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On The Positive Pell Equation $y^2 = 5x^2 + 4$

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Abstract:*The binary quadratic equation represented by the positive Pellian* $y^2 = 5x^2 + 4$ *isanalyzed for its distinct integer solutions. A few interesting relations among the solutions are given. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbola, parabolas and special Pythagorean triangle.*

Keywords:*Binary quadratic, hyperbola,intergral solutions, parabola, pell equation.*
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I. Introduction

The binary quadratic equation of the form $y^2 = Dx^2 + 1$, where D is non-squarePositive integer has been studied by various mathematicians for its non-trivial integral solutions when D takes different integral values [1-4]. For an extensive review of various problems, one may refer [5-20]. In this communication, yet another interesting hyperbola given by $y^2 = 5x^2 + 4$considered and infinitely many integer solutions are obtained. A few interesting properties among the solutions are presented.

II. Method Of Analysis

Consider the binary quadratic equation

$y^2 = 5x^2 + 4$(1)

whose smallest positive integer solution is$x_0 = 1, y_0 = 3$

To obtain the other solutions of (1), consider the pell equation $y^2 = 5x^2 + 1$whosesolution is given by

$\tilde{x}_n = \frac{1}{2\sqrt{5}} g_n, \ \tilde{y}_n = \frac{1}{2} f_n$

where,

$f_n = \left(9 + 4\sqrt{5}\right)^{n+1} + \left(9 - 4\sqrt{5}\right)^{n+1},$

$$g_n = \left(9 + 4\sqrt{5}\right)^{n+1} - \left(9 - 4\sqrt{5}\right)^{n+1}, n = 0,1,2 \ldots$$

Applying Brahamagupta lemma between $(x_0, y_0)$and$(\tilde{x}_n, \tilde{y}_n)$, the other integer solutions of (1) are given by

$x_{n+1} = \frac{1}{2} f_n + \frac{3}{2\sqrt{5}} g_n,$

$y_{n+1} = \frac{3}{2} f_n + \frac{\sqrt{5}}{2} g_n$

The recurrence relations satisfied by the solutions x and y are given by

$x_{n+1} - 18x_{n+2} + x_{n+3} = 0$

$y_{n+1} - 18y_{n+2} + y_{n+3} = 0$

Some numerical examples of $x \ and \ y$ satisfying (1) are given in Table(1) below

Table1: Examples

n	x_n	y_n
0	1	3
1	21	47
2	377	843
3	6765	15127
4	121393	271443

From the above table, we observe some interesting relations among the solutions which are presented below

(1) x_n and y_n are always odd

(2) Each of the following expressions is a nasty number

- $\frac{9x_{2n+3} - 141x_{2n+2} + 48}{4}$
- $\frac{3x_{2n+4} - 843x_{2n+2} + 288}{24}$
- $\frac{3y_{2n+3} - 105x_{2n+2} + 36}{3}$
- $\frac{9y_{2n+4} - 5655x_{2n+2} + 1932}{161}$
- $\frac{141x_{2n+4} - 252\,9x_{2n+3} + 48}{4}$
- $\frac{47y_{2n+2} - 5x_{2n+3} + 36}{3}$
- $141y_{2n+3} - 315x_{2n+3} + 12$
- $\frac{47y_{2n+4} - 1885x_{2n+3} + 36}{3}$
- $\frac{2529y_{2n+2} - 15x_{2n+4} + 1932}{161}$
- $281y_{2n+3} - 35x_{2n+4} + 12$
- $2529y_{2n+4} - 5655x_{2n+4} + 12$
- $\frac{63y_{2n+2} - 3y_{2n+3} + 48}{4}$
- $\frac{377y_{2n+2} - y_{2n+4} + 288}{24}$
- $\frac{1131y_{2n+3} - 63y_{2n+4} + 108}{4}$

