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Generalized Trapezoids for Granular Representation of Information

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Abstract: *Trapezoidal fuzzy numbers [1] can be used for representing fuzzy sets in most situations. It provides computational ease while it has generality. Its membership function [9] is piecewise linear. However while modelling nonlinear situations trapezoids may not suffice. So to deal with such cases generalized trapezoids are introduced. It is shown that the set of generalized trapezoids is closed under the usual arithmetic operations.*
***Key words**: fuzzy sets, trapezoidal fuzzy number, α- cuts, non linear, granular information.*

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I. Introduction

Trapezoids are ideal for modelling fuzzy sets [4,5,7] in most situations. In his paper Yager[6] clearly works out their properties and brings out their advantages. However in real life non linearity often crops up. In such cases we need a 'generalized version of trapezoid'. That motivates the following definition.

DEFINITION 1: GENERALIZED TRAPEZOID
Let a fuzzy set A be defined by its membership function

$$A(x) = \begin{cases} f_1(x) & x \le l \\ h & l \le x \le u \\ f_2(x) & x \ge u \end{cases} \tag{1}$$

such that A(x) is continuous, $0 < h \le 1$, $f_1: (-\infty, l] \to [0, h]$ is monotonically increasing, $f_1(x) = 0$ for x ∈ $(-\infty, \omega_1)$; $f_2: [u, \infty) \to [0, h]$ is monotonically decreasing, $f_2(x) = 0$ for x∈ (ω_2, ∞). Then such a fuzzy set will be a generalized trapezoid.

Note1: In other words the graph of A(x) looks like a trapezoid with curved sides.

Note2: The definition closely resembles the characterization of a fuzzy number. The only differences are [5]

(i) the definition of fuzzy number allows jump discontinuities in the membership function[2,9] while here we have a membership function which is continuous .

(ii) a fuzzy number should be normal while here A(x) need not be normal.

From the definition the following results can be easily deduced:

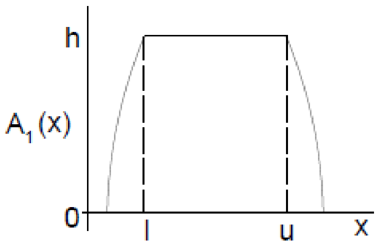
LEMMA 1: A normal generalized trapezoid is a continuous fuzzy number.

LEMMA 2: A continuous fuzzy number is a normal generalized trapezoid.

Examples of Generalized Trapezoids

(i)

$$A_1(x) = \begin{cases} (1 + \cos(p_3\pi\,(x - l)))/2 \ when\ x \,\epsilon\, [\, l - 1/(p_3)\,,\ l] \\ h \qquad , \ l \le x \le u \\ (1 + \cos(p_3\pi\,(x - u)))/2 \ when\ x \,\epsilon\, [\, u\,,\ u + 1/(p_3)\,] \\ 0 \qquad elsewhere. \end{cases}$$



(ii) $A_2(x) = \begin{cases} \frac{x^2}{l^2} & when\ 0 \le x \le l \\ 1 & when\ \ l < x \le u \\ \frac{(p-x)^3}{(p-u)^3} & when\ u < x \le p \\ 0 & elsewhere \end{cases}$

(iii) $A_3(x) = \begin{cases} k_1 e^{-\frac{1}{x-w_1}} & when\ w_1 < x \le l \\ 1 & when\ l < x \le u \\ k_2 e^{-\frac{1}{x-w_2}} & when\ u < x < w_2 \\ 0 & elsewhere \end{cases}$

where $k_1 = e^{\frac{1}{l-w_1}}$ and $k_2 = e^{\frac{1}{u-w_2}}$.

II. Preliminary Definitions:

Definition 2.1: Fuzzy sets:
A fuzzy set on a domain U is defined by its membership function from U to [0, 1] is meant a function
A: U → [0,1]. 'A' is called the membership function, A(x) is called membership grade of x in A. we also write it as A = { (x, A(x)) : x ∈ U} [7,5,4]
Definition 2.2: $\alpha - cuts$ of a fuzzy sets:
Given a fuzzy set A on U, a domain and a number α in I ,such that $0 < \alpha \le 1$, then associate a crisp set with A , denoted by A_α , defined as A_α={ x ∈ U/ A(x) ≥ α } is called the $\alpha - cut$ of A [5].For each α, we obtain an α-cut of A
Definition 2.3:
Generally, a generalized fuzzy number A is described as any fuzzy subset of the real line R, whose membership function $\mu_A(x)$ satisfies the following conditions,[8]
(i) $\mu_A(x)$ is a continuous mapping from R to the closed interval [0, ω], $0 < \omega \le 1$,
(ii) $\mu_A(x) = 0$, for all x ϵ $(-\infty, a]$,
(iii) $\mu_L(x) = L(x)$ is strictly increasing on [a, b],
(iv) $\mu_A(x) = \omega$, for all [b, c], as ω is a constant and $0 < w \le 1$,
(v) $\mu_R(x) = R(x)$ is strictly decreasing on [c, d],
(vi) $\mu_A(x) = 0$, for all x ϵ $[d, \infty)$,
where a,b,c,d are real numbers such that a < b ≤ c < d.

