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Sample-and-Hold Functions for Solving Linear Volterra Integral-Algebraic Equations

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Abstract: In this paper, Sample-and-Hold functions are implemented to give approximate solutions for linear volterra integral-algebraic equations. The proposed method will transform the problem to a linear lower triangular system of algebraic equations using the operation matrix associated with the Sample-and-Hold functions. Convergence result and tested examples are given in order to check the validity and efficiency of the proposed method.
Keyword:Sample-and-Hold functions, Integral equation, Integral-algebraic equations.

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I. Introduction

Couple system of integral algebraic equations (IAEs) including of the first and second kind volterra integral algebraic equations ,These systems are also called as singular system of integral equations , naturally appear in mathematical model processes , e.g. the kernel identification subject in heat conduction and viscoelasticity [1], development of a chemical reaction within a small cell [2], the two dimensional biharmonic equation in a semi-infinite strip [3], dynamic procedure in chemical reactors [4] and Kirchhoff's laws. (For more applications see [5,6] and references therein.)

In this paper, Sample-and-Hold functions will be used to solve linear volterra integral equations (LVIEs) with variable coefficients as given below:

$$\mathrm{A}(t)\mathrm{Y}(t) = \mathrm{G}(t) + \int_0^t \mathrm{K}(t,r)\mathrm{Y}(r)\,\mathrm{d}r\,, \qquad 0 \le t \le 1\,, \qquad \dots(1)$$

Where $\mathrm{A}(t) = [a_{i,j}(t)]$, $i,j = 1,2, \dots , n$.
$\mathrm{Y}(t) = [f_1(t)\ f_2(t) \dots f_n(t)]^T$,
$\mathrm{G}(t) = [g_1(t)\ g_2(t) \dots g_n(t)]^T$,
$\mathrm{K}(t,r) = [k_{i,j}(t,r)]$. $i,j = 1,2, \dots , n$

Since$\mathrm{A}(t)$, $\mathrm{G}(t)$, $\mathrm{K}(t,r)$ are known as functions and $\mathrm{Y}(t)$ is unknown . If $\det\mathrm{A}(t) = 0$, this system is denoted as VolterraIntegral-Algebraic Equations (IAEs). Under the condition $\det\mathrm{A}(t) = 0$,the system can have a number of solutions or no solution at all. Sufficient conditions for the existence of unique continuous solution have been introduced in [7].

A little numerical methods are found, to solve these systems such as polynomial spline collocation method and its convergence results [8], Legendre collocation method [9], Jacobi collocation method including the matrix-vector multiplication representation [10], Multistep methods based on Adams quadratures rules and extrapolation formulas [11], Piecewise constant orthogonal functions such as Walsh functions [12,13], block-pulse functions [14], Haar functions [15,16] .

This paper is ordered as following: in section 2, summary of Sample-and-Hold functions and their properties have been described. In section 3, the proposed method for solving volterra integral-algebraic equations have been presented. In section 4, we display illustrative examples,finally a conclusion have been drawnin section 5.

II. Generalized Sample-and-Hold functions (SHFs)

For any $\mathrm{Y}(t) \in L^2[0, A)$, $\mathrm{Y}(t)$ can be represented by a Sample-and-Hold functions.Consider the interval [0, A) and by considering:

$y_i(t) \approx y(ih)$, $i = 1, \dots, m$

Where$h$ = $\frac{A}{m}$, $y_i(t)$ is the generosity of the function y(t) at time t and y(ih) is the sample of function y(t) at the point t = ih .
The m-set of SHFs,$S_m(t)$, consisting of m element functions, and is defined as following [17]:

$$S_i(t) = \begin{cases} 1, & for \quad (i-1)h \le t < ih \\ 0, & elsewhere \end{cases}$$

Where i = 1, …, m.

2.1 Properties:

The Sample-and-Hold functions are similar to Block Pulse functions in aspects. Further it'seasy to satisfy the following properties and by same way of [14].
(i)- $S_m(t)S_m^T(t)$ = diag(S_m(t)), where $S_m(t)$= $[\ S_1(t) S_2(t)\ \dots\ S_m(t)\ ]^T$.
(ii)-$S_m^T(t)S_m(t) = 1$.
(iii)-$S_m^T(t)S_m(t)U = \tilde{U}S_m(t)$, $\tilde{U}$ = diag(U), where U be an m-vector. …(2)
(iv)- $S_m^T(t)JS_m(t) = \hat{J}^T S_m(t)$, for every m×m matrix J and $\hat{J}$is an m-vector with entries equal to the diagonal elements of J. …(3)

