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Fixed Point Theorem In Menger Space Using Integral Functions

*V. H. Badshah[1], Suman Jain[2] and Subhash Mandloi[1]
[1]School of Studies in Mathematics, Vikram University, Ujjain (M.P.) 456010 India
[2]Department of Mathematics, Govt. College, Kalapipal (M.P.) India
Corresponding Author: V. H. Badshah

Abstract: *The present paper deals with a new fixed point theorem in Menger space. This is in line with research in fixed point theory using any kind of coincidentally commuting mappings and integral functions. Examples have also been given in support of our result.*

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I. Introduction:

Due to the wide applicability of fixed point theory, it is considered as a major branch of non-linear functional analysis. Fixed point theory is an interesting area, with a tremendous numbers of applications in various fields of Mathematics such as the theory of differential and integral equations, boundary value problem and variational inequalities etc. There have been a number of generalizations of metric space. One such generalization is Menger space initiated by Menger [4]. It is a probabilistic generalization in which we assign to any two points x and y, a distribution function $F_{x,y}$. Schweizer and Sklar [6] studied this concept and gave some fundamental results on this space. Sehgal and Bharucha-Reid [7] obtained a generalization of Banach Contraction Principle on a complete Menger space which is backbone in the field of fixed point theory in Menger space. The notion of compatible mapping in a Menger space has been introduced by Mishra [5]. Many mathematicians gave fruitful results in Menger space which help us to study fixed point theory. In the sequel, our result is a milestone in the field of fixed point theory and helps the Mathematicians to elaborate new fixed point theory.

II. Preliminaries

For terminologies, notations and properties of probabilistic metric spaces, refer to [6] and [7].

Definition 2.1. [2] A mapping $\mathcal{F} : R \to R^+$ is called a distribution if it is non-decreasing left continuous with $\inf\{\mathcal{F}(t) : t \in R\} = 0$ and $\sup\{\mathcal{F}(t) : t \in R\} = 1$.

We shall denote by L the set of all distribution functions while H will always denote the specific distribution function defined by

$$H(t) = \begin{cases} 0, & t \le 0, \\ 1, & t > 0. \end{cases}$$

Definition 2.2. [3] A triangular norm * (shortly t-norm) is a binary operation on the unit interval [0, 1] such that for all a, b, c, d ∈ [0, 1] the following conditions are satisfied :

(t-1) $a * 1 = a$;
(t-2) $a * b = b * a$;
(t-3) $a * b \le c * d$ whenever $a \le c$ and $b \le d$;
(t-4) $a * (b * c) = (a * b) * c$.

Examples of t-norms are $a * b = \max\{a + b - 1, 0\}$ and $a * b = \min\{a, b\}$.

Definition 2.3. [6] A probabilistic metric space (PM-space) is an ordered pair $(X,\mathcal{F})$ consisting of a non empty set X and a function $\mathcal{F} : X \times X \to L$, where L is the collection of all distribution functions and the value of $\mathcal{F}$ at $(u, v) \in X \times X$ is represented by $F_{u,v}$. The function $F_{u,v}$ assumed to satisfy the following conditions :

(PM-1) $F_{u,v}(x) = 1$, for all $x > 0$, if and only if $u = v$;
(PM-2) $F_{u,v}(0) = 0$;
(PM-3) $F_{u,v} = F_{v,u}$;
(PM-4) If $F_{u,v}(x) = 1$ and $F_{v,w}(y) = 1$ then $F_{u,w}(x + y) = 1$,

for all $u, v, w \in X$ and $x, y > 0$.

A *Menger space* is a triplet $(X, \mathcal{F}, *)$ where $(X, \mathcal{F})$ is a PM-space and $*$ is a t-norm such that the inequality

(PM-5) $F_{u,w}(x + y) \geq F_{u,v}(x) * F_{v,w}(y)$, for all $u, v, w \in X$ and $x, y \geq 0$.

