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Δ^m-statistically pre-Cauchy sequences of fuzzy numbers

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***Abstract**: In this paper, the concept of _m-statistically pre-Cauchy sequences of fuzzy numbers is introduced. A necessary and sufficient condition for the sequence X = (Xk) of fuzzy numbers to be statistically pre-Cauchy is also discussed. Some more results are also established. AMS classification: 40A35;46A45;46S40;03E72.*
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I. 1 Introduction

The theory of sequences of fuzzy numbers was first introduced by Matloka [14]. Then, after all, Nanda [17], Mursaleen and Başarır [15] and others constructed various types of sequence spaces of fuzzy numbers.

The notion of statistical convergence which is closely related to the concept of natural density or asymptotic density of a subset of the set of natural numbers $\mathbb{N}$ was first introduced by Fast [9] and Schoenberg [20] independently. For a set $A \subset \mathbb{N}$, the density of the subset A is defined by

$$\delta(A) = \lim_{n\to\infty} \frac{1}{n} \mid \{k \in A : k \leq n\} \mid,$$

where $\mid \{k \in A : k \leq n\} \mid$ denote the number of elements in the set. Over the years and under different names, statistical convergence was discussed in number theory, trigonometric series and summability theory. From the point of view of sequence spaces, the concept of statistical convergence was generalized and developed by $\check{S}$al$\acute{a}$t [19], Fridy [10], Connor [3] and many others.

The existing literature on statistical convergence was restricted to real or complex sequences, but Nuray and Savas¸ [18] extended the idea of statistical convergence to sequences of fuzzy numbers. Later on, Kwon [13], Bilgin [1] and many others extended the idea of statistical convergence to the sequences of fuzzy numbers. Using this concept, Dutta and Tripathy [6], Dutta [7] and others discussed various concepts of statistically pre-Cauchy sequences of fuzzy numbers.

A sequence $x = (x_k)$ is statistically pre-Cauchy if

$$\lim_n n^{-2}|\{(j,k) : |x_k - x_j| \geq \varepsilon, j, k \leq n\}| = 0.$$

In [2], the notion of statistically pre-Cauchy sequences was introduced and shown that statistically convergent sequences are always statistically pre-Cauchy. On the other hand under certain general conditions statistically pre-Cauchy condition implies statistical convergence of a sequence.. Using this concept Das and Savas¸ [4], Khan et al. [11], Yamanc_ and G□urdal [22] and many others studied different types of sequence spaces of real numbers.

The difference sequence spaces $\ell_\infty(\Delta), c(\Delta)$ and $c_0(\Delta)$, consisting of all real valued sequences $x = (x_k)$ such that $\Delta x = (x_k - x_{k+1})$ in the sequence spaces ℓ_∞, c and c_0 were defined by Kızmaz [12]. Later on using the idea of mth order difference sequence, these spaces were generalized by Et and Çolak where the mth order difference sequence is defined as, for $m \in \mathbb{N}$, $\Delta^0 x = (x_k)$, $\Delta x = (x_k - x_{k+1})$, $\Delta^m x = (\Delta^m x_k) = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1})$ and so that

$$\Delta^m x_k = \sum_{i=0}^{m} (-1)^i \begin{pmatrix} m \\ i \end{pmatrix} x_{k+i}.$$

The purpose of this paper is to introduce the concept of Δ^m-statistically pre-Cauchy sequences of fuzzy numbers using a modulus function. A necessary and sufficient condition is proved for a sequence of fuzzy numbers to be Δ^m-statistically pre-Cauchy.

II. Definitions and Preliminaries

Let D denote the set of all closed and bounded intervals $A = [a_1, a_2]$ on the real line $\mathbb{R}$. For $A, B \in D$, we define

$$d(A, B) = \max\{|a_1 - b_1|, |a_2 - b_2|\}$$

where $A = [a_1, a_2]$ and $B = [b_1, b_2]$. It is known that (D, d) is a complete metric space [5].