2.2 Operational Matrix of the Integration for SHFs

The integral $\int_0^t S_m(\tau)d\tau$is expanded in terms of SHFs and by arranging the coefficients in matrix form,we have

$$\int_0^t S_m(\tau)d\tau = P_m S_m(t).$$

Where P is given by [18]:

$$P_m = \frac{h}{2}\begin{pmatrix} 1 & 2 & 2 & 2 & & 2 & 2 & 2 \\ 0 & 1 & 2 & 2 & \cdots & 2 & 2 & 2 \\ 0 & 0 & 1 & 2 & & 2 & 2 & 2 \\ 0 & 0 & 0 & 1 & & 2 & 2 & 2 \\ & \vdots & & & \ddots & & \vdots & \\ 0 & 0 & 0 & 0 & & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & & 0 & 0 & 1 \end{pmatrix}_{m\times m}$$

2.3 Function approximation

A square integrable time function $y(t)$ of Lebesgue measure may also be extended into an m-term Sample-and-Hold functions series in t ∈ [0,A) as

$$y(t) = \sum_{i=1}^{m} y_i S_i(t) = Y_m^T S_m(t) = S_m^T(t) Y_m \qquad \dots (4)$$

Where $Y_m \triangleq [y_1 y_2\ \dots\ y_m\ ]^T$,
Where $y_i = y(ih)$ the i^{th} sample of the function$y(t)$. Actually, y_i's are the sample of $y(t)$with the sampling period h .

2.4 Convergence Analysis

Assume $y \in L^2[0, A)$. Let S_m is called the set of all linear combination of Sample-and-Hold Functions S_i, where i = 1,2, …, m . We defined partial sum of SHFs of y as $Z_m = \sum_{i=1}^{m} y_i S_i(t)$, where S_i is the i^{th} Sample-and-Hold functions.
By using orthogonality property,

$$\|Z_m\|^2 = \langle \sum_{i=1}^{m} y_i S_i(t)\ , \sum_{i=1}^{m} y_i S_i(t) \rangle = \sum_{i=1}^{m} |y_i|^2 \|S_i\|^2$$

Also,$\langle\, y - Z_m\ , S_i \rangle = \langle y\ , S_i \rangle - \langle Z_m\ , S_i \rangle = y_i - y_i = 0$,where i=1,2,…,$m$, since $\langle\, y - Z_m\ , S_i \rangle = 0$, we get $y - Z_m = 0$ then $\|\, y - Z_m \| = 0$, since$\langle y\ , S_i \rangle = \int_0^1 y S_i(t)dt$, where i = 1, 2, …, m .
Pythagorean theorem state,

$$\|y\|^2 = \|y - Z_m\|^2 + \|Z_m\|^2. \qquad \dots (5)$$

From this we have, $\|Z_m\|^2 = \sum_{i=1}^{m} |y_i|^2 \|S_i\|^2 \le \|y\|^2$.
The infinite series $\sum_{i=1}^{\infty} |y_i|^2 \|S_i\|^2$ has non-negative terms and it's partial sum are bounded above by $\|y\|^2$.It's convergence and satisfies Bessel's inequality $\sum_{i=1}^{\infty} |y_i|^2 \|S_i\|^2 \le \|y\|^2$.
Since $S_m \subseteq S_{m+1}$, the norm $\|\, y - Z_m \|$ are decreasing and hence $\lim_{m\to\infty} \|\, y - Z_m \| = 0$, this gurantens that the numerical solution converges to exact solution.
Further, from equation (5), we have $\|y\|^2 = \lim_{m\to\infty} \|\, y - Z_m \|^2 = \sum_{i=1}^{\infty} |y_i|^2 \|S_i\|^2$.

III. The Approach

Let us consider the system of linear volterra integral-algebraic equations with variable coefficients (1), which can be written as:

$$\sum_{q=1}^{n} a_{p,q}(t)y_q(t) = g_p(t) + \sum_{q=1}^{n}\int_0^t k_{p,q}(t,r)y_q(r)dr, \quad 0 \le \mathrm{t} \le 1, \ldots (6)$$

Where $p = 1, 2, \ldots, n$ and $r = \frac{\mathrm{A}}{m}$.