Definition 2.4: A normal fuzzy number A with shape function [8,9]

$$\mu_A = \begin{cases} \left(\frac{x-a}{b-a}\right)^n, & x \in [a,b) \\ \omega, & x \in [b,c] \\ \left(\frac{d-x}{d-c}\right)^n, & x \in (c,d] \\ 0, & \text{otherwise} \end{cases}$$

where n > 0, will be denoted by

$$A = (a, b, c, d)_n.$$

If A be non-normal fuzzy number, it will be denoted by

$$A = (a, b, c, d; \omega)_n.$$

If n = 1, we simply write A = (a, b, c, d), which is known as a normal trapezoidal fuzzy number [1]

III. Arithmetic Operations On Generalized Trapezoids

Next we examine the effect of arithmetic operations on generalized trapezoids. Let us recall that arithmetic operations on fuzzy numbers [3] are defined in terms of arithmetic operations on their α-cuts [5].For a fuzzy number an α-cut is always an interval. So interval arithmetic has to be used.

Let $*$ denote any of the four basic operations addition, subtraction, multiplication or division. Then the interval operation may be defined by [5]

$$[a,b] * [d,e] = \{f * g | a \leq f \leq b, d \leq g \leq e\} \quad (2)$$

From above definition it can be verified that

$$[a,b] + [d,e] = [a+d, b+e] \quad (3)$$

$$[a,b] - [d,e] = [a-e, b-d] \quad (4)$$

$$[a,b].[d,e] = [\min(ad, ae, bd, be), \max(ad, ae, bd, be)] \quad (5)$$

and if $0 \notin [d,e]$ then $\frac{[a,b]}{[d,e]} = [\min\left(\frac{a}{d}, \frac{a}{e}, \frac{b}{d}, \frac{b}{e}\right), \max\left(\frac{a}{d}, \frac{a}{e}, \frac{b}{d}, \frac{b}{e}\right)]$ (6)

So if A and B are fuzzy numbers, then A*B is defined via its alpha cut as follows:

$$^{\alpha}(A * B) = {}^{\alpha}(A) * {}^{\alpha}(B) \quad (7)$$

Therefore we may write

$$A * B = \cup_{\alpha \in [0,1]} {}^{\alpha}(A * B) \quad (8)$$

LEMMA 3: If k is a crisp number and A is a fuzzy number $^{\alpha}(kA) = k\, ^{\alpha}(A)$.

Proof: Let $^{\alpha}(A) = [l, u]$ and $*$ denote multiplication.

By definition , $^{\alpha}(kA) = {}^{\alpha}(k) * {}^{\alpha}(A) = k\, ^{\alpha}(A)$.

That is $^{\alpha}(kA) = [kl, ku]$. (9)

Klir and Yuan[5] give the following theorem:

THEOREM 1: Let $*$ denote addition, subtraction, multiplication or division and let A, B denote continuous fuzzy numbers. Then the fuzzy set A*B defined as above is a continuous fuzzy number.

Using the above theorem we prove:

THEOREM 2: Let $*$ denote addition, subtraction, multiplication or division and let C, D be generalized trapezoids. Then C*D is also a generalized trapezoid.

Proof: Consider fuzzy sets A and B defined by

$$A(x) = \frac{C(x)}{h_1} \text{ and } B(x) = \frac{D(x)}{h_2} \quad (10)$$

where h_1 and h_2 are the heights of C and D respectively.

Then it is obvious that A and B are normal generalized trapezoids. So by lemma 1, we may conclude that A and B are continuous fuzzy numbers.

By theorem 1, A*B is a continuous fuzzy number. Hence $C * D = h_1 h_2 (A * B)$ is a continuous fuzzy number. Hence by lemma 2, $C * D$ is a generalized trapezoid.

DEFINITION 2: SYMMETRIC GENERALIZED TRAPEZOID

A generalized trapezoid satisfying definition 1 is said to be symmetric about the interval [l,u] if $f_1(l - x) = f_2(u + x)$ for all $x \geq 0$.

Examples: A_1 is an example of symmetric trapezoid. A_3 is also symmetric if $l - w_1 = w_2 - u$.

Note: We can of course have generalized trapezoids which are not symmetric. A_2 is not symmetric.