A fuzzy number X is a fuzzy set on $\mathbb{R}$, that is, a mapping $X : \mathbb{R} \to [0, 1]$ which has the following properties:

(i) X is normal, that is, there exists a $t_0 \in \mathbb{R}$ such that $X(t_0) = 1$;

(ii) X is fuzzy convex, that is, for $t_1, t_2 \in \mathbb{R}$ and $0 \leq \lambda \leq 1$, $X(\lambda t_1 + (1-\lambda)t_2) \geq \min\{X(t_1), X(t_2)\}$;

(iii) X is upper semicontinuous;

(iv) supp X $=\{t \in \mathbb{R} : X(t) > 0\}$ is compact.

The set of all upper-semicontinuous, normal, convex fuzzy numbers with compact support is denoted by $L(\mathbb{R})$. Throughout this paper, by a fuzzy number we mean that the number belongs to $L(\mathbb{R})$.

The set $\mathbb{R}$ of all real numbers can be embedded in $L(\mathbb{R})$. For $r \in \mathbb{R}$, $\overline{r} \in L(\mathbb{R})$ is defined by $\overline{r}(t) = 1$ for $r = t$ and 0 for $r \neq t$.

The α-level set X^α of $X \in L(\mathbb{R})$ is defined as

$$X^\alpha = \begin{cases} t : X(t) \geq \alpha & \text{if } \alpha \in (0,1], \\ t : X(t) > 0 & \text{if } \alpha = 0. \end{cases}$$

The set $L(\mathbb{R})$ forms a linear space under addition $X + Y$ and scalar multiplication λX, where $X, Y \in L(\mathbb{R})$ and $\lambda \in \mathbb{R}$, in terms of α- level sets as defined below:

$$[X+Y]^\alpha = [X]^\alpha + [Y]^\alpha \text{ and } [\lambda X]^\alpha = \lambda [X]^\alpha \quad \text{for each } 0 \leq \alpha \leq 1.$$

For each $\alpha \in [0,1]$, the set X^α is a closed, bounded and nonempty interval of $\mathbb{R}$.

Let $X, Y \in L(\mathbb{R})$ and the α-level set of fuzzy numbers X and Y be $X^\alpha = [\underline{X}^\alpha, \overline{X}^\alpha]$ and $Y^\alpha = [\underline{Y}^\alpha, \overline{Y}^\alpha]$, $\alpha \in (0,1]$. Then a partial ordering " $\leq$ " in $L(\mathbb{R})$ is defined by $X \leq Y$ if and only if $\underline{X}^\alpha \leq \underline{Y}^\alpha$ and $\overline{X}^\alpha \leq \overline{Y}^\alpha$ for all $\alpha \in (0,1]$.

Let $\overline{d} : L(\mathbb{R}) \times L(\mathbb{R}) \to \mathbb{R}$ be defined by

$$\overline{d}(X,Y) = \sup_{0 \leq \alpha \leq 1} d(X^\alpha, Y^\alpha).$$

Then $\overline{d}$ defines a metric on $L(\mathbb{R})$ and $(L(\mathbb{R}), \overline{d})$ is a complete metric space [5].

Definition 2.1. *(Matloka [14]) A sequence $X = (X_k)$ of fuzzy numbers is a function from the set $\mathbb{N}$ of all positive integers into $L(\mathbb{R})$. Thus, a sequence of fuzzy numbers (X_k) is a correspondence between the set of positive integers and the set of fuzzy numbers. that is, to each positive integer k, there corresponds a fuzzy number X_k. The fuzzy number X_k is called the kth term of the sequence $X = (X_k)$. Let $w(F)$ denote the set of all sequences of fuzzy numbers.*

Definition 2.2. *(Matloka [14]) The sequence $X = (X_k)$ of fuzzy numbers is said to be bounded, if there exists fuzzy numbers X_0 and Y_0 such that $X_0 \leq X_k \leq Y_0$ for all $k \in \mathbb{N}$ and convergent to the fuzzy number X_0, written as $\lim_{k \to \infty} X_k = X_0$, if for every $\varepsilon > 0$, there exists a positive integer k_0 such that $\overline{d}(X_k, X_0) < \varepsilon$ for all $k \geq k_0$. Let $\ell_\infty(F)$ and $c(F)$ denote the set of all bounded and convergent sequences of fuzzy numbers, respectively.*

