IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 13, Issue 4 Ver. III (Jul. – Aug. 2017), PP 57-60
www.iosrjournals.org

On Quasi Generalized Topological Simple Groups

*C. Selvi, R. Selvi
Research scholar, Department of mathematics, Sriparasakthi college for women, India.
Assistant Professor, Department of mathematics, Sriparasakthi college for women, India.
Corresponding Author: C. Selvi, R. Selvi

Abstract: *In this paper we introduce the concept of quasi $\mathcal{G}$-topological simple group. Also some basic properties, theorems and examples of a quasi $\mathcal{G}$-topological simple groups are investigated. Moreover we studied the important result, If the mapping between two quasi $\mathcal{G}$-topological simple groups is $\mathcal{G}$-continous at the identity element, then f is $\mathcal{G}$-continous.*

Keywords: *Quasi topological group, $\mathcal{G}$-open set, $\mathcal{G}$-continous, Quasi $\mathcal{G}$-topological simple group.*

Date of Submission: 16-08-2017 Date of acceptance: 05-09-2017

I. Introduction

Csaszar[6], Introduced the notion of generalized neighbourhood system and generalized topological space. Also Csaszar[6], Investigated the generalized continous mappings. In this paper we introduce the new concept of quasi $\mathcal{G}$-topological simple group. Quasi $\mathcal{G}$-topological simple group have both topological and algebraic structures such that the translation mappings and the inversion mapping are $\mathcal{G}$-continous with respect to the generalized topology. Also some basic results are studied and discussed.

II. Preliminaries

Definition: 2.1 [3] Let X be any set and let $\mathcal{G} \subseteq$ P(X) be a subfamily of power set of X. Then $\mathcal{G}$ is called a generalized topology if $\phi \in \mathcal{G}$ and for any index set $I, \bigcup_{i\in I} O_i \in \mathcal{G}, \mathrm{O}_i \in \mathcal{G}$, i ∈ I.
Definition: 2.2 [3] The elements of $\mathcal{G}$ are called $\mathcal{G}$-open sets. Similarly, generalized closed set (or) $\mathcal{G}$-closed, is defined as complement of a $\mathcal{G}$-open set.
Definition: 2.3 [3] Let X and Y be two $\mathcal{G}$-topological space. A mapping $f: X \to Y$ is called a $\mathcal{G}$-continous on X if for any $\mathcal{G}$-open set O in $Y, f^{-1}(O)$ is $\mathcal{G}$-open in X.
Definition : 2.4 [3] The bijective mapping f is called a $\mathcal{G}$-homeomorphism from X to Y if both f and f^{-1} are $\mathcal{G}$-continous. If there is a $\mathcal{G}$-homeomorphism between X and Y, then they are said to be $\mathcal{G}$-homeomorphic. It is denoted by $X \cong_{\mathcal{G}} Y$.
Definition : 2.5 [3] Collection of all $\mathcal{G}$-interior points of $A \subset X$ is called $\mathcal{G}$-interior of A. It denoted by $Int_{\mathcal{G}}(A)$. By definiton it obvious that $Int_{\mathcal{G}}(A) \subset$ A.
Note: 2.6 [3] *(i).* $\mathcal{G}$-interior of A, $Int_{\mathcal{G}}(A)$ is equal to union of all $\mathcal{G}$-open sets contained in A.
(ii). $\mathcal{G}$-closure of A as intersection of all $\mathcal{G}$-closed sets containing A. It is denoted by $Cl_{\mathcal{G}}(A)$.
Definition: 2.7 [3] Let $(G, *)$ is a group and given $x \in G, L_x: G \to G$ defined by $L_x(y) = x * y$ and $R_x: G \to G$ defined by $R_x(y) = y * x$, denote left and right translation by x, respectively.
Definition: 2.8 [1] A quasi topological group G, is a group which is also a topological space if the following conditions are satisfied,
(i). Left translation $L_x: G \to G,\ x \in G$ and right translation $R_x: G \to G,\ x \in G$ are continous and
(ii). The inverse mapping $i: G \to G$ defined by $i(x) = x^{-1}, x \in G$ is continous.
Definition: 2.9 [20] A group G is called a simple group if it has no nontrivial normal subgroup of G.

