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Dynamical Behaviour of a Two Dimensional Chemical Oscillator

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Abstract: *This study addresses the dynamic behaviour of a chemical oscillator. Of particular interest is the study of the associated Hopf bifurcations. Centre manifold theory and normal form theory are employed in this work to analyse the bifurcation and the stability of the Brusselator. Analysis reveals that there is Hopf bifurcation for the oscillator and the equilibrium loses stability and bifurcates into periodic solutions with limit cycles associated with the Brusselator that are always stable.*
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I. Introduction

It is well known that many chemical systems often exhibit oscillations and thus the study of the stability of these systems is important [1,4]. One of the most widely studied non-linear oscillators is the Brusselator system given by [1,2].

$$A \to X, \qquad B + X \to Y + D \tag{1.1}$$

where A and B are input chemicals, D and Y are outputs chemicals. The ability of nonlinear oscillators to synchronize to external influences has been studied and this provides the basis of analytical investigations of multiple oscillations in a chemical network.

Bifurcation and stability analysis on oscillators has been an interesting topic for several decades and many results have been reported in the literature (see, for instance, reference [4] and [7]). Stephen [6] et al undertook a detailed investigation of a coupled oscillator by fixing ratio exchange coefficients. Recently, the double-pendulum system has been reinvestigated in detail to show that such a system can indeed exhibit rich dynamical behaviour, in the vicinity of various compound critical points.

In 2006 A.B Gumel et developed a second-order method for the numerical solution of the initial-value problems. Their approach was extended to solve a class of non-linear reaction- diffusion equations in two space dimensions known as the "Brusselator" systems. The algorithm was implemented in parallel using two processors, each solving a linear algebraic system as opposed to solving non-linear systems, which is often required when integrating non-linear partial differential equations.

A generic co-dimension 1 bifurcation [7] of the oscillator is studied in detail following the general approach: (1) reduction of a generic parameter-dependent system to the normal form up to terms of a certain order. Relevant normal form and genericity conditions of a bifurcation appear naturally at this step by means of polynomial changes of variables with unknown coefficients that are then fixed at particular values to simplify the equations, (2) analysis of the bifurcations and (3) demonstration that higher order terms do not affect the local bifurcation diagram. In this work we will be concerned with the first two approaches.

A projection method the key to this project found in [7] for center manifold computation that avoids the transformation of the system into its eigenbasis is discussed. Normal form theory is then employed to obtain bifurcation solutions and their associated stability conditions explicitly.

The centre manifold theory helps to reduce the dimension of a system. The centre manifold theorem found in [5] says the dynamics of the system can be projected onto the center manifold without loosing any significant aspects of the dynamics. The dynamics projected onto the center manifold can be transformed to normal forms by a non-linear transformation of the phase space variables. The geometrical consequence of the normal form is to decide on stability questions and determining bifurcations.

II. Bifurcation Theory

Bifurcation theory considers a structure in phase portrait of a fixed point, a periodic orbit or an invariant torus and studies its behavior as a function of the parameter α. At the bifurcation point .i.e. α_0, the structure may change its stability, split into new structure or merge with other structures. By using Taylor's series approximation of the functions and an understanding of the differences that may be eliminated by a change of co-ordinates it is possible to catalog the bifurcation of dynamical systems. The bifurcation of a

hyperbolic fixed point x_0 of a system family $F(x,\alpha)$ can be characterized by the eigenvalues of the first derivatives of the system $DF_x(x_0)$ computed at the bifurcation point.

III. Simple Bifurcation Conditions.

Consider the system

$$\dot{x} = f(x,\alpha) \qquad x \in \mathrm{R}^n, \alpha \in \mathrm{R}^1 \tag{3.1}$$

where f is smooth with respect to both x and α. Let $x = x_0$ be a hyperbolic equilibrium in the system for $\alpha = \alpha_0$. Under a small parameter variation the equilibrium moves slightly but remains hyperbolic. There are generically, only two ways in which the hyperbolicity condition can be violated.