Lemma 2.1. *(Talo and Başar [21]) Let $X, Y, Z, V \in L(\mathbb{R})$ and $k \in \mathbb{R}$. Then,*

(i) $\overline{d}(kX, kY) = |k|\overline{d}(X,Y)$

(ii) $\overline{d}(X+Z, Y+Z) = \overline{d}(X,Y)$

(iii) $\overline{d}(X+Z, Y+V) \leq \overline{d}(X,Y) + \overline{d}(Z,V)$

(iv) $|\overline{d}(X,\overline{0}) - \overline{d}(Y,\overline{0})| \leq \overline{d}(X,Y) \leq \overline{d}(X,\overline{0}) + \overline{d}(Y,\overline{0})$

Definition 2.3. *(Nakano [16]) A real valued function $f : [0,\infty) \to [0,\infty)$ is called a modulus if*

(i) $f(x) \geq 0$ *for all* x,

(ii) $f(x) = 0$ *if and only if* $x = 0$,

(iii) $f(x+y) \leq f(x) + f(y)$ *for all* $x,\ y \geq 0$,

(iv) f *is an increasing function,*

(v) f *is continuous from the right at* 0.

III. Δ^m-statistically pre-Cauchy sequences of fuzzy numbers

The idea of statistical convergence depends on the density of subsets of the set $\mathbb{N}$ of natural numbers.

Definition 3.1. *(Nuray and Savaş [18]) The sequence $X = (X_k)$ of fuzzy numbers is said to be statistically convergent to the fuzzy number L, if for every $\varepsilon > 0$,*

$$\lim_{n\to\infty} \frac{1}{n} \left|\left\{k \le n : \overline{d}(X_k, L) \ge \varepsilon\right\}\right| = 0$$

where the vertical bars indicate the number of elements in the enclosed set. In this case, we write $S - \lim X_k = L$ and $S(F)$ denotes the set of all statistically convergent sequences of fuzzy numbers.

Definition 3.2. *The sequence $X = (X_k)$ of fuzzy numbers is said to be statistically pre-Cauchy, if for every $\varepsilon > 0$,*

$$\lim_{n\to\infty} \frac{1}{n^2} \left|\left\{i, j \le n : \overline{d}(X_i, X_j) \ge \varepsilon\right\}\right| = 0.$$

Let S^F denotes the set of all statistically pre-Cauchy sequences of fuzzy numbers.

Definition 3.3. *The sequence $X = (X_k)$ of fuzzy numbers is said to be Δ^m-statistically convergent to the fuzzy number L, if for every $\varepsilon > 0$,*

$$\lim_{n\to\infty} \frac{1}{n} \left|\left\{k \le n : \overline{d}(\Delta^m X_k, L) \ge \varepsilon\right\}\right| = 0.$$

In this case, we write $S(\Delta^m) - \lim X_k = L$ and $S(\Delta^m)(F)$ denotes the set of all Δ^m-statistically convergent sequences of fuzzy numbers.

Definition 3.4. *The sequence $X = (X_k)$ of fuzzy numbers is said to be Δ^m-statistically Cauchy, if for every $\varepsilon > 0$, there exist a positive integer k_0 (depending upon ε only) such that*

$$\lim_{n\to\infty} \frac{1}{n} \left|\left\{k \le n : \overline{d}(\Delta^m X_k, \Delta^m X_{k_0}) \ge \varepsilon\right\}\right| = 0.$$

Definition 3.5. *The sequence $X = (X_k)$ of fuzzy numbers is said to be Δ^m-statistically pre-Cauchy, if for every $\varepsilon > 0$,*

$$\lim_{n\to\infty} \frac{1}{n^2} \left|\left\{i, j \le n : \overline{d}(\Delta^m X_i, \Delta^m X_j) \ge \varepsilon\right\}\right| = 0.$$

In this case, we write $S(\Delta^m) - \lim X_k = L$ and $S^F(\Delta^m)$ denotes the set of all Δ^m-statistically pre-Cauchy sequences of fuzzy numbers.