Writing the coefficient $a_{p,q}$, y_q, g_p and $k_{p,q}$ in terms of Sample-and-Hold functions yields:

$$a_{p,q}(t) \simeq \mathrm{A}_{\mathrm{p,q}}^{\mathrm{T}}\mathrm{S}_{\mathrm{m}}(t) = \mathrm{S}_{\mathrm{m}}^{\mathrm{T}}(t)\mathrm{A}_{\mathrm{p,q}},$$
$$y_q(t) \simeq \mathrm{Y}_{\mathrm{q}}^{\mathrm{T}}\mathrm{S}_{\mathrm{m}}(t) = \mathrm{S}_{\mathrm{m}}^{\mathrm{T}}(t)\mathrm{Y}_{\mathrm{q}},$$
$$g_p(t) \simeq \mathrm{G}_{\mathrm{p}}^{\mathrm{T}}\mathrm{S}_{\mathrm{m}}(t) = \mathrm{S}_{\mathrm{m}}^{\mathrm{T}}(t)\mathrm{G}_{\mathrm{p}},$$
$$k_{p,q}(t,r) \simeq \mathrm{S}_{\mathrm{m}}^{\mathrm{T}}(t)\mathrm{K}_{p,q}\mathrm{S}_{\mathrm{m}}(r),$$

Where $\mathrm{Y}_{\mathrm{q}} = [\, y_{q0}y_{q1} \dots y_{qn}]^{\mathrm{T}}$, $\mathrm{K}_{p,q} = \left[\kappa_{p,q}^{(i,j)}\right]$ and $\mathrm{A}_{\mathrm{p,q}} = \left[a_{p,q}^{(i)}\right]$ with i, j= 1,… , m.The p^{th}equations, from the system (6) can be extended in m-terms of Sample-and-Hold functions expansion as follows,

$$\sum_{q=1}^{n} \mathrm{S}_{\mathrm{m}}^{\mathrm{T}}(t)\mathrm{A}_{\mathrm{p,q}}\,\mathrm{Y}_{\mathrm{q}}^{\mathrm{T}}\mathrm{S}_{\mathrm{m}}(t) \simeq \mathrm{G}_{\mathrm{p}}^{\mathrm{T}}\mathrm{S}_{\mathrm{m}}(t) + \sum_{q=1}^{n}\int_0^t \mathrm{S}_{\mathrm{m}}^{\mathrm{T}}(t)\mathrm{K}_{p,q}\mathrm{S}_{\mathrm{m}}(r)\,\mathrm{S}_{\mathrm{m}}^{\mathrm{T}}(r)\mathrm{Y}_q\,\mathrm{dr}$$
$$= \mathrm{G}_{\mathrm{p}}^{\mathrm{T}}\mathrm{S}_{\mathrm{m}}(t) + \mathrm{S}_{\mathrm{m}}^{\mathrm{T}}(t)\sum_{q=1}^{n}\int_0^t \mathrm{K}_{p,q}\mathrm{S}_{\mathrm{m}}(r)\,\mathrm{S}_{\mathrm{m}}^{\mathrm{T}}(r)\mathrm{Y}_q\,\mathrm{dr}$$
$$= \mathrm{G}_{\mathrm{p}}^{\mathrm{T}}\mathrm{S}_{\mathrm{m}}(t) + \mathrm{S}_{\mathrm{m}}^{\mathrm{T}}(t)\sum_{q=1}^{n}\mathrm{K}_{p,q}\int_0^t \mathrm{S}_{\mathrm{m}}(r)\,\mathrm{S}_{\mathrm{m}}^{\mathrm{T}}(r)\mathrm{Y}_q\,\mathrm{dr}$$

According to equation (2), we have:

$$\mathrm{S}_{\mathrm{m}}^{\mathrm{T}}(t)\sum_{q=1}^{n}\mathrm{A}_{\mathrm{p,q}}\,\mathrm{Y}_{\mathrm{q}}^{\mathrm{T}}\mathrm{S}_{\mathrm{m}}(t) \simeq \mathrm{G}_{\mathrm{p}}^{\mathrm{T}}\mathrm{S}_{\mathrm{m}}(t) + \mathrm{S}_{\mathrm{m}}^{\mathrm{T}}(t)\sum_{q=1}^{n}\mathrm{K}_{p,q}\int_0^t \widetilde{\mathrm{Y}}_q\mathrm{S}_{\mathrm{m+1}}(r)\,\mathrm{dr}$$
$$= \mathrm{G}_{\mathrm{p}}^{\mathrm{T}}\mathrm{S}_{\mathrm{m}}(t) + \mathrm{S}_{\mathrm{m}}^{\mathrm{T}}(t)\sum_{q=1}^{n}\mathrm{K}_{p,q}\widetilde{\mathrm{Y}}_q\int_0^t \mathrm{S}_{\mathrm{m}}(r)\,\mathrm{dr}$$

$$\mathrm{S}_{\mathrm{m}}^{\mathrm{T}}(t)\sum_{q=1}^{n}\mathrm{A}_{\mathrm{p,q}}\,\mathrm{Y}_{\mathrm{q}}^{\mathrm{T}}\mathrm{S}_{\mathrm{m}}(t) \simeq \mathrm{G}_{\mathrm{p}}^{\mathrm{T}}\mathrm{S}_{\mathrm{m}}(t) + \mathrm{S}_{\mathrm{m}}^{\mathrm{T}}(t)\sum_{q=1}^{n}\mathrm{K}_{p,q}\widetilde{\mathrm{Y}}_q\,\mathrm{P}\quad\mathrm{S}_{\mathrm{m}}(t)$$