LEMMA 4: Any generalized trapezoid A is a symmetric generalized trapezoid about [l,u] if and only if each of its alpha cut is symmetric about [l,u].

Proof: Assume A has the following membership function

$$A(x) = \begin{cases} f_1(x) & x \leq l \\ h & l \leq x \leq u \\ f_2(x) & x \geq u \end{cases} \quad \text{(by (1))}$$

such that A(x) is continuous, $0 < h \leq 1$, $f_1: (-\infty, l] \to [0, h]$ is monotonically increasing, $f_1(x) = 0$ for $x \in (-\infty, \omega_1)$; $f_2: [u, \infty) \to [0, h]$ is monotonically decreasing, $f_2(x) = 0$ for $x \in (\omega_2, \infty)$ and further

$$f_1(l - x) = f_2(u + x)$$

Fix the value of α. Let the $\alpha -$ cut of A be [a, b]. It is obvious that $[l, u] \subseteq [a, b]$.

Let $\alpha \leq h$. By definition of α −cut it is necessary $that\ A(a) \geq \alpha$. That is $f_1(a) \geq \alpha$. Let $f_1(a) = \beta \geq \alpha$ and $a = l - x$. Then $f_2(u + x) = f_1(l - x) = \beta$. Hence $f_2(u + x) \geq \alpha$. Hence $u + x \in [a, b]$. So have $u + x \leq b$. We claim $u + x = b$. If $u + x < b = u + y(say)$ then using $f_2(b) \geq \alpha$ we can say $f_1(l - y) = f_2(u + y) \geq \alpha$, which means $l - y < a$ belongs to the$\alpha - cut$. That contradicts our initial assumption that the $\alpha -$ cut of A is [a,b]. So $u + x = b$.

So for each value of $\alpha \in (0, h)$ we have ${}^{\alpha}(A)$ is symmetric about $[l, u]$ and of the form $[l - x, u + x]$.

Conversely if each of the $\alpha -$ cut of A is symmetric about [l,u] since

$$A = \bigcup_{\alpha \in [0,1]} {}^{\alpha}(A)$$

we can argue as follows. Suppose $f_1(l - x) = \beta$.Let β −cut of A be of form [l-y,u+y].Since(l-x) belongs to the β −cut of A it is necessary $l - y \leq l - x$. So $y \geq x$.So $u + y \geq u + x$.

Also $f_2(u + y) \geq \beta$.But f_2 is monotonically decreasing. Hence $u + y \geq u + x$ implies

$f_2(u + x) \geq f_2(u + y)$ which proves that $f_2(u + x) \geq \beta$.

In short

$$f_2(u + x) \geq f_1(l - x)$$

Similarly we can prove that

$$f_2(u + x) \leq f_1(l - x)$$

Combining both inequalities

$$f_2(u + x) = f_1(l - x) \qquad (11)$$

So A is a symmetric generalized trapezoid about [l,u].

THEOREM 3: Let $*$ denote addition or subtraction and let A, B be symmetric generalized trapezoids. Then A*B is also symmetric generalized trapezoid.

Let A, B be symmetric generalized trapezoids. Then A+B is symmetric generalized trapezoid.

Proof: Assume A and B have the following membership functions[5]:

$$A(x) = \begin{cases} f_1(x) & x \leq l \\ h & l \leq x \leq u \\ f_2(x) & x \geq u \end{cases} \qquad \text{(by (1))}$$

such that A(x) is continuous, $0 < h \leq 1$, $f_1: (-\infty, l] \rightarrow [0, h]$ is montonically increasing, $f_2: [u, \infty)] \rightarrow [0, h]$ is monotonically decreasing and further $f_1(l - x) = f_2(u + x)$.

$$B(x) = \begin{cases} g_1(x) & x \leq L \\ H & L \leq x \leq U \\ g_2(x) & x \geq U \end{cases} \qquad \text{(by (1))}$$

such that B(x) is continuous, $0 < H \leq 1$, $g_1: (-\infty, L] \rightarrow [0, H]$ is montonically increasing, $g_2: [U, \infty)] \rightarrow [0, H]$ is monotonically decreasing and further $g_1(L - x) = g_2(U + x)$.

By previous lemma 2, for each value of $\alpha \in [0, h]$ we have ${}^{\alpha}(A)$ is symmetric about $[l, u]$ and of the form $[l - x, u + x]$ and for each value of $\alpha \in [0, H]$ we have ${}^{\alpha}(B)$ is symmetric about [L,U] and of the form $[L - w, U + w]$

Let $0 \leq \alpha \leq \min(h, H)$. Let us calculate ${}^{\alpha}(A * B) = {}^{\alpha}(A) * {}^{\alpha}(B)$ for each of the operations.