Theorem 3.1. *If a sequence is Δ^m-statistically convergent sequence, then the sequence is Δ^m-statistically pre-Cauchy.*

Proof: The proof follows using the technique of theorem 2 of Connor et al. [2].

Remark 3.1. *In general, a sequence which is Δ^m-statistically pre-Cauchy is not Δ^m-statistically convergent.*

Proof: This result follows from the following example.

Let $f(x) = x,\ p_k = 1$ for all $k \in \mathbb{N}$ and consider the sequence $X = (X_k)$ as follows:
When k is odd,

$$X_k = \begin{cases} t+7 & \text{if } -7 \le t \le -6 \\ -t-5) & \text{if } -6 \le t \le -5 \\ 0 & \text{otherwise} \end{cases}$$

and when k is even,

$$X_k = \begin{cases} t-5 & \text{if } 5 \le t \le 6 \\ 7-t & \text{if } 6 \le t \le 7 \\ 0 & \text{otherwise} \end{cases}$$

The corresponding α-level sets of $X_k,\ \alpha \in [0,1]$ are

$$[X_k]^\alpha = \begin{cases} [-7+\alpha, -5-\alpha] & \text{if } k \text{ is odd} \\ [5+\alpha, 7-\alpha] & \text{if } k \text{ is even} \end{cases}$$

which implies the corresponding α-level sets of $\Delta X_k,\ \alpha \in [0,1]$ are

$$[\Delta^m X_k]^\alpha = \begin{cases} [2^m(-7+\alpha), 2^m(-5-\alpha)] & \text{if } k \text{ is odd} \\ [2^m(5+\alpha), 2^m(7-\alpha)] & \text{if } k \text{ is even} \end{cases}$$

which implies $\frac{1}{n^2}\left|\{k \le n : \overline{d}(\Delta^m X_k, L) \ge \varepsilon\}\right| \to 0$ as $n \to \infty$ where L is the fuzzy number such that $[L]^\alpha = [-7+\alpha, -5-\alpha]$. But the sequence $X = (X_k)$ is not Δ^m-statistically convergent to L.

Theorem 3.2. *Let a sequence of fuzzy numbers $X = (X_k)$ is Δ^m-statistically pre-Cauchy. If $(\Delta^m X_k)$ has a subsequence $(\Delta^m X_{n_k})$ which converges to X_0 and $0 < \liminf \frac{|\{n_k \le n : k \in \mathbb{N}\}|}{n} < \infty$, then, $(\Delta^m X_k)$ is statistically convergent to X_0.*

Proof. Let $\varepsilon > 0$ be given. Since $\Delta^m X_{n_k} \to X_0$, we can choose $n_0 \in \mathbb{N}$ such that $\overline{d}(\Delta^m X_{n_k}, X_0) < \varepsilon$ for all $n_k \ge n_0$.
Let $A = \{n_k : n_k > n_0, k \in \mathbb{N}\}$ and $A(\varepsilon) = \{k : \overline{d}(\Delta^m X_k, X_0) \ge \varepsilon\}$. Now

$$\begin{aligned} \frac{1}{n^2}|\{(j,k) : \overline{d}(\Delta^m X_j, \Delta^m X_k) \ge \varepsilon, j,k \le n\}| &\ge \frac{1}{n^2}\sum_{j,k \le n} \chi_{A(\varepsilon)\times A}(j,k) \\ &= \frac{1}{n}|\{n_k \le n : n_k \in A\}| \\ &\quad \times \frac{1}{n}|\{k \le n : \overline{d}(\Delta^m X_k, X_0) \ge \varepsilon\}|. \end{aligned}$$

Since, $\liminf \frac{|\{n_k \le n : k \in \mathbb{N}\}|}{n} > 0$ and $X = (X_k)$ is Δ^m-statistically pre-Cauchy sequence, so we have $\frac{1}{n^2}|\{(j,k) : \overline{d}(\Delta^m X_j, \Delta^m X_k) \ge \varepsilon, j,k \le n\}| \to 0$ as $n \to \infty$ which implies $X = (X_k)$ is Δ^m-statistically convergent to X_0. □