By using equation (3) we obtain:

$$\widehat{\mathrm{A}}_{\mathrm{p,q}}^{\mathrm{T}}\mathrm{S}_{\mathrm{m}}(t) \simeq \mathrm{G}_{\mathrm{p}}^{\mathrm{T}}\mathrm{S}_{\mathrm{m}}(t) + \widehat{\mathrm{Y}}_{\mathrm{p}}^{\mathrm{T}}\mathrm{S}_{\mathrm{m}}(t)$$

Hence

$$\widehat{\mathrm{A}}_{\mathrm{p}}^{\mathrm{T}} \simeq \mathrm{G}_{\mathrm{p}}^{\mathrm{T}} + \widehat{\mathrm{Y}}_{\mathrm{p}}^{\mathrm{T}}$$

Or

$$\widehat{\mathrm{A}}_{\mathrm{p}} \simeq \mathrm{G}_{\mathrm{P}} + \widehat{\mathrm{Y}}_{\mathrm{P}} \ldots (7)$$

Where $\widehat{\mathrm{Y}}_{\mathrm{P}}$ is an m-vector with element equal to the diagonal entries of the m×m matrix $\sum_{q=1}^{n}\mathrm{K}_{p,q}\widetilde{\mathrm{Y}}_q\mathrm{P}$ and $\widehat{\mathrm{A}}_{\mathrm{p}}$ is an m-vector with element equal to the diagonal entries of the m×m matrix$\sum_{q=1}^{n}\mathrm{A}_{\mathrm{p,q}}\,\mathrm{Y}_{\mathrm{q}}^{\mathrm{T}}$. since $\widetilde{\mathrm{Y}}_{\mathrm{q}} = \mathrm{diag}(\mathrm{Y}_{\mathrm{q}})$. Now we can calculate $\widehat{\mathrm{Y}}_{\mathrm{P}}$ as

$$\widehat{\mathrm{Y}}_{\mathrm{P}} = \begin{bmatrix} \frac{h}{2}\sum_{q=1}^{n}\kappa_{p,q}^{(1,1)}y_{q1} \\ h\sum_{q=1}^{n}\kappa_{p,q}^{(2,1)}y_{q1} + \frac{h}{2}\sum_{q=1}^{n}\kappa_{p,q}^{(2,2)}y_{q2} \\ h\sum_{q=1}^{n}\kappa_{p,q}^{(3,1)}y_{q1} + h\sum_{q=1}^{n}\kappa_{p,q}^{(3,2)}y_{q2} + \frac{h}{2}\sum_{q=1}^{n}\kappa_{p,q}^{(3,3)}y_{q3} \\ \vdots \\ h\sum_{q=1}^{n}\kappa_{p,q}^{(m,1)}y_{q1} + h\sum_{q=1}^{n}\kappa_{p,q}^{(m,2)}y_{q2} + \cdots + \frac{h}{2}\sum_{q=1}^{n}\kappa_{p,q}^{(m,m)}y_{qm} \end{bmatrix}$$

Which can be written as:

$$\widehat{\mathrm{Y}}_{\mathrm{P}}=\begin{bmatrix}\frac{h}{2}\sum_{q=1}^{n}\kappa_{p,q}^{(1,1)} & 0 & 0 & \cdots & 0\\ h\sum_{q=1}^{n}\kappa_{p,q}^{(2,1)} & \frac{h}{2}\sum_{q=1}^{n}\kappa_{p,q}^{(2.2)} & 0 & \dots & 0\\ h\sum_{q=1}^{n}\kappa_{p,q}^{(3,1)} & h\sum_{q=1}^{n}\kappa_{p,q}^{(3,2)} & \frac{h}{2}\sum_{q=1}^{n}\kappa_{p,q}^{(3,3)} & \dots & 0\\ \vdots & \vdots & & \vdots & \\ h\sum_{q=1}^{n}\kappa_{p,q}^{(m,1)} & h\sum_{q=1}^{n}\kappa_{p,q}^{(m,2)} & h\sum_{q=1}^{n}\kappa_{p,q}^{(m,3)} & \dots & \frac{h}{2}\sum_{q=1}^{n}\kappa_{p,q}^{(m,m)}\end{bmatrix}_{m\times m}\begin{bmatrix}y_{q1}\\ y_{q2}\\ \vdots\\ y_{qm}\end{bmatrix}_{m\times 1}$$