III. Quasi Generalized Topological Simple Groups

Definition: 3.1 A quasi $\mathcal{G}$-topological simple group G, is a simple group which is also a $\mathcal{G}$-topological space if the following conditions are satisfied,
(i). Left translation $L_x: G \to G,\ x \in G$ and Right translation $R_x: G \to G,\ x \in G$ are $\mathcal{G}$-continous and
(ii). The inverse mapping $i: G \to G$ defined by $i(x) = x^{-1}, x \in G$ is $\mathcal{G}$-continous .
Example: 3.2 Any group of prime order with indiscrete or discrete $\mathcal{G}$-topology is a quasi $\mathcal{G}$-topological simple group.

Example: 3.3 Let $G = \left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right\}$ be a trivial simple group under addition and we define a generalized topology on G by $\mathcal{G} = \left\{\phi, \left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right\}\right\}$. Clearly $(G, +, \mathcal{G})$ quasi $\mathcal{G}$-topological simple group.

Example: 3.4 $G = \{1, w, w^2\}$, $where\ w^3 = 1$, is a simple group under multiplication. Now we define a generalized on G by $\mathcal{G} = \{\phi, G, \{w\}\}$. Then the inverse mapping i is $\mathcal{G}$-continous at the points $1, w^2$ and not $\mathcal{G}$-continous at the point w. In right translation mapping, R_1 is $\mathcal{G}$-continous at each point of G, R_w is $\mathcal{G}$-continous at the points w, w^2 and not $\mathcal{G}$-continous at the point 1 and R_{w^2} is $\mathcal{G}$-continous at the point $1, w$ and not $\mathcal{G}$-continous at the point w^2. Similarly we can prove left translation(L_x).

Theorem: 3.5 Let $(G, *, \mathcal{G})$ be a quasi $\mathcal{G}$-topological simple group and β_e be the collection of all $\mathcal{G}$-open neighbourhood at identity e of G. Then

(i). For every $U \in \beta_e$, there is an element $V \in \beta_e$ such that $V^{-1} \subseteq U$.

(ii). For every $U \in \beta_e$, there is an element $V \in \beta_e$ such that $V * x \subseteq U$ and $x * V \subseteq U$, for each $x \in U$.

Proof: *(i).* Since $(G, *, \mathcal{G})$ is a quasi $\mathcal{G}$-topological simple group. Therefore, for every $U \in \beta_e$, there exists $V \in \beta_e$ such that $i(V) = V^{-1} \subseteq U$, because the inverse mapping $i: G \rightarrow G$ is $\mathcal{G}$-continous.

(ii). Since $(G, *, \mathcal{G})$ is a quasi $\mathcal{G}$-topological simple group. Thus for each $\mathcal{G}$-open set U containing x, there exists $V \in \beta_e$ such that $R_x(V) = V * x \subseteq U$. Similarly, $L_x(V) = x * V \subseteq U$.

Theorem: 3.6 Let G be a quasi $\mathcal{G}$-topological simple group and g be any element of G. Then the right translation(R_g) and left translation(L_g) of G by g is a $\mathcal{G}$-homeomorphism of the space G onto itself.

Proof: First we prove that R_g is a bijection. Assume that $y \in G$, then the element yg^{-1} maps to y. Therefore R_g is surjective.

Assume that $R_g(x) = R_g(y)$.

$\Rightarrow xg = yg$.

$\Rightarrow x = y$. Hence R_g is 1-1. Since G is a quasi $\mathcal{G}$-topological simple group, R_g is $\mathcal{G}$-continous.

Consider ${R_g}^{-1}$ which maps xg to x, this is equivalent to the map from x to xg^{-1}. Therefore ${R_g}^{-1}(x) = R_{g^{-1}}(x)$. Since $R_{g^{-1}}(x)$ is $\mathcal{G}$-continous, ${R_g}^{-1}(x)$ is $\mathcal{G}$-continous. Similarly we will prove that the left translation (L_g). Hence the theorem.

Theorem: 3.7 Let G be a quasi $\mathcal{G}$-topological simple group and U be any $\mathcal{G}$-open set in G. Then

(i). $a * U$ and $U * a$ is $\mathcal{G}$-open in G for all $a \in G$.