Theorem 3.3. *Let $X = (X_k)$ be a sequence of fuzzy numbers such that $(\Delta^m X_k)$ is bounded. Then, X is said to be Δ^m-statistically pre-Cauchy if and only if $\lim_{n\to\infty} \frac{1}{n^2} \sum_{i,j\le n} f(\overline{d}(\Delta^m X_i, \Delta^m X_j)) = 0$ for every bounded modulus function f.*

Proof . Let $\lim_{n\to\infty} \frac{1}{n^2} \sum_{i,j\le n} f(\overline{d}(\Delta^m X_i, \Delta^m X_j)) = 0$. Given $\varepsilon > 0$ and for any $n \in \mathbb{N}$, we have

$$\begin{aligned}
\frac{1}{n^2} \sum_{i,j\le n} f(\overline{d}(\Delta^m X_i, \Delta^m X_j)) &= \frac{1}{n^2} \sum_{\substack{i,j\le n,\\ \overline{d}(\Delta^m X_i, \Delta^m X_j)\ge\varepsilon}} f(\overline{d}(\Delta^m X_i, \Delta^m X_j)) \\
&\quad + \frac{1}{n^2} \sum_{\substack{i,j\le n,\\ \overline{d}(\Delta^m X_i, \Delta^m X_j)<\varepsilon}} f(\overline{d}(\Delta^m X_i, \Delta^m X_j)) \\
&\ge \frac{1}{n^2} \sum_{\substack{i,j\le n,\\ \overline{d}(\Delta^m X_i, \Delta^m X_j)\ge\varepsilon}} f(\overline{d}(\Delta^m X_i, \Delta^m X_j)) \\
&\ge f(\varepsilon)\frac{1}{n^2} \left|\left\{k \le n : \overline{d}(\Delta^m X_i, \Delta^m X_j) \ge \varepsilon\right\}\right|
\end{aligned}$$

which implies $\frac{1}{n^2} \left|\left\{i, j \le n : \overline{d}(\Delta^m X_i, \Delta^m X_j) \ge \varepsilon\right\}\right| \to 0$ as $n \to \infty$. Thus, X is Δ^m-statistically pre-Cauchy sequence.

Conversely, let $X = (X_k)$ is Δ^m-statistically pre-Cauchy sequence and $\varepsilon > 0$ be given. Choose $\delta > 0$ such that $f(\delta) < \frac{\varepsilon}{2}$. Since f is a bounded modulus function, so there exists an integer B such that $f(\overline{d}(\Delta^m X_i, \Delta^m X_j)) < B$. Now for each $n \in \mathbb{N}$, consider

$$\begin{aligned}
\frac{1}{n^2} \sum_{i,j\le n} f(\overline{d}(\Delta^m X_i, \Delta^m X_j)) &= \frac{1}{n^2} \sum_{\substack{i,j\le n,\\ \overline{d}(\Delta^m X_i, \Delta^m X_j)<\delta}} f(\overline{d}(\Delta^m X_i, \Delta^m X_j)) \\
&\quad + \frac{1}{n^2} \sum_{\substack{i,j\le n,\\ \overline{d}(\Delta^m X_i, \Delta^m X_j)\ge\delta}} f(\overline{d}(\Delta^m X_i, \Delta^m X_j)) \\
&\le f(\delta) + B\frac{1}{n^2} \left|\left\{i, j \le n : \overline{d}(\Delta^m X_i, \Delta^m X_j) \ge \delta\right\}\right| \\
&\le \frac{\varepsilon}{2} + B\frac{1}{n^2} \left|\left\{i, j \le n : \overline{d}(\Delta^m X_i, \Delta^m X_j) \ge \delta\right\}\right|
\end{aligned}$$

Since X is Δ^m-statistically pre-Cauchy sequence, so we have

$$\frac{1}{n^2} \sum_{i,j\le n} f(\overline{d}(\Delta^m X_i, \Delta^m X_j)) \le \frac{\varepsilon}{2B} \text{ for all } n \ge n_0.$$

Hence, we have,

$$\frac{1}{n^2} \sum_{i,j\le n} f(\overline{d}(\Delta^m X_i, \Delta^m X_j)) \le \varepsilon \text{ for all } n \ge n_0.$$

that is,

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{i,j\le n} f(\overline{d}(\Delta^m X_i, \Delta^m X_j)) = 0.$$

Theorem 3.4. *Let f be a bounded modulus function, then, the sequence $X = (X_k)$ of fuzzy numbers is Δ^m-statistically convergent to L if and only if* $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f(\overline{d}(\Delta^m X_k, L)) = 0$.