Hence

$$\widehat{\mathrm{Y}}_{\mathrm{P}}=\begin{bmatrix}\frac{1}{2m}\mathrm{K}^{(1,1)} & 0 & 0 & \cdots & 0\\ \frac{1}{m}\mathrm{K}^{(2,1)} & \frac{1}{2m}\mathrm{K}^{(2,2)} & 0 & \dots & 0\\ \frac{1}{m}\mathrm{K}^{(3,1)} & \frac{1}{m}\mathrm{K}^{(3,2)} & \frac{1}{2m}\mathrm{K}^{(3,3)} & \dots & 0\\ \vdots & \vdots & & \vdots & \\ \frac{1}{m}\mathrm{K}^{(m,1)} & \frac{1}{m}\mathrm{K}^{(m,2)} & \frac{1}{m}\mathrm{K}^{(m,3)} & \dots & \frac{1}{2m}\mathrm{K}^{(m,m)}\end{bmatrix}_{m\times m}\begin{bmatrix}\mathrm{Y}^{(1)}\\ \mathrm{Y}^{(2)}\\ \vdots\\ \mathrm{Y}^{(m)}\end{bmatrix}_{m\times 1}$$

And

$$\widehat{\mathrm{A}}_{\mathrm{p}}=\begin{bmatrix}\sum_{q=1}^{n}a_{p,q}^{(1)}\,y_{q1}\\ \sum_{q=1}^{n}a_{p,q}^{(2)}\,y_{q2}\\ \vdots\\ \sum_{q=1}^{n}a_{p,q}^{(m)}\,y_{qm}\end{bmatrix}$$

Substituting equation (7) into equation (1), also replace $\simeq$ with equality, we obtains,

$$\begin{bmatrix}\mathrm{A}^{(1)}\\ \mathrm{A}^{(2)}\\ \vdots\\ \mathrm{A}^{(m)}\end{bmatrix}\begin{bmatrix}\mathrm{Y}^{(1)}\\ \mathrm{Y}^{(2)}\\ \vdots\\ \mathrm{Y}^{(m)}\end{bmatrix}=\begin{bmatrix}\mathrm{G}^{(1)}\\ \mathrm{G}^{(2)}\\ \vdots\\ \mathrm{G}^{(m)}\end{bmatrix}+\begin{bmatrix}\frac{1}{2m}\mathrm{K}^{(1,1)} & 0 & 0 & \cdots & 0\\ \frac{1}{m}\mathrm{K}^{(2,1)} & \frac{1}{2m}\mathrm{K}^{(2,2)} & 0 & \dots & 0\\ \frac{1}{m}\mathrm{K}^{(3,1)} & \frac{1}{m}\mathrm{K}^{(3,2)} & \frac{1}{2m}\mathrm{K}^{(3,3)} & \dots & 0\\ \vdots & \vdots & & \vdots & \\ \frac{1}{m}\mathrm{K}^{(m,1)} & \frac{1}{m}\mathrm{K}^{(m,2)} & \frac{1}{m}\mathrm{K}^{(m,3)} & \dots & \frac{1}{2m}\mathrm{K}^{(m,m)}\end{bmatrix}\begin{bmatrix}\mathrm{Y}^{(1)}\\ \mathrm{Y}^{(2)}\\ \vdots\\ \mathrm{Y}^{(m)}\end{bmatrix}$$

This can be written as:

$$\mathrm{A}^{(i)}\mathrm{Y}^{(i)}=\mathrm{G}^{(i)}+\frac{1}{2m}\mathrm{K}^{(i,i)}\mathrm{Y}^{(i)}+\frac{1}{m}\sum_{\mathrm{j}=1}^{\mathrm{i}-1}\mathrm{K}^{(i,j)}\mathrm{Y}^{(j)}\,,\qquad i=1,2,\dots,m\,,\qquad \dots(8)$$

Where $\mathrm{Y}^{(i)}=[y_{1i}y_{2i}\ \dots y_{ni}]^{\mathrm{T}}$, $\mathrm{A}^{(i)}=[a_{p,q}^{(i)}]$ and $\mathrm{K}^{(\mathrm{i,j})}=[\kappa_{p,q}^{(i,j)}]forp,q$ =1,2,…,n. By simplifying Eq. (8), we get:

$$\mathrm{Y}^{(\mathrm{i})}=[\mathrm{A}^{(\mathrm{i})}-\frac{1}{2\mathrm{m}}\mathrm{K}^{(\mathrm{i,i})}]^{-1}\left[\mathrm{G}^{(\mathrm{i})}+\frac{1}{\mathrm{m}}\sum_{\mathrm{j}=1}^{\mathrm{i}-1}\mathrm{K}^{(\mathrm{i,j})}\mathrm{Y}^{(\mathrm{j})}\right],i=1,2,\dots,m\,,\qquad \dots(9)$$