(3) Each of the following expression is a cubical integer

- $(3x_{3n+4} - 47x_{3n+3}) + 3(3x_{n+2} - 47x_{n+1})$
- $20736(3x_{3n+5} - 843x_{3n+3}) + 62208(3x_{n+3} - 843x_{n+1})$
- $324(3y_{3n+4} - 105x_{3n+3}) + 972(3y_{n+2} - 105x_{n+1})$
- $103684(3y_{3n+5} - 1885x_{3n+3}) + 311052(3y_{n+3} - 1885x_{n+1})$
- $(47x_{3n+5} - 843x_{3n+4}) + 3(47x_{n+3} - 843x_{n+2})$
- $324(47y_{3n+3} - 5x_{3n+4}) + 972(47y_{n+1} - 5x_{n+2})$
- $4(47y_{3n+4} - 105x_{3n+4}) + 12(47y_{n+2} - 105x_{n+2})$
- $324(47y_{3n+5} - 1885x_{3n+4}) + 972(47y_{n+3} - 1885x_{n+2})$
- $103684(843y_{3n+3} - 5x_{3n+5}) + 311052(843y_{n+1} - 5x_{n+3})$
- $324(843y_{3n+4} - 105x_{3n+5}) + 972(843y_{n+2} - 105x_{n+3})$
- $4(843y_{3n+5} - 1885x_{3n+5}) + 12(843y_{n+3} - 1885x_{n+3})$
- $(21y_{3n+3} - y_{3n+4}) + 3(21y_{n+1} - y_{n+2})$
- $20736(377y_{3n+3} - y_{3n+5}) + 62208(377y_{n+1} - y_{n+3})$
- $(377y_{3n+4} - 21y_{3n+5}) + 3(377y_{n+2} - 21y_{n+3})$

(4) Relation among the solution

- $x_{n+3} = 18x_{n+2} - x_{n+1}$
- $4y_{n+1} = x_{n+2} - 9x_{n+1}$
- $4y_{n+2} = 9x_{n+2} - x_{n+1}$

- $4y_{n+3} = 161x_{n+2} - 9x_{n+1}$
- $72y_{n+1} = x_{n+3} - 161x_{n+1}$
- $8y_{n+2} = x_{n+3} - x_{n+1}$
- $72y_{n+3} = 161x_{n+3} - x_{n+1}$
- $9y_{n+1} = y_{n+2} - 20x_{n+1}$
- $9y_{n+3} = 161y_{n+2} + 20x_{n+1}$
- $161y_{n+1} = y_{n+3} - 360x_{n+1}$
- $4y_{n+1} = 9x_{n+3} - 161x_{n+2}$
- $4y_{n+2} = x_{n+3} - 9x_{n+2}$
- $4y_{n+3} = 9x_{n+3} - x_{n+2}$
- $9y_{n+2} = 20x_{n+2} + y_{n+1}$
- $y_{n+3} = y_{n+1} + 40x_{n+2}$
- $y_{n+3} = 9y_{n+2} + 20x_{n+2}$
- $161y_{n+2} = 20x_{n+3} + 9y_{n+1}$
- $161y_{n+3} = 360x_{n+3} + y_{n+1}$
- $9y_{n+3} = y_{n+2} + 20x_{n+3}$
- $y_{n+3} = 18y_{n+2} - y_{n+1}$

Remarkable Observations

Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbolas which are presented in Table2 below.

Table 2: Hyperbolas

S. No.	(X,Y)	Hyperbola
1	$(21x_{n+1} - x_{n+2}, 3x_{n+2} - 47x_{n+1})$	$Y^2 - 5X^2 = 256$
2	$(377x_{n+1} - x_{n+3}, 3x_{n+3} - 843x_{n+1})$	$Y^2 - 5X^2 = 82944$
3	$(47x_{n+1} - y_{n+2}, 3y_{n+2} - 105x_{n+1})$	$Y^2 - 5X^2 = 1296$
4	$(843x_{n+1} - y_{n+3}, 3y_{n+3} - 1885x_{n+1})$	$Y^2 - 5X^2 = 414736$
5	$(377x_{n+2} - 21x_{n+3}, 47x_{n+3} - 843x_{n+2})$	$Y^2 - 5X^2 = 256$
6	$(3x_{n+2} - 21y_{n+1}, 47y_{n+1} - 5x_{n+2})$	$Y^2 - 5X^2 = 1296$
7	$(47x_{n+2} - 21y_{n+2}, 47y_{n+2} - 105x_{n+2})$	$Y^2 - 5X^2 = 16$
8	$(843x_{n+2} - 21y_{n+3}, 47y_{n+3} - 1885x_{n+2})$	$Y^2 - 5X^2 = 1296$
9	$(3x_{n+3} - 377y_{n+1}, 843y_{n+1} - 5x_{n+3})$	$Y^2 - 5X^2 = 414736$
10	$(47x_{n+3} - 377y_{n+2}, 843y_{n+2} - 105x_{n+3})$	$Y^2 - 5X^2 = 1296$
11	$(843x_{n+3} - 377y_{n+3}, 843y_{n+3} - 1885x_{n+3})$	$Y^2 - 5X^2 = 16$
12	$(3y_{n+2} - 47y_{n+1}, 21y_{n+1} - y_{n+2})$	$5Y^2 - X^2 = 1280$
13	$(3y_{n+3} - 843y_{n+1}, 377y_{n+1} - y_{n+3})$	$5Y^2 - X^2 = 414720$
14	$(47y_{n+3} - 843y_{n+2}, 377y_{n+2} - 21y_{n+3})$	$5Y^2 - X^2 = 1280$

Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabolas which are presented in the Table 3 below.

Table 3: Parabolas

S. No.	(X,Y)	Parabolas
1	$(21x_{n+1} - x_{n+2}, 3x_{2n+3} - 47x_{2n+2} + 16)$	$5X^2 = 8Y - 256$
2	$(377x_{n+1} - x_{n+3}, 3x_{2n+4} - 843x_{2n+2} + 288)$	$5X^2 = 144Y - 82944$
3	$(47x_{n+1} - y_{n+2}, 3y_{2n+3} - 105x_{2n+2} + 36)$	$5X^2 = 18Y - 1296$
4	$(843x_{n+1} - y_{n+3}, 3y_{2n+4} - 1885x_{2n+2} + 644)$	$5X^2 = 322Y - 414736$
5	$(377x_{n+2} - 21x_{n+3}, 47x_{2n+4} - 843x_{2n+3} + 16)$	$5X^2 = 8Y - 256$

6	$(3x_{n+2} - 21y_{n+1}, 47y_{2n+2} - 5x_{2n+3} + 36)$	$5X^2 = 18Y - 1296$
7	$(47x_{n+2} - 21y_{n+2}, 47y_{2n+3} - 105x_{2n+3} + 4)$	$5X^2 = 2Y - 16$
8	$(843x_{n+2} - 21y_{n+3}, 47y_{2n+4} - 1885x_{2n+3} + 36)$	$5X^2 = 18Y - 1296$
9	$(3x_{n+3} - 377y_{n+1}, 843y_{2n+2} - 5x_{2n+4} + 644)$	$5X^2 = 322Y - 414736$
10	$(47x_{n+3} - 377y_{n+2}, 843y_{2n+3} - 105x_{2n+4} + 36)$	$5X^2 = 18Y - 1296$
11	$(843x_{n+3} - 377y_{n+3}, 843y_{2n+4} - 1885x_{2n+4} + 4)$	$5X^2 = 2Y - 16$
12	$(3y_{n+2} - 47y_{n+1}, 21y_{2n+2} - y_{2n+3} + 16)$	$X^2 = 40Y - 1280$
13	$(3y_{n+3} - 843y_{n+1}, 377y_{2n+2} - y_{2n+4} + 288)$	$X^2 = 720Y - 414720$
14	$(47y_{n+3} - 843y_{n+2}, 377y_{2n+3} - 21y_{2n+4} + 16)$	$X^2 = 40Y - 1280$

Consider$p = x + y, q = x$. Observe that$p > q > 0$. Treatp, qas the generators of the Pythagorean triangle $T(X, Y, Z)$,where $X = 2pq, Y = p^2 - q^2, Z = p^2 + q^2$-

Let $AandP$denotes the area andperimeter of the Pythagorean triangle.

Then the following interesting relations are observed.

a) $-2X + 5Y - 3Z = 8$

b) $2Z - 7X + 20\frac{A}{P} = 8$

c) $xy = \frac{2A}{P}$

III. Conclusion

In this paper, we have presented infinitely many integer solutions for the hyperbolarepresented by the positive pell equation $y^2 = 5x^2 + 4$. As the binary quadratic Diophantine equation are rich in variety, one may search for the other choices of negative pell equations and determine their integer solutions along with suitable properties.

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