(i) Addition : Then,

$$\begin{aligned} [l - x, u + x] + [L - w, U + w] &= [(l - x) + (L - w), (u + x) + (U + w)] \qquad \text{(by (3))} \\ &= [l + L - (x + w), u + U + (x + w)] \end{aligned}$$

So for every α in [0,min(h,H)] we have ${}^{\alpha}(A + B)$ to be symmetric about $[l + L, u + U]$. Hence

$$A + B = \bigcup_{\alpha \in [0,1]} {}^{\alpha}(A + B)$$

is also symmetric about $[l + L, u + U]$.

Hence A+B is a symmetric generalized trapezoid.

(ii)Subtraction: Here

$$\begin{aligned} [l - x, u + x] - [L - w, U + w] &= [(l - x) - (U + w), (u + x) - (L - w)] \qquad \text{(by (4))} \\ &= [l - U - (x + w), u - L + (x + w)] \end{aligned}$$

So for every α in [0,min(h,H)] we have ${}^{\alpha}(A - B)$ to be symmetric about $[l - U, u - L]$. Hence

$$A - B = \bigcup_{\alpha \in [0,1]} {}^{\alpha}(A - B)$$

is also symmetric about $[l - U, u - L]$. Hence, A- B is symmetric generalized trapezoid.

THEOREM 4: If k is a crisp number and A is a symmetric generalized trapezoid. Then kA is also a symmetric generalized trapezoid.

Proof: If A is symmetric about [l,u] by previous lemma for each value of $\alpha \in (0,h)$, we have ${}^{\alpha}(A)$ is symmetric about $[l,u]$ and of the form $[l-x, u+x]$.
So by lemma 3

$${}^{\alpha}(kA) = k\,{}^{\alpha}(A) = k[l-x, u+x]$$

If $k \geq 0$

$$k[l-x,u+x] = [k(l-x), k(u+x)]$$
$$= [kl-kx, ku+kx]$$

.
is symmetric about $[kl, ku]$
Hence ,

$$kA = \bigcup_{\alpha\in[0,1]} {}^{\alpha}(kA)$$

is also symmetric about $[kl, ku]$
If $k < 0$

$$k[l-x,u+x] = [k(u+x), k(l-x)]$$
$$= [ku+kx, kl-kx]$$

is symmetric about $[ku, kl]$.
Hence,

$$kA = \bigcup_{\alpha\in[0,1]} {}^{\alpha}(kA) \tag{16}$$

is also symmetric about $[ku, kl]$

DEFINITION 3:
If A is a generalized trapezoid such that there exists a positive number ϵ with $Supp(A) \subset (\epsilon, \infty)$ then A is said to be a positive generalized trapezoid.

THEOREM 5: If k is a positive crisp number and A is a positive symmetric generalized trapezoid. Then kA is also a positive symmetric generalized trapezoid.
Proof:
By theorem 4, kA is a symmetric generalized trapezoid .If A is positive then there exists an $\epsilon > 0$ such that $Supp(A) \subset (\epsilon, \infty)$ .Then $Supp(A) \subset (k\epsilon, \infty)$ where $k\epsilon > 0$.Hence kA is also a positive symmetric generalized trapezoid.

THEOREM 6: Let A, B be positive symmetric generalized trapezoids. Then A+B is positive symmetric generalized trapezoid.
Proof: By theorem 3, A+B is symmetric. If there exist a positive numbers ϵ_1, ϵ_2 with $Supp(A) \subset (\epsilon_1, \infty)$ and $Supp(B) \subset (\epsilon_2, \infty)$ it is clear $Supp(A+B) \subset (\epsilon_1 + \epsilon_2, \infty)$ .Then A+B is positive. Hence A+B is positive symmetric generalized trapezoid.

THEOREM 7: If $w_1, w_2 \dots, w_n$ are positive crisp numbers and $A_1, A_2 \dots A_n$ are positive symmetric generalized trapezoids, then $w_1 A + w_2 A_2 + \cdots w_n A_n$ also a positive symmetric generalized trapezoid.
Proof:
Result follows from theorems 5 and 6

IV. Conclusion

Generalized trapezoids are very useful when ordinary trapezoids do not quite fit the real life situation. So they provide additional choices to the mathematician. Also note that product of two trapezoids need not be a trapezoid. But theorems 2 and 3 show that generalized trapezoids has some useful properties. This is an added advantage when working with generalized trapezoids. Generalized trapezoids can be ordered using their centroids. For symmetric generalized trapezoid symmetric about [l,u] the centroid will be the midpoint $\frac{l+u}{2}$. Hence ordering symmetric generalized trapezoids and applying OWA operations[6] should be relatively easy.

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