Proof . Let $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f(\overline{d}(\Delta^m X_k, L)) = 0$. Given $\varepsilon > 0$ and for any $n \in \mathbb{N}$, we have

$$\begin{aligned}
\frac{1}{n}\sum_{k=1}^{n} f(\overline{d}(\Delta^m X_k, L)) &= \frac{1}{n}\sum_{\substack{k=1,\\ \overline{d}(\Delta^m X_k,L)\geq\varepsilon}}^{n} f(\overline{d}(\Delta^m X_k, L)) \\
&\quad + \frac{1}{n}\sum_{\substack{k=1,\\ \overline{d}(\Delta^m X_k,L)<\varepsilon}}^{n} f(\overline{d}(\Delta^m X_k, L)) \\
&\geq \frac{1}{n}\sum_{\substack{k=1,\\ \overline{d}(\Delta^m X_k,L)\geq\varepsilon}}^{n} f(\overline{d}(\Delta^m X_k, L)) \\
&\geq f(\varepsilon)\frac{1}{n}\left|\{k \leq n : \overline{d}(\Delta^m X_k, L) \geq \varepsilon\}\right|
\end{aligned}$$

which implies $\frac{1}{n}\left|\{k \leq n : \overline{d}(\Delta^m X_k, L) \geq \varepsilon\}\right| \to 0$ as $n \to \infty$. Thus, X is Δ^m-statistically convergent to L.

Conversely, let $X = (X_k)$ is Δ^m-statistically convergent to L and $\varepsilon > 0$ be given. Choose $\delta > 0$ such that $f(\delta) < \frac{\varepsilon}{2}$. Since f is a bounded modulus function, so there exists an integer B such that $f(\overline{d}(\Delta^m X_k, L) < G$. Now for each $n \in \mathbb{N}$, consider

$$\begin{aligned}
\frac{1}{n}\sum_{k=1}^{n} f(\overline{d}(\Delta^m X_k, L)) &= \frac{1}{n}\sum_{\substack{k=1,\\ \overline{d}(\Delta^m X_k,L)<\delta}}^{n} f(\overline{d}(\Delta^m X_k, L)) \\
&\quad + \frac{1}{n}\sum_{\substack{k=1,\\ \overline{d}(\Delta^m X_k,L)\geq\delta}}^{n} f(\overline{d}(\Delta^m X_k, L)) \\
&\leq f(\delta) + G\frac{1}{n}\left|\{k \leq n : \overline{d}(\Delta^m X_k, L) \geq \delta\}\right| \\
&\leq \frac{\varepsilon}{2} + G\frac{1}{n}\left|\{k \leq n : \overline{d}(\Delta^m X_k, L) \geq \delta\}\right|
\end{aligned}$$

Since, X is Δ^m-statistically convergent sequence, so we have

$$\frac{1}{n}\sum_{k=1}^{n} f(\overline{d}(\Delta^m X_k, L)) \leq \frac{\varepsilon}{2G} \text{ for all } n \geq n_0.$$

Hence, we have,

$$\frac{1}{n}\sum_{k=1}^{n} f(\overline{d}(\Delta^m X_k, L)) \leq \varepsilon \text{ for all } n \geq n_0.$$

that is,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f(\overline{d}(\Delta^m X_k, L)) = 0.$$

Lemma 3.1. *Let $X = (X_k)$ be a sequence of fuzzy numbers such that $(\Delta^m X_k)$ is bounded and f be a bounded modulus function. Then, we have*

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{i,j\le n}\overline{d}(\Delta^m X_i, \Delta^m X_j) = 0$$

if and only if

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{i,j\le n} f(\overline{d}(\Delta^m X_i, \Delta^m X_j)) = 0.$$