If$\mathrm{A}(t)$ is singular, $[\mathrm{A}^{(\mathrm{i})}-\frac{1}{2\mathrm{m}}\mathrm{K}^{(\mathrm{i,i})}]$transform to be nonsingular and hence equation (9) given Sample-and-Hold coefficientsrecursively. Employing this coefficients with Y(t) = $[\mathrm{Y}^{(0)}\ \mathrm{Y}^{(1)}\ \dots\ \mathrm{Y}^{(\mathrm{m}-1)}]\mathrm{S}_{\mathrm{m}}(t)$, now the numerical solution can be facility calculated.

IV. Illustrative Examples:

In this section, we represent three illustrative examples in order to utilize the efficiency and accuracy of the proposed method.

Example (1):

Assume the system of linear volterra integral equations with variable coefficients.

$$\begin{pmatrix} 1 & t \\ -2t & 1 \end{pmatrix}\begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix} = \begin{pmatrix} sin(t) + t\,cos(t) \\ cos(t) - 2t\,sin(t) \end{pmatrix} + \int_0^t \begin{pmatrix} t^2 cos(r) & -t^2 sin(r) \\ sin(t)cos(r) & -sin(t)sin(r) \end{pmatrix}\begin{pmatrix} f_1(r) \\ f_2(r) \end{pmatrix} dr \ldots (10)$$

Where the exact solution is found in [19] as $f_1(t) = sin\,t$, $f_2(t) = cos\,t$.

Following tables (1 - 3) represent a comparison between the approximatesolution ofequation (10) using the proposed method and Block Pulse functions (BPFs)[14] with the exact solution.

Table (1): Comparison between the approximatesolution of equation (10) using the proposed method andBlock Pulse method withthe exact solution when m=32.

t	f_1 (SHFs) m=32	f_1 (BPFs) m=32	Exact f_1	Absolute Error f_1(SHFs)	Absolute Error f_1(BPFs)
0.4	3.952E-001	3.894E-001	3.894E-001	5.749E-003	8.87E-003
0.7	6.584E-001	6.422E-001	6.442E-001	1.423E-002	2.02E-003
0.9	7.872E-001	7.649E-001	7.833E-001	3.87E-003	1.839E-002

t	f_2 (SHFs) m=32	f_2 (BPFs) m=32	Exact f_2	Absolute Error f_2(SHFs)	Absolute Error f_2 (BPFs)
0.4	918.609E-003	896.107E-003	9.211E-001	2.452E-003	2.495E-002
0.7	752.629E-003	614.549E-003	7.648E-001	1.221E-002	1.503E-001
0.9	616.702E-003	357.368E-003	6.216E-001	4.908E-003	2.642E-001

Table (2)

Comparison between the approximate solution of equation (10) using the proposed method and Block Pulse method with the exact solution when m=128.

t	f_1 (SHFs) m=128	f_1 (BPFs) m=128	Exact f_1	Absolute Error f_1 (SHFs)	Absolute Error f_1 (BPFs)
0.4	3.952E-001	3.913E-001	3.894E-001	5.749E-003	1.9E-003
0.7	6.466E-001	6.393E-001	6.442E-001	2.387E-003	4.877E-003
0.9	7.872E-001	7.716E-001	7.833E-001	3.87E-003	1.174E-002

t	f_2 (SHFs) m=128	f_2 (BPFs) m=128	Exact f_2	Absolute Error f_2(SHFs)	Absolute Error f_2 (BPFs)
0.4	918.609E-003	889.143E-003	9.211E-001	2.452E-003	3.192E-002
0.7	762.825E-003	619.258E-003	7.648E-001	2.017E-003	1.456E-001
0.9	616.702E-003	339.387E-003	6.216E-001	4.908E-003	2.822E-001

Table (3)

Comparison between the approximatesolution of equation (10) using the proposed method and Block Pulse method with the exact solution when m=256.

t	f_1 (SHFs) m=256	f_1 (BPFs) m=256	Exact f_1	Absolute Error f_1 (SHFs)	Absolute Error f_1 (BPFs)
0.4	3.916E-001	3.895E-001	3.894E-001	2.158E-003	1.092E-004
0.7	6.442E-001	6.408E-001	6.442E-001	2.387E-003	3.439E-003
0.9	7.848E-001	7.705E-001	7.833E-001	1.455E-003	1.283E-002

t	f_2 (SHFs) m=256	f_2 (BPFs) m=256	Exact f_2	Absolute Error f_2 (SHFs)	Absolute Error f_2 (BPFs)
0.4	920.146E-003	890.346E-003	9.211E-001	9.152E-004	3.072E-002
0.7	762.825E-003	616.909E-003	7.648E-001	2.017E-003	1.479E-001
0.9	619.772E-003	342.384E-003	6.216E-001	1.838E-003	2.792E-001