1). When a simple real eigenvalue approaches zero and we have $\lambda_1 = 0$

2). A pair of simple complex eigenvalues reaches the imaginary axes and we have $\lambda_{1,2} = \pm i\omega_0,\ \omega_0 > 0$ for some value of the parameter.

The bifurcation associated with the appearance of $\lambda_1 = 0$ is called a fold bifurcation which is possible when $n \geq 1$. The bifurcation corresponding to the presence of $\lambda_{1,2} = \pm i\omega_0,\ \omega_0 > 0$ is called a Hopf (Andronov – Hopf) bifurcation which is possible when $n \geq 2$.

Consider an autonomous system (3.1), and suppose that for all sufficiently small $|\alpha|$ the system has a family of equilibria $x_0(\alpha)$ and its Jacobian matrix $A(\alpha) = f_x(x_0(\alpha),\alpha)$ has one pair of complex eigenvalues

$$\lambda_{1,2}(\alpha) = \mu(\alpha) \pm i\omega(\alpha)$$

that becomes purely imaginary when $\alpha = 0$ i.e. $\mu(0) = 0$ and $\omega(0) = \omega_0 > 0$.

Then generically as α passes through α = 0 the equilibrium changes stability and a unique limit cycle bifurcates from it. For an example, consider a system with $n = 2$.

$$\dot{x}_1 = f(x_1, x_2, \alpha)$$

$$\dot{x}_2 = f(x_1, x_2, \alpha)$$

The following two conditions must be satisfied for the system to be represented in normal form.

1). $\dfrac{d}{dx}\operatorname{Re}\lambda_{1,2}(\alpha)\big|_{\alpha=0} \neq 0$

2). $l_1(0) \neq 0$. (An explicit formula for $l_1(0)$ will be derived in section (v)).

The first condition known as transversality, means that the pair of complex conjugate eigenvalues $\lambda_{1,2}(\alpha)$ crosses the imaginary axis with a non-zero speed. The second condition (non-degeneracy) implies that a certain combination of Taylor's coefficient of the right-hand side of the system, up to and including third order coefficients does not vanish.

If the above conditions are satisfied then the system is topologically equivalent to the normal form

$$\dot{x} = \beta x_1 - x_2 + \sigma x_1\left(x_1^2 + x_2^2\right)$$

$$\dot{x}_2 = x_1 + \beta x_2 + \sigma x_2\left(x_1^2 + x_2^2\right) \tag{3.2}$$

Where $x = (x_1, x_2)^T, \beta \in \mathrm{R}$ and $\sigma = \operatorname{Sign}\ l_1(0) = \pm 1$. When σ is negative the Hopf bifurcation is supercritical and when σ is positive the Hopf bifurcation is subcritical.

IV. Generic Hopf Bifurcation

We shall now prove that any generic two-dimensional system undergoing a Hopf bifurcation can be transformed in the form of equation (3.2) with a possible difference in the sign of the cubic terms. Consider a system

$$\dot{x} = f(x,\alpha), x = (x_1, x_2)^T, x \in \mathrm{R}^2, \alpha \in \mathrm{R}^1 \tag{4.1}$$

with smooth function *f* which has at $\alpha = 0$ the equilibrium $x = 0$ with eigenvalues $\lambda_{1,2} = \pm i\omega_0, \omega_0 > 0$. The system has a unique equilibrium $x_0(\alpha)$ in some neighborhood of the origin for all sufficiently small $|\alpha|$

since $\lambda = 0$ is not an eigenvalue of the Jacobian matrix. We can perform a coordinate shift placing this equilibrium at the origin and assume that with no loss of generality, $x = 0$ is the equilibrium point of the system for $|\alpha|$ sufficiently small. Thus the system can be written as

$$\dot{x} = A(\alpha)x + F(x,\alpha) \tag{4.2}$$

where F is a smooth vector function whose components $F_{1,2}$ have Taylor expansions in x starting with at least quadratic terms.