(ii). For any subset A of G, the sets $U * A$ and $A * U$ are $\mathcal{G}$-open in G.

Proof: Let $x \in U * a$. We want to show that x is a $\mathcal{G}$-interior point of $U * a$. Let $x = u * a$ for some $u \in U = U * a * a^{-1}$. Then $u = x * a^{-1}$. We know that $R_{a^{-1}}: G \rightarrow G$ is $\mathcal{G}$-continous. Then for every $\mathcal{G}$-open set containing $R_{a^{-1}}(x) = x * a^{-1} = u$, there exists a $\mathcal{G}$-open set M_x containing x such that $R_{a^{-1}}(M_x) \subseteq U$.

$\Rightarrow M_x * a^{-1} \subseteq U$.

$\Rightarrow M_x \subseteq U * a$.

$\Rightarrow x$ is a $\mathcal{G}$-interior point of $U * a$. Therefore $U * a$ is $\mathcal{G}$-open in G. Similarly we can prove that $a * U$ is $\mathcal{G}$-open G.

(ii). By above result, $U * a$ is $\mathcal{G}$-open, for all $a \in G$. Then $U * A = \bigcup_{a \in A} U * a$ also $\mathcal{G}$-open in G. Similarly we can prove that $A * U$ is $\mathcal{G}$-open in G.

Theorem: 3.8 Suppose that a subgroup H of a quasi $\mathcal{G}$-topological simple group G contains a non-empty $\mathcal{G}$-open subset of G. Then H is $\mathcal{G}$-open in G.

Proof: Let U be a non-empty $\mathcal{G}$-open subset of G with $U \subset H$. For every $g \in H$, the set $L_g(U) = U * g$ is $\mathcal{G}$-open in G, then $H = \cup_{g \in H} U * g$ is $\mathcal{G}$-open in G.

Theorem: 3.9 Every quasi $\mathcal{G}$-topological simple group G has $\mathcal{G}$-open neighbourhood at the identity element e consisting of symmetric $\mathcal{G}$-neighbourhoods.

Proof: For an arbitrary $\mathcal{G}$-open neighbourhood U of the identity e, if $V = U \cap U^{-1}$, then $V = V^{-1}$, the set V is an $\mathcal{G}$-open neighbourhood of e , which implies that V is a symmetric $\mathcal{G}$- neighbourhood and $V \subset U$.

Theorem: 3.10 Let $f: G \rightarrow H$ be a homomorphism of quasi $\mathcal{G}$-topological simple groups.If f is $\mathcal{G}$-continous at the neutral element e_G of G, then f is $\mathcal{G}$-continous.

Proof: Let $x \in G$ be arbitrary and suppose that W is an $\mathcal{G}$-open neighbourhood of $y = f(x)$ in H. Since the left translation L_y in H is a $\mathcal{G}$-continous mapping , there exists an $\mathcal{G}$-open neighbourhood V of the neutral element e_H in H such that $L_y(V) = yV \subseteq W$. Since f is $\mathcal{G}$-continous at e_G of G, then $f(U) \subset V$, for some $\mathcal{G}$-open neighbourhood U of e_G in G. Since $L_x: G \rightarrow G$ is $\mathcal{G}$-continous, then xU is an $\mathcal{G}$-open neighbourhood of x in G.

Now we have
$$\begin{aligned} f(xU) &= f(x)f(U) \\ &= y\,f(U) \\ &\subseteq yV \\ &\subseteq W. \end{aligned}$$
Hence f is $\mathcal{G}$-continous at the point $x \in G$.