Example (2) :
Assume the system of linear volterra integral-algebraic equations with variable coefficients.

$$\begin{pmatrix} 0 & 0 \\ t & -2t \end{pmatrix}\begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix} = \begin{pmatrix} -t^2 \\ t - \dfrac{5t^3}{3} + \dfrac{7t^4}{6} \end{pmatrix} + \int_0^t \begin{pmatrix} 3r & 2t+1 \\ 2(t+r) & 2r(t+r) \end{pmatrix}\begin{pmatrix} f_1(r) \\ f_2(r) \end{pmatrix} dr \quad \dots(11)$$

Where the exact solution is found in [14] as $f_1(t) = 1 + t$, $f_2(t) = -t$.
Following tables (4 - 6) represent a comparison between the approximate solution ofequation (11) using the proposed method and Block Pulse functions (BPFs) [14] with the exact solution.

Table (4)
Comparison between the approximate solution of equation (11) using the proposed method and Block Pulse method with the exact solution when m=32.

t	***f₁* (SHFs) m=32**	***f₁* (BPFs) m=32**	**Exact *f₁***	**Absolute Error *f₁* (SHFs)**	**Absolute Error *f₁* (BPFs)**
0.4	1.376E+000	9.324E-001	1.4E+000	2.374E-002	4.676E-001
0.7	1.684E+000	9.009E-001	1.7E+000	1.645E-002	7.991E-001
0.9	1.869E+000	8.368E-001	1.9E+000	3.133E-002	1.063E+000

t	***f₂* (SHFs) m=32**	***f₂* (BPFs) m=32**	**Exact *f₂***	**Absolute Error *f₂* (SHFs)**	**Absolute Error *f₂* (BPFs)**
0.4	-390.789E-003	507.834E-003	-4E-001	9.211E-003	1.078E-001
0.7	-696.109E-003	-592.340E-003	-7E-001	3.891E-003	1.077E-001
0.9	-880.548E-003	-558.420E-003	-9E-001	1.945E-002	3.416E-001

Table (5)
Comparison between the approximate solution of equation (11) using the proposed method and Block Pulse method with the exact solution when m=128.

t	***f₁* (SHFs) m=128**	***f₁* (BPFs) m=128**	**Exact *f₁***	**Absolute Error *f₁* (SHFs)**	**Absolute Error *f₁* (BPFs)**
0.4	1.399E+000	9.543E-001	1.4E+000	1.253E-003	4.457E-001
0.7	1.694E+000	1.232E+000	1.7E+000	5.652E-003	4.681E-001
0.9	1.897E+000	1.518E+000	1.9E+000	3.188E-003	3.816E-001

t	***f₂* (SHFs) m=128**	***f₂* (BPFs) m=128**	**Exact *f₂***	**Absolute Error *f₂* (SHFs)**	**Absolute Error *f₂* (BPFs)**
0.4	-402.257E-003	-643.604E-003	-4E-001	2.257E-003	2.436E-001
0.7	-697.425E-003	-1.047E+000	-7E-001	2.575E-003	3.468E-001
0.9	-899.718E-003	-1.424E+000	-9E-001	2.817E-004	5.244E-001

Table (6)
Comparison between the approximate solution of equation (11) using the proposed method and Block Pulse method with the exact solution when m=256.

t	***f₁* (SHFs) m=256**	***f₁* (BPFs) m=256**	**Exact *f₁***	**Absolute Error *f₁* (SHFs)**	**Absolute Error *f₁* (BPFs)**
0.4	1.399E+000	8.721E-001	1.4E+000	1.423E-003	5.279E-001
0.7	1.699E+000	8.222E-001	1.7E+000	1.276E-003	8.778E-001
0.9	1.898E+000	7.719E-001	1.9E+000	2.388E-003	1.128E+000

t	***f₂* (SHFs) m=256**	***f₂* (BPFs) m=256**	**Exact *f₂***	**Absolute Error *f₂* (SHFs)**	**Absolute Error *f₂* (BPFs)**
0.4	-400.361E-003	-543.706E-003	-4E-001	3.608E-004	1.437E-001
0.7	-700.270E-003	-631.770E-003	-7E-001	2.7E-004	6.823E-002
0.9	-899.082E-003	9.177E-004	-9E-001	9.177E-004	3.142E-001

Example (3) :
Assume the system of linear volterra integral-algebraic equations with variable coefficients.