The Jacobian matrix $A(\alpha)$ can be written as

$$A(\alpha) = \begin{bmatrix} a(\alpha) & b(\alpha) \\ c(\alpha) & d(\alpha) \end{bmatrix}$$

Its eigenvalues are the roots of the characteristic equation

$\lambda^2 - \sigma\lambda + \Delta = 0,$

where $\sigma = a(\alpha) + d(\alpha) = trA(\alpha)$,and $\Delta = a(\alpha) + d(\alpha) - b(\alpha)c(\alpha) = \det A(\alpha)$

So $\lambda_{1,2}(\alpha) = \frac{1}{2}\left[\sigma(\alpha) \pm \sqrt{\sigma^2(\alpha) - 4\Delta(\alpha)}\right]$

The Hopf bifurcation condition implies $\sigma(0) = 0,\ \Delta(0) = \omega_0^2 > 0$

For small $|\alpha|$ we can introduce $\mu(\alpha) = \frac{1}{2}\sigma(\alpha), \omega(\alpha) = \frac{1}{2}\sqrt{4\Delta(\alpha) - \sigma^2(\alpha)}$

and therefore obtain the following representation for the eigenvalues as

$\lambda_1(\alpha) = \lambda(\alpha), \lambda_2(\alpha) = \bar{\lambda}(\alpha)$ where $\lambda(\alpha) = \mu(\alpha) + i\omega(\alpha), \mu(0) = 0, \omega(0) = \omega_0 > 0.$

The following two lemmas and whose prove are found in [7] help to represent (4.1) into normal form and then analyze the bifurcations arising from it.

Lemma 4.1:

Introducing a complex variable z, system (iv) can be written for sufficiently small $|\alpha|$ *as a single equation*

$$\dot{z} = \lambda(\alpha)z + g(z,\bar{z},\alpha) \tag{4.3}$$

where $g = O(|z|^2)$ *is a smooth function of* $(z,\bar{z},\alpha)$.

Remark: We can write g as a formal Taylor's series in two complex variables, z and $\bar{z}$ as

$$g(z,\bar{z},\alpha) = \sum_{k+l\geq 2} \frac{1}{k!l!} g_{kl}(\alpha) z^k \bar{z}^l$$

where

$$g_{kl}(\alpha) = \frac{\partial^{k+1}}{\partial z^k \partial \bar{z}^l} \left\langle p(\alpha), F\left(z\, q(\alpha) + \bar{z}\, \bar{q}(\alpha), \alpha\right)\right\rangle\Big|_{z=0}$$

for $k + l \geq 2 \qquad k,l = 0,1\ldots$

Suppose that at $\alpha = 0$, the function $F(x,\alpha)$ from (2.3.2) is represented as

$$F(x,0) = \frac{1}{2}B(x,y) + \frac{1}{6}C(x,y,u) + O\left(\|x\|^4\right)$$

where $B(x,y)$ and $C(x,y,u)$ are symmetrical multilinear vector functions of $(x,y,u) \in R^2$. In coordinates, we have

$$B_i(x,y) = \sum_{j,k=1}^{2} \frac{\partial^2 F_i(\xi,0)}{\partial\xi_j \partial\xi_k}\Big|_{\xi=0} x_j y_k \quad i = 1,2$$

$$C_i(x,y,u) = \sum_{j,k=1}^{2} \frac{\partial^3 F_i(\xi,0)}{\partial\xi_j \partial\xi_k \partial\xi_l}\Big|_{\xi=0} x_j y_k u_l \quad i = 1,2 \tag{4.4}$$

Then,

$B(zq+\bar{z}\,\bar{q}, zq+\bar{z}\,\bar{q}) = z^2B(q,q)+2z\bar{z}B(q,\bar{q})+\bar{z}^2B(\bar{q},\bar{q})$ where

$q=q(0), \quad p=p(0)$, the Taylor coefficients $g_{kl}, k+l=2$ of quadratic terms in $g(z,\bar{z},0)$ can be expressed as $g_{20}=\langle p, B(q,q)\rangle, g_{11}=\langle p, B(q,\bar{q})\rangle, g_{02}=\langle p, B(\bar{q},\bar{q})\rangle$