Theorem: 3.11 Suppose that G, H and K are quasi $\mathcal{G}$-topological simple groups and that $\phi: G \to H$ and $\psi: G \to K$ are homomorphism Such that $\psi(G) = K$ and $Ker\ \psi \subset Ker\ \phi$. Then there exists homomorphism $f: K \to H$ such that $\phi = f\ o\ \psi$. In addition, for each $\mathcal{G}$-neighbourhood U of the identity element e_H in H, there exists a $\mathcal{G}$-neighbouhood V of the identity element e_k in K such that $\psi^{-1}(\mathrm{V}) \subset \phi^{-1}(\mathrm{U})$, then f is $\mathcal{G}$-continous.
Proof: Algebraic part of the theorem is well known. Suppose U is a $\mathcal{G}$-neighbourhood of e_H in H. By assumption, there exists a $\mathcal{G}$-neighbouhood V of the identity element e_k in K such that , $W = \psi^{-1}(\mathrm{V}) \subset \phi^{-1}(\mathrm{U})$.
$\Rightarrow \phi(W) = \varphi(\psi^{-1}(\mathrm{V})) \subset \phi(\phi^{-1}(\mathrm{U}))$
$\Rightarrow \phi(W) = f(V) \subset \mathrm{U}$. Hence f is $\mathcal{G}$-continous at the identity element of K. Therefore by above theorem, f is $\mathcal{G}$-continous.
Corollary: 3.12 Let $\phi: G \to H$ and $\psi: G \to K$ be $\mathcal{G}$-continous homomorphism of a quasi $\mathcal{G}$-topological simple groups G, H and K Such that $\psi(G) = K$ and $Ker\ \psi \subset Ker\ \phi$. If the homomorphism ψ is $\mathcal{G}$-open, then there exists a $\mathcal{G}$-continous homomorhism, $f: K \to H$ such that $\phi = f\ o\ \psi$.
Proof: The existence of a homomorphism $f: K \to H$ such that $\phi = f\ o\ \psi$. Take an arbitrary $\mathcal{G}$-open set V in H. Then $f^{-1}(V) = \psi(\phi^{-1}(V))$. Since ϕ is $\mathcal{G}$-continous and ψ is an $\mathcal{G}$-open map, $f^{-1}(V)$ is $\mathcal{G}$-open in K. Therefore f is $\mathcal{G}$-continous.
Theorem: 3.13 Let G be a quasi $\mathcal{G}$-topological simple group and H is a normal subgroup of G. Then $\overline{H}$ also a normal subgroup of G.
Proof: Now we have to prove that $g\overline{H}g^{-1} \in \overline{H}\ \forall\ g \in G$.
Since H is a normal subgroup of G, $gHg^{-1} \in H\ \forall\ g \in G$.
Now $\overline{gHg^{-1}} \subset \overline{H}\ \forall\ g \in G$.
$\Rightarrow g\overline{H}g^{-1} \subset \overline{H}\ \forall g \in G$.
$\Rightarrow g\overline{H}g^{-1} \in \overline{H}, \forall g \in G$. Therefore $\overline{H}$ is a normal subgroup of G.
Corrollary: 3.14 Let G be a quasi $\mathcal{G}$-topological simple group and $Z(G)$ be the centre of G. Then $\overline{Z(G)}$ is a normal subgroup of G.
Proof: proof follows from the above theorem.
Corollary: 3.15 Let G and H be a quasi $\mathcal{G}$-topological simple groups. If $f: G \to H$ is a homomorphism mapping ,then $\overline{kerf}$ is a normal subgroup of G.
Theorem: 3.16 Let G and H be quasi $\mathcal{G}$-topological simple groups with neutral elements e_G and e_H, respectively, and let p be a $\mathcal{G}$-continous homomorphism of G onto H such that, for some non-empty subset U of G, the set $p(U)$ is $\mathcal{G}$-open in H and the restriction of p to U is an $\mathcal{G}$-open mapping of U onto $p(U)$. Then the homomorphism p is $\mathcal{G}$-open.
Proof: It suffices to show that $x \in G$, where W is an $\mathcal{G}$-open neighbourhood of x in G, then $p(W)$ is a $\mathcal{G}$-open neighbourhood of $p(x)$ in H. Fix a point y in U, and let L be the left translation of G by yx^{-1}. Then L is a $\mathcal{G}$-homeomorphism of G onto itself such that ,

$$L_{yx^{-1}}(x) = yx^{-1}$$
$$= y.$$

So $V = U \cap L(W)$ is an $\mathcal{G}$-open neighbourhood of y in U. Then $p(V)$ is $\mathcal{G}$-open subset of H. consider the left translation h of H by the inverse to $p(yx^{-1})$.
Now clearly, $(h\ o\ p\ o\ l) = h(p(l(x)))$