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix} = \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix} + \int_0^t \begin{pmatrix} t^3 + r + 1 & cos(3r) + 1 \\ t + r + 2 & sin(3r) + 2 \end{pmatrix}\begin{pmatrix} f_1(r) \\ f_2(r) \end{pmatrix} dr \quad \dots(12)$$

With $g_1(t) = 1 - (1 + t + t^3)\sin t - \frac{1}{3}(3 + \cos 3t)(\sin^2 \frac{3t}{2})$

$$g_2(t) = 1 - \cos t - 2(1+t)\sin t + \frac{1}{12}(-8 - 6t + 8\cos 3t + \sin 6t)$$

Where the exact solution is found in [9] as $f_1(t) = cos\ t$, $f_2(t) = \sin 3t$.

Following tables (7 - 9) represent a comparison between the approximate solution ofequation (12) using the proposed method and Block Pulse functions (BPFs) [14]with the exact solution .

Table (7)

Comparison between the approximate solution of equation (12) using the proposed method and Block Pulse method with the exact solution when m=32.

t	f_1 (SHFs) m=32	f_1 (BPFs) m=32	Exact f_1	Absolute Error f_1 (SHFs)	Absolute Error f_1 (BPFs)
0.4	9.329E-001	8.765E-001	9.211E-001	1.185E-002	4.455E-002
0.7	7.700E-001	5.080E-001	7.648E-001	5.166E-003	2.568E-001
0.9	6.413E-001	1.137E-001	6.216E-001	1.972E-002	5.079E-001

t	f_2 (SHFs) m=32	f_2 (BPFs) m=32	Exact f_2	Absolute Error f_2 (SHFs)	Absolute Error f_2 (BPFs)
0.4	1.615E+000	766.728E-003	9.32E-001	6.827E-001	1.653E-001
0.7	1.526E+000	1.215E+000	8.632E-001	6.625E-001	3.522E-001
0.9	1.207E+000	1.835E+000	4.274E-001	7.8E-001	1.408E+000

Table (8)

Comparison between the approximate solution of equation (12) using the proposed method and Block Pulse method with the exact solution when m=128.

t	f_1 (SHFs) m=128	f_1 (BPFs) m=128	Exact f_1	Absolute Error f_1 (SHFs)	Absolute Error f_1 (BPFs)
0.4	9.225E-001	8.676E-001	9.211E-001	1.454E-003	5.342E-002
0.7	7.674E-001	5.146E-001	7.648E-001	2.584E-003	2.502E-001
0.9	6.233E-001	8.309E-002	6.216E-001	1.689E-003	5.385E-001

t	f_2 (SHFs) m=128	f_2 (BPFs) m=128	Exact f_2	Absolute Error f_2 (SHFs)	Absolute Error f_2 (BPFs)
0.4	253.911E-003	793.656E-003	9.32E-001	6.781E-001	1.384E-001
0.7	152.386E-003	1.218E+000	8.632E-001	7.108E-001	3.549E-001
0.9	-431.397E-003	1.913E+000	4.274E-001	8.588E-001	1.486E+000

Table (9)

Comparison between the approximate solution of equation (12) using the proposed method and Block Pulse method with the exact solution when m=256.

t	f_1 (SHFs) m=256	f_1 (BPFs) m=256	Exact f_1	Absolute Error f_1 (SHFs)	Absolute Error f_1 (BPFs)
0.4	9.221E-001	8.691E-001	9.211E-001	1.024E-003	5.193E-002
0.7	7.651E-001	5.114E-001	7.648E-001	2.955E-004	2.535E-001
0.9	6.231E-001	8.833E-002	6.216E-001	1.449E-003	5.333E-001

t	f_2 (SHFs) m=256	f_2 (BPFs) m=256	Exact f_2	Absolute Error f_2 (SHFs)	Absolute Error f_2 (BPFs)
0.4	1.615E+000	787.902E-003	9.32E-001	6.833E-001	1.441E-001
0.7	155.532E-003	1.221E+000	8.632E-001	7.077E-001	3.577E-001
0.9	1.243E+000	1.899E+000	4.274E-001	8.154E-001	1.471E+000

V. Conclusion

In this manuscript, we have used Sample-and-hold functions method for the numerical solutions of linear volterra integral-algebraic equations. The method is calculable on SHFs and operational matrix of integration. The method trade with system of linear volterra integral equations with variable coefficients and volterra integral-algebraic equations with same ease. The systems of algebraic equations are transform to linear lower triangular systems. Numerical results, which assure optical results, demonstrate the effectiveness and applicability of the method.Further, the main advantage is that the attending method, which is facile and direct, needs less computational effort thanother numerical methods.

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