Lemma 4.2: (Poincare normal form for the Hopf bifurcation)
The equation

$$\dot{z}=\lambda z+\sum_{2\le k+l\le 3}\frac{1}{k!l!}g_{kl}z^k\bar{z}^l+0\left(|z|^4\right) \tag{4.5}$$

where $\lambda=\lambda(\alpha)=\mu(\alpha)+i\omega(\alpha), \mu(0), \omega(0)=\omega_0>0$ *and* $g_{ij}(\alpha)$ *can be transformed by an invertible parameter – dependent change of complex coordinate, smoothly depending on the parameter,*

$$z=w+\frac{h_{20}}{2}w^2+h_{11}w\bar{w}+\frac{h_{02}}{2}\bar{w}^2+\frac{h_{30}}{6}w^3+\frac{h_{12}}{2}w\bar{w}^2+\frac{h_{03}}{6}\bar{w}^3,$$

for all sufficiently small $|\alpha|$*, into an equation with only the resonant cubic term*

$$\dot{w}=\lambda w+C_1w^2\bar{w}+0\left(|w|^4\right) \tag{4.5}$$

where $C_1=C_1(\alpha)$*, by setting* $h_{20}=\frac{g_{20}}{\lambda}, \quad h_{11}=\frac{g_{11}}{\bar{\lambda}}, \quad h_{02}=\frac{g_{02}}{2\bar{\lambda}-\lambda}$.

V. The Normal Form of Hopf Bifurcation

Consider a system of two differential equations depending on one parameter α,

$$\dot{x}_1=\alpha x_1-x_2-x_1\left(x_1^2+x_2^2\right)$$
$$\dot{x}_2=x_1+\alpha x_2-x_2\left(x_1^2+x_2^2\right) \tag{5.1}$$

The system has the equilibrium $x_1=x_2=0$ for all α with Jacobian matrix

$$A(0)=f_x(0,0)=\begin{pmatrix}\alpha & -1\\ 1 & \alpha\end{pmatrix}$$

having eigenvalues $\lambda_{1,2}=\alpha\pm i$. Introducing the complex variable

$$z=x_1+ix_2, \quad \bar{z}=x_1-ix_2, \quad |z|^2=z\bar{z}=x_1^2+x_2^2.$$

Replacing $\dot{x}_1$ and $\dot{x}_2$ in (2.4.1) we obtain

$$\dot{z}=\alpha(x_1+ix_2)+i(x_1+ix_2)-(x_1+ix_2)\left(x_1^2+x_2^2\right)$$

and the system can be written in complex form as

$$\dot{z}=(\alpha+i)z-z|z|^2 \tag{5.2}$$

In polar coordinates, let $z=re^{i\theta}$ so that $\dot{z}=re^{i\theta}\left(\alpha+i-r^2\right)$

which gives the polar form of system (5.1) as

$$\dot{r}=r\left(\alpha-r^2\right) \quad \dot{\theta}=1 \tag{5.3}$$

Since the equation in r and θ are uncoupled, the bifurcations of these phase portraits of the system as α passes through zero can easily be analyzed using the polar form.

The first equation has the equilibrium point $r=0$ for all values α. The equilibrium is asymptotically stable for $\alpha<0$, stable for $\alpha=0$ and unstable for $\alpha>0$. Moreover, there is additional stable equilibrium point, a circle whose radius $r_0=\sqrt{\alpha}$ for $\alpha>0$. This equilibrium is surrounded for a $\alpha>0$ by an isolated closed orbit (limit cycle) that is unique and stable. All orbits starting outside or inside the cycle except at the origin tend to the circle as $t\to+\infty$ and this is known as supercritical Hopf bifurcation.

The second equation describes a rotation with constant speed.