$$= h(p(y))$$
$$= p(xy^{-1})p(y)$$
$$= p(xy^{-1}y)$$
$$= p(x).$$

Hence $h\left(p(l(W))\right) = p(W)$. Clearly h is a $\mathcal{G}$-homeomorphism of H onto itself. Since $p(V)$ is $\mathcal{G}$-open in H, $h(p(V))$ is also $\mathcal{G}$-open in H. Therefore $p(W)$ contains the $\mathcal{G}$-open neighbourhood $h(p(V))$ of $p(x)$ in H. Hence $p(W)$ is a $\mathcal{G}$-open neighbourhood of $p(x)$ in H.
Definition: 3.17 Let H be a subgroup of quasi $\mathcal{G}$-topological simple group G. Then H is called neutral in G if every $\mathcal{G}$-neighbourhood U of the identity e_G in G, there exists a $\mathcal{G}$-neighbourhood V of e_G such that $VH \subset HU$.
Theorem: 3.18 Let H be a subgroup of quasi $\mathcal{G}$-topological simple group G. Suppose that, for every $\mathcal{G}$-open neighbourhood U of the identity e_G in G, there exists an $\mathcal{G}$-open neighbourhood V of e_G in G such that $xVx^{-1} \subset U$ whenever $x \in G$. Then H is neutral in G.
Proof: Given a $\mathcal{G}$-neighbourhood U of e_G in G. Take an $\mathcal{G}$-open neighbourhood V of e_G satisfying,
$xVx^{-1} \subset U, \forall\ x \in G$
$\Rightarrow xV \subset Ux, \forall\ x \in G$
$\Rightarrow HV \subset UH, \forall\ x \in G$. Then H is neutral in G.

References

[1]. A.V.Arhangel'skii, M.Tkachenko, Topological Groups and Related Structures, At- lantis press/world Scientific, Amsterdampairs, 2008.
[2]. C.Selvi, R.Selvi, On Generalized Topological Simple Groups, Ijirset Vol.6, Issue 7, July (2017).
[3]. Muard Hussain, Moiz Ud Din Khan, Cenap Ozel, On generalized topological groups, Filomat 27:4(2013),567-575
[4]. Dylan spivak, Introduction to topological groups, Math(4301).
[5]. J. R. Munkres, Topology, a first course, Prentice-Hall, Inc., Englewood cliffs, N.J.,1975.
[6]. .A.Csaszar, generalized topology, generalized continuity, Acta Math. Hungar. 96(2002) 351-357.
[7]. A.Csaszar, γ-connected sets, Acta Math..Hungar.101 (2003) 273-279.
[8]. A.Csaszar, A separarion axioms for generalized topologies, Acta Math.Hungar.104 (2004) 63-69.
[9]. A.Csaszar, Product of generalized topologies, Acta Math.Hungar.123 (2009) 127-132.
[10]. W.K.Min, Weak continuity on generalized topological spaces, Acta Math.Hungar. 124 (2009)73-81.
[11]. L.E.De Arruda Saraiva, Generalized quotient topologies, Acta Math.Hungar. 132 (2011) 168-173.
[12]. R.Shen, Remarks on products of generalized topologies, Acta Math.Hungar.124 (2009)363-369.
[13]. Volker Runde, A Taste of topology, Springer(2008).
[14]. Taqdir Hussain, Introduction to Topological groups, Saundres(1966).
[15]. David Dummit and Richard Foote, Abstract Algebra(3^{rd} edition), Wiley(2003).
[16]. Morris Kline, Mathematical Thought from Ancient to modern times, Oxford University Press(1972).
[17]. Muhammad Siddique Bosan, Moiz Ud Din Khan and Ljubisa D.R. Kocinac, On s-Topological Groups, Mathematica Moravica,Vol. 18-2(2014),35-44 .
[18]. Pierre Ramond, Group theory: A physicists survey, Cambridge(2010).
[19]. Robert Bartle, The Elements of Integration and Lebesgue Measure,Wiley(1995).
[20]. Joseph A. Gallian, Contemporary Abstract Algebra, Narosa(fourth edition).

C. Selvi. “On Quasi Generalized Topological Simple Groups.” IOSR Journal of Mathematics (IOSR-JM), vol. 13, no. 4, 2017, pp. 57–60.