refers to a compact subspace of a topological space.

Definition.1: A metric space X is compact if every open cover of X has a finite sub cover.

Definition.2: A metric space X is sequentially compact if every sequence of points in X has a convergent subsequence converging to a point in X.

Definition.3: Assuming the axiom of choice, the following are equivalent:

(a) A topological space X is Compact.
(b) Every open cover of X has a finite sub cover.
(c) X has a sub-base such that every cover of the space by members of the sub-base has a finite sub cover.
(d) Any collection of closed subsets of X with the finite intersection property has nonempty intersection.
(e) Every net on X has a convergent subnet.
(f) Every filter on X has a convergent refinement.
(g) Every ultra filter on X converges to at least one point.
(h) Every infinite subset of X has a complete accumulation point.

Definition.4: For any metric space $(x, \rho)$the following are equivalent:

(a) $\rho(x, y)$, is compact.
(b) $\rho(x, y)$, is complete and totally bounded.
(c) $\rho(x, y)$, is sequentially compact i.e every sequence in X has a convergent subsequence whose limit is in X
(d) $\rho(x, y)$, is limit point compact;that is,every infinite subset of X has at least one limit point in X
(e) $\rho(x, y)$, is an image of a continuous function from the Cantor set.

Properties of Compact Spaces

A continuous image of a compact space is compact. This implies the extreme value theorem: a continuous real-valued function on a nonempty compact space is bounded above and attains its super mum. As a sort of converse to the above statement, the pre-image of a compact space under a proper map is compact.

Compact Spaces and its operation.

A closed subset of a compact space is compact and a finite union of compact set is compact .The product of any collection of compact spaces is compact.(This is Tychonoffs Theorem, which is equivalent to the axiom of choice.

Every topological space X is an open dense subspace of a compact space having at most one point more than X by the Alexandroff one point compactification. By the same construction, every locally compact Hausdorff space X is an open dense subspace of a compact Hausdorff space having at most one point more than X.

Ordered compact spaces:
A nonempty compact subset of the real numbers has a greatest element and a least element .Let X be a simply ordered set endowed with the order of topology, Then X is compact if and only if X is a complete lattice (i.e all subsets have Suprema and infima).

Definition.5: A metric space M is called complete if every Cauchy sequence of points in M has a limit alternatively, a complete metric space in which every Cauchy sequence is convergent.

Theorem.6: Let f and g are two continuous mappings from the complete metric space X into itself. If for given $\varepsilon > 0$, there exists a $\delta > 0$ such that,

$\varepsilon \le \max \{ \rho(x, y), \rho(x, f(x)), \rho(y, g(y)), \rho(x, g(y)), \rho(y, f(x)) \} < \varepsilon + \delta \Rightarrow \rho\{ f(x), f(y)\}$
$\le\alpha \max\{\rho(x, y), \rho(x, f(x)), \rho(y, g(y)), \rho(x, g(y)), \rho(y, f(x)) \}$ .... (1)

where $\varepsilon < \alpha < \varepsilon < \delta \quad \forall\ x, y \in X$ and $0 < \alpha < 1$. then f and g have a common fixed point which is unique.

Proof : Let us take x_0 be an arbitrary fixed point of X. We now define $f(x_0) = x_1$, $x_1 \in X$, $x_2 = g(x_1)$ $x_3 = f(x_2)$. In general $x_{n+1} = f(x_n)$ and $x_{n+2} = g(x_{n+1})$ where n = 0, 1, 2, 3..................(2)

Now we have from the inequality(1),

$\rho(x_1, x_2) = \rho \{ f (x_0), g(x_1) \}$
$\le \alpha \max \{\rho(x_0, x_1), \rho(x_0, f(x_0)), \rho(x_1, g(x_1)), \rho(x_0, g(x_1)), \rho(x_1, f(x_0))\}$
$\le \alpha \max \{ \rho(x_0, x_1), \rho(x_0, x_1), f (x_1, x_2), \rho(x_0, x_2), \rho(x_1, x_1) \}$
$\Rightarrow \rho(x_1, x_2) \le \alpha\rho (x_0, x_1)$...(3)

Again $\rho(x_2, x_3) = \rho \{ g(x_1) f(x_2) \}$
$= \rho \{ f(x_2) g(x_1) \}$
$\le\alpha \max \{\rho(x_2, x_1), \rho(x_2, f(x_2)), \rho(x_1, g(x_1)), \rho(x_2, g(x_1)), \rho(x_1, f(x_2))\}$
$\le \alpha \max \{\rho(x_1, x_2), \rho(x_2, x_3), \rho (x_1, x_2), \rho(x_2, x_2), \rho(x_1, x_3))\}. \le \alpha\rho (x_1, x_2)$
$\Rightarrow \quad \rho(x_2, x_3) \le \alpha\rho (x_1, x_2) \le \alpha\alpha\rho (x_0, x_1)$
$\Rightarrow \quad \rho(x_2, x_3) \le \alpha^2\rho (x_0, x_1)$ ....(4)

Similarly $\rho(x_3, x_4) \le \alpha^3\rho (x_0, x_1)$...(5)

Continuing this process, we get,

$\rho(x_{n+1},x_{n+2})\le\alpha^{n+1}\rho(x_0,x_1)$...(6)

Since α is lies between ε and $\varepsilon + \delta$ which are so small then α is so small, and hence $\{x_n\}$ is a Cauchy sequence.