A system having nonlinear terms with opposite sign to that of (5.1) has the complex form $\dot{z} = (\alpha + i)z + z|z|^2$ and can be analyzed in the same way. Contrary to system (5.1), there is unstable limit cycle which disappears when α crosses from negative to positive values. The equilibrium at the origin has the same stability for $\alpha \neq 0$ as in system (5.1) i.e. stable for $\alpha < 0$ and unstable for $\alpha > 0$. Its stability at the critical parameter point is opposite to that of (5.1). All orbits starting outside or inside the cycle tends away from it and this is known as subcritical Hopf bifurcation.

VI. System Equations and Static Analysis

From the law of mass action we have the following system of two nonlinear kinetic equations associated with equation (1.1), given by Tyson [2].

$$\begin{aligned} \dot{z}_1 &= f_1(z, A, B) = A - Bz_1 + z^2 z - z_1 \\ \dot{z}_2 &= f_2(z, A, B) = Bz_1 - z_1^2 z_2 \end{aligned} \tag{6.1}$$

where $z = (z_1, z_2)^T = (X, Y)^T$, with A and B real constants.

The only equilibrium of the system determined by setting $z = 0$ is $z_c = \left(A, \frac{B}{A}\right)^T$.

Performing a coordinate shift $z = y + z_c$, to place this equilibrium at the origin yields

$$\begin{aligned} \dot{y}_1 &= (B-1)y_1 + A^2 y_2 + \frac{B}{A} y_1^2 + 2Ay_1 y_2 + y_1^2 y_2 \\ \dot{y}_2 &= -By_1 - A^2 y_2 - \frac{B}{A} y_1^2 - 2Ay_1 y_2 + y_1^2 y_2 \end{aligned} \tag{6.2}$$

The unique equilibrium of system (6.2) is $y = 0$ and the characteristic polynomial of the linearized system evaluated at the equilibrium is

$$\left[\lambda^2 + \left(A^2 - B + 1\right)\lambda + A^2\right] = 0 \tag{6.3}$$

In order for the equilibrium $y = 0$ to be stable, the conditions

$$A^2 - B + 1 = 0, A \neq 0 \tag{6.4}$$

must be satisfied.

Suppose that the parameters *A* and *B* are real and positive as in the case of any chemical system, the condition (6.4) is satisfied and defines a region in the *A*-*B* space under the curve excluding *B*-axis, where the Brusselator is stable. If parameters A and B are chosen too close to the stability boundary of the curve, then we have a bifurcation model.

VII. Analysis of the dynamic system

Fix $A > 0$ and take B as a bifurcation parameter. The Jacobian matrix evaluated at the equilibrium point $y = 0$ is

$$A(B) = \begin{bmatrix} B-1 & A^2 \\ -B & -A^2 \end{bmatrix}.$$

The eigenvalues $\lambda_{1,2}$ satisfy the characteristic equation

$$\lambda^2 - \left(B - 1 - A^2\right)\lambda + A^2 = 0. \tag{7.1}$$

Let $\mu(B) = \frac{1}{2}\left(B - \left(1 + A^2\right)\right)$ then $\lambda_{1,2} = \mu(B) \pm \sqrt{\mu^2(B) - A^2}$

If $\mu^2(B) < A^2$, then the roots of (6.2) form a complex conjugate pair

$\lambda_1 = \bar{\lambda}_2 = \mu(B) + i\omega$, where $\omega = \sqrt{\mu^2(B) - A^2}$. At $B = 0$ we have $\mu(B_0) = 0$ for $B_0 = 1 + A^2$ and $\omega^2(B_0) = A^2 > 0$ and Hopf bifurcation condition is satisfied.

The transversality condition is easy to verify as $\mu'(B_0) = \frac{1}{2} > 0$.