Proof. Assume that $\lim_{n\to\infty}\frac{1}{n^2}\sum_{i,j\le n}\overline{d}(\Delta^m X_i, \Delta^m X_j) = 0$.
Let $V = \sup_k \overline{d}(\Delta^m X_k, \overline{0})$ and define $f(x) = (1+2V)\frac{x}{1+x}$. Then,

$$f(\overline{d}(\Delta^m X_i, \Delta^m X_j)) = (1+2V)\frac{\overline{d}(\Delta^m X_i, \Delta^m X_j)}{1+\overline{d}(\Delta^m X_i, \Delta^m X_j)} \le (1+2V)\overline{d}(\Delta^m X_i, \Delta^m X_j).$$

Hence, we have,

$$\frac{1}{n^2}\sum_{i,j\le n} f(\overline{d}(\Delta^m X_i, \Delta^m X_j)) \le (1+2V)\frac{1}{n^2}\sum_{i,j\le n}\overline{d}(\Delta^m X_i, \Delta^m X_j).$$

Since, $\lim_{n\to\infty}\frac{1}{n^2}\sum_{i,j\le n}\overline{d}(\Delta^m X_i, \Delta^m X_j) = 0$ which implies $\lim_{n\to\infty}\frac{1}{n^2}\sum_{i,j\le n} f(\overline{d}(\Delta^m X_i, \Delta^m X_j)) = 0$.
Conversely, assume that $\lim_{n\to\infty}\frac{1}{n^2}\sum_{i,j\le n} f(\overline{d}(\Delta^m X_i, \Delta^m X_j)) = 0$. Now,

$$\begin{aligned} f(\overline{d}(\Delta^m X_i, \Delta^m X_j)) &= (1+2V)\frac{\overline{d}(\Delta^m X_i, \Delta^m X_j)}{1+\overline{d}(\Delta^m X_i, \Delta^m X_j)} \\ &\ge \frac{(1+2V)\overline{d}(\Delta^m X_i, \Delta^m X_j)}{1+\overline{d}(\Delta^m X_i, \overline{0})+\overline{d}(\Delta^m X_j, \overline{0})} \\ &\ge \frac{(1+2V)\overline{d}(\Delta^m X_i, \Delta^m X_j)}{1+2V} \\ &= \overline{d}(\Delta^m X_i, \Delta^m X_j) \end{aligned}$$

Hence, we have $\frac{1}{n^2}\sum_{i,j\le n}\overline{d}(\Delta^m X_i, \Delta^m X_j) \le \frac{1}{n^2}\sum_{i,j\le n} f(\overline{d}(\Delta^m X_i, \Delta^m X_j))$.
Since, $\lim_{n\to\infty}\frac{1}{n^2}\sum_{i,j\le n} f(\overline{d}(\Delta^m X_i, \Delta^m X_j)) = 0$ which implies $\lim_{n\to\infty}\frac{1}{n^2}\sum_{i,j\le n}\overline{d}(\Delta^m X_i, \Delta^m X_j) = 0$.

Theorem 3.5. *Let $X = (X_k)$ be a sequence of fuzzy numbers such that $(\Delta^m X_k)$ is bounded. Then, for any bounded modulus function f, the following statement holds:*
$X = (X_k)$ is Δ^m-statistically pre-Cauchy if and only if $\lim_{n\to\infty}\frac{1}{n^2}\sum_{i,j\le n}\overline{d}(\Delta^m X_i, \Delta^m X_j) = 0$.

Proof. Using Lemma 3.1 and Theorem 3.3, we get the required result.

Theorem 3.6. *Let $X = (X_k)$ be a sequence of fuzzy numbers such that $(\Delta^m X_k)$ is bounded. Then, the following statement holds:*
X is Δ^m-statistically convergent to L if and only if $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \overline{d}(\Delta^m X_k, L) = 0.$

Proof. Consider the modulus function $f(x) = (1 + V + L)\frac{x}{1+x}$ where $V = \sup_k \overline{d}(\Delta^m X_k, \overline{0})$. Then, using the technique of Theorem 3.4 and Lemma 3.1, we get the required result.

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