Since X is complete there exists a point z such that $\lim_{n\to\alpha} x_n = z$. Hence $\{x_n\}$ converges to a point z.

Now using the inequality (1) we have,

$\rho\{ f(z), x_{n+2} \} = \rho \{ f(z, g(x_{n+1})\}$
$\le \alpha \max \{\rho(z, x_{n+1}), \rho(z, f(z)), \rho((x_{n+1}), g(x_{n+1})), \rho(z, g(x_{n+1})), \rho(x_{n+1}), f(z))\}$
$\le \alpha \max \{f (z, x_{n+1}), \rho(z, f(z)), \rho (x_{n+1}, x_{n+2}), \rho(z, x_{n+2}), \rho(x_{n+1}, f(z))\}$.

Letting $n \to \alpha$ we get, $\rho(f(z), z) \le \alpha\rho (f(z), z)$
$\Rightarrow \quad \rho\{ f(z), z) (1 - \alpha) \le 0$
$\Rightarrow \quad f(z) = z$...(7)

Similarly we can show that $g(z) = z$ is a common fixed point of f and g.

Let us suppose that $w(w \ne z)$ is another common fixed point of f and g, then we have,

$\rho(z, w) = \rho (fz, gw)$
$\le \alpha \max \{ \rho(z, w), \rho (z, f(z)), \rho(w, g(w)), \rho(z, g(w)), \rho(w, f(z)) \}$
$\le \alpha \max \{ \rho(z, w), \rho (z, z), \rho(w, w), \rho(z, w), \rho(w, z) \}$
$\le \alpha \max \{ \rho(z, w), 0, 0, \rho (z, w), \rho(z, w) \} \le \alpha\rho(z, w)$
$\Rightarrow \quad \rho (z, w) (1 - \alpha) \le 0$
$\Rightarrow \quad z = w$, Hence this complete the proof of the theorem.

Edelstein M. [5] proves the following theorem.

Theorem.7: If T is a mapping of the compact metric space X into itself, satisfying the inequality
$\rho(Tx, Ty) < \rho(x, y)$for all distinct x, y in X then T has a unique fixed point.

More recently **Fisher B**. [2] has proved the following theorem.

Theorem.8: If T is a continuous mapping of a compact metric space X into itself, satisfying the inequality

$\rho(Tx, Ty) < ½ \{ \rho(x, Tx) + \rho(y, Ty)\}$

for all distinct x, y in X, then T has a unique fixed point.

Theorem.9: If T is a continuous mapping of a compact metric space X into itself, satisfying the inequality

$\rho(Tx, Ty) < ½ \{\rho(x, Ty) + \rho(y, Tx)\}$

for all distinct x, y in X, then T has a unique fixed point.

We will now prove the following theorem which includes each of these theorems as special cases:

Theorem.10: If T is a continuous mapping of a compact metric space X into itself, satisfying the inequality

$\rho(Tx, Ty) < a\rho(x, y) + b\{ \rho(x, Tx) + \rho(y, Ty)\}+c\{ \rho(x, Ty) + \rho(y, Tx)\}$

for all distinct x, y∈ in X, where

$a + 2(b + c) = 1, b + c < 1, a + 2c \leq 1, c \geq 0$, then T has a unique fixed point

Proof: Define a function f on X , $f(x) = \rho(x, Tx)$, for all x∈X.

Since ρ and T are continuous functions, it follows that f is a continuous function on X. Since X is compact there exists a point z∈X such that, $f(z) = \inf \{ f(x) : x \in X\}$

If, $f(z) \neq 0$, it follows that $Tz \neq z$ and so, $f(Tz) = \rho(Tz, T^2z)$

$< a\rho\{ (z, Tz) + b\{ \rho(z, Tz) + \rho(Tz, T^2z)\} + c\rho(z, T^2z)$

$\leq (a+b+c) \rho(z, Tz) + (b+c) \rho(Tz, T^2z)$.

Since $c \geq 0$. It follows that since $b+c < 1$,

$$\Rightarrow \qquad \rho(Tz, T^2z) < \frac{a + b + c}{1 - b - c} \rho(z, Tz) = \rho(z, Tz)$$

i.e, $f(Tz) < f(z)$.

This contradicts the definition of z so that we must have Tz=z and z is then a fixed point of T.

Now suppose that T has a second distinct fixed point z.

Then, $\rho(z, z') = \rho(Tz, Tz')$

$< a\rho\{ (z, z') + b\{ \rho(z, Tz) + \rho(z', Tz')\} + c\{ \rho(z, Tz')+\rho(z', Tz)\}= (a + 2c) \rho(z, z')$

giving a contradiction.

Since $a+2c \leq 1$. It follows that the fixed point is unique, completing the proof of the theorem.

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