To check for non-degeneracy condition, we need to compute the first lyapunov coefficient. The system (6.2) can be written as

$$\dot{y} = Ay + F(y), \tag{7.2}$$

where $A = A(B_0)$ and F(y) is a smooth function whose components have Taylor's expansion starting with quadratic terms represented from (4.5) as

$$B(\xi,\eta) = \begin{bmatrix} \frac{2B}{A}\xi_1\eta_1 + 2A(\xi_1\eta_2 + \xi_2\eta_1) \\ -2\frac{B}{A}\xi_1\eta_1 - 2A(\xi_1\eta_2 + \xi_2\eta_1) \end{bmatrix} \tag{7.3}$$

and

$$C(\xi,\eta,\zeta) = \begin{bmatrix} 2\xi_1\eta_1\zeta_2 + 2(\xi_1\eta_1\zeta_2 + \xi_1\zeta_2\eta_1) \\ -2\xi_1\eta_1\zeta_2 - 2(\xi_1\eta_1\zeta_2 + \xi_1\zeta_2\eta_1) \end{bmatrix} \tag{7.4}$$

Let $q \in \mathrm{R}^2$ be an eigenvector of $A(B_0)$ corresponding to the eigenvalue λ_1 so that

$A(B_0)q = \lambda_1(B_0)q$. That is $\begin{bmatrix} A^2 & A^2 \\ -1-A^2 & -A^2 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = iA\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$. Solving, $q = \begin{bmatrix} -\left(\frac{iA + A^2}{A^2+1}\right), & 1 \end{bmatrix}^T$.

Let p be the eigenvector of the matrix $A^T(B_0)$ corresponding to the eigenvalue λ_2,

then $A^T(B_0)p = \lambda_2(B_0)p$ and $p = \begin{bmatrix} \frac{-iA + A^2}{A^2}, & 1 \end{bmatrix}^T$.

Normalizing p with respect to q

$$\begin{aligned} < p,q > &= \bar{p}_1 q_1 + \bar{p}_2 q_2 \\ &= \left(\frac{iA + A^2}{A^2}\right)\left(\frac{-iA - A^2}{A^2+1}\right) + 1 \\ &= \frac{2(1 - Ai)}{(A^2+1)}. \end{aligned}$$

Thus $$p = \frac{1+A}{2(1+iA)}\left(\frac{-iA + A^2}{A^2}, \quad 1\right)^T$$

$$= \begin{bmatrix} \frac{-i(A^2+1)}{2A}, & \frac{1 - iA}{2} \end{bmatrix}^T.$$

From equation (7.3)

$$B(q,q) = \begin{bmatrix} \frac{2B}{A}q_1^2 + 4Aq_1q_2 \\ \frac{-2B}{A}q_1^2 - 4Aq_1q_2 \end{bmatrix}$$

and $$g_{20} = < p, B(q,q) > = \bar{p}_1 B_1(q,q) + \bar{p}_2 B_2(q,q) = A - i$$

$$B(q,\bar{q}) = \begin{bmatrix} \frac{2B}{A} q_1\bar{q}_1 + 2A\left(q_1\bar{q}_2 + \bar{q}_1 q_2\right) \\ -\frac{2B}{A} q_1\bar{q}_1 - 2A\left(q_1\bar{q}_2 + \bar{q}_1 q_2\right) \end{bmatrix}$$

$$g_{11} =< p, B(q,\bar{q}) >= \bar{p}_1 B_1(q,\bar{q}) + \bar{p}_2 B_2(q,\bar{q})$$

$$= \frac{\left(A^2+1\right)\left(A-1\right)}{A^2+1}.$$

$$B(\bar{q},\bar{q}) = \begin{bmatrix} \frac{2B}{A}\bar{q}_1^2 + 4A\bar{q}_1\bar{q}_2 \\ \frac{-2B}{A}\bar{q}_1^2 - 4A\bar{q}_1\bar{q}_2 \end{bmatrix}$$

$$g_{02} =< p, B(\bar{q},\bar{q}) >= \bar{p}_1 B_1(\bar{q},\bar{q}) + \bar{p}_2 B_2(\bar{q},\bar{q})$$

$$= A + i$$

Similarly from (7.4) $C(q,q,\bar{q}) = \begin{bmatrix} 2q_1^2\bar{q}_2 + 4q_1\bar{q}_1 q_2 \\ -2q_1^2 q_2 - 4q_1\bar{q}_1 q_2 \end{bmatrix}$ and

$$g_{21} =< p, C(q,q,\bar{q}) >= \bar{p}_1 C_1(q,q,\bar{q}) + \bar{p}_2 C_2(q,q,\bar{q})$$

$$= \frac{-A(3A-i)}{(A^2+1)}.$$

Computing the first lyapunov coefficient we get

$$l_1(0) = \frac{1}{2\omega_0^{\,2}} \mathrm{Re}(ig_{20}g_{11} + \omega_0 g_{21})$$

$$= -\frac{2+A^2}{2A(A^2+1)} < 0$$

Hence non-degeneracy condition is satisfied and the system exhibit supercritical Hopf bifurcation. By lemma 4.1 the equation (6.2) is restricted to the center manifold

$$\dot{z} = \lambda z + \frac{1}{2}(A-i)z^2 + \frac{(A^2+1)(A-1)}{(A^2+1)} z\bar{z} + \frac{1}{2}(A+i)\bar{z}^2 + \frac{1}{2}A\frac{(i-3A)}{(A^2+1)} + \ldots \qquad (7.5)$$

By lemma 4.2 the quadratic and cubic terms are removed except the resistance term containing $z^2\bar{z}$ whose coefficient at the bifurcation parameter value is

$$C_1 = \frac{i}{2A}\left[g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2 \right] + \frac{1}{2}g_{21}$$

$$= \frac{-3A^2 - 6A - i(4A^4 - 7A^2 + 4)}{24(A^2+1)}.$$

Therefore (7.5) becomes

$$\dot{z} = (\frac{1}{2}\mu(B) + i)Az + \left[-\frac{(2+A^2)}{8(A^2+1)} - i\frac{(4A^4 - 7A^2 + 4)}{24(A^2+1)} \right] z^2\bar{z} + \ldots$$

and written in polar form becomes

$$\dot{r} = \frac{1}{2}A\mu(B)r - \frac{(2+A^2)}{8(A^2+1)} r^3$$

$$\dot{\theta} = A - \frac{(4A^4 - 7A^2 + 4)}{24(A^2+1)} r^2$$

(7.6)

where $\mu(B)$ is treated as a small parameter, r and θ are the amplitude and the phase of the periodic motion respectively. Setting $\dot{r} = 0$ results in two steady state solutions

$$a) \quad r = 0 \qquad\qquad b) \quad r^2 = \frac{4A(A^2+1)}{(2+A^2)}\mu(B) \tag{7.7}$$

Solution (a) $r = 0$ represents the initial equilibrium solution $y_1 y_2 = 0$. When $A > 0$ the bifurcating solution (b) exists for $\mu(B) > 0$.

The stability of the steady solutions is determined by the Jacobian of the first equation of (7.7(a)), $J = d\dot{r}/dr$. When $J < 0 (> 0)$ the solution is stable (unstable).

The stability conditions are given by

	Equilibrium solution	Periodic solution
$\mu(B) > 0$	Unstable	Stable
$\mu(B) < 0$	Stable	Unstable.

This shows that the equilibrium losses stability when $\mu(B)$ is varied to cross the critical point $\mu_c = \mu(B_0) = 0$ and bifurcates into a family of periodic solutions. For periodic solutions it is clear that when $\mu(B) > 0 (A > 0)$ the system undergoes supercritical Hopf bifurcation i.e. the solutions given are always stable. This confirms the results in [4].

VIII. Conclusion and Recommendations

A chemical system has been studied in detail to show the dynamic behavior and the stability conditions for the steady state solutions are also given explicitly in terms of the system parameters. Critical stability boundaries along which incipient bifurcations takes place leading to periodic solutions are also explicitly obtained. All the results are derived using center manifold theory and normal form theory.

Numerical simulations for example Gauss-Seidel (GS1), Euler and Runge-Kutta forth order can be used to verify the analytic result of the initial value problem (IVP).

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