Proposition 2.1. [7] If (X,d) is a metric space then the metric d induces a mapping $X \times X \to L$ defined by $F_{p,q}(x) = H(x - d(p, q))$, for all $p, q \in X$ and $x > 0$. Further, if the t-norm $*$ is $a * b = \min\{a, b\}$ for all $a, b \in [0, 1]$, then $(X, \mathcal{F}, *)$ is a Menger space. It is complete if (X, d) is complete.

The space $(X, \mathcal{F}, *)$ so obtained is called the *induced Menger space.*

Definition 2.4. [5] A sequence $\{x_n\}$ in a Menger space X is said to be *convergent* and converges to a point x in X if and only if for each $\varepsilon > 0$ and $\lambda > 0$, there is an integer $M(\varepsilon, \lambda)$ such that $F_{x_n,x}(\varepsilon) > 1 - \lambda$, for all $n \geq M(\varepsilon, \lambda)$.

Further the sequence $\{x_n\}$ is said to be *Cauchy sequence* if for $\varepsilon > 0$ and $\lambda > 0$, there is an integer $M(\varepsilon, \lambda)$ such that $F_{x_n,x_m}(\varepsilon) > 1- \lambda$, for all $m, n \geq M(\varepsilon, \lambda)$.

A Menger space is said to be *complete* if every Cauchy sequence in X converges to a point in X.

Definition 2.5. [5] Self maps S and T of a Menger space $(X, \mathcal{F}, *)$ are said to be *compatible* if $F_{STx_n,TSx_n}(t) \to 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $Sx_n, Tx_n \to u$, for some u in X, as $n \to \infty$.

Definition 2.6. [1] Self maps A and S of a Menger space $(X, \mathcal{F}, *)$ are said to be occasionally weakly compatible (owc) if and only if there is a point x in X which is coincidence point of A and S at which A and S commute.

Lemma 2.1. [Lebesgue Dominated Convergence Theorem] If a sequence $\{f_n\}$ of Lebesgue measurable function converges almost everywhere to f and if there exists an integrable function $g \geq 0$ such that $|f_n(x)| \leq g(x)$ for every n, then $|f(x)| \leq g(x)$ and $\lim_{n\to\infty} \int f_n(x)\,d\mu = \int f(x)\,d\mu$.

Definition 2.7. A pair of self mappings (f, g) of a Menger space $(X, \mathcal{F}, *)$ is said to be any kind of coincidentally commuting mappings if and only if there is a sequence $\{x_n\}$ in X satisfying $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u$, for some $u \in X$ and $fgu = gfu$ at this point.

Example 2.1. Let $(X, \mathcal{F}, *)$ be a Menger space, where X = [0, 2] with a t – norm defined by $a * b = \min\{a, b\}$ for all $a, b \in X$ and

$$F_{x,y}(t) = \begin{cases} \frac{t}{t+|x-y|} & , \text{ if } t > 0 \\ 0 & , \text{ if } t = 0 \end{cases} \quad \text{for all } x, y \in X.$$

Define $f, g : [0, 2] \to [0, 2]$ by

$$f(x) = \begin{cases} 2, & \text{if } x \in [0,1] \\ \frac{x}{2}, & \text{if } x \in (1,2] \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 2, & \text{if } x \in [0,1] \\ \frac{x+3}{5}, & \text{if } x \in (1,2] \end{cases}.$$

Consider the sequence $\{x_n\} = \left(2 - \frac{1}{2n}\right)$. Clearly $f(1) = g(1) = 2$ and $f(2) = g(2) = 1$.

Also $fg(1) = gf(1) = 1$ and $fg(2) = gf(2) = 2$. Thus, f and g are coincidentally commuting mappings.

Now $fx_n = \left(1 - \frac{1}{4n}\right)$ and $gx_n = \left(1 - \frac{1}{10n}\right)$.

Therefore, $fx_n \to 1$, $gx_n \to 1$, $fg(x_n)=2$, $gf(x_n) = \left(\frac{4}{5} - \frac{1}{20n}\right)$ & $\lim_{n\to\infty} F_{fgx_n,gfx_n}(t) = \frac{t}{t+\frac{6}{5}} \neq 1$,

so f and g are not compatible maps on X but they are any kind of coincidentally commuting mappings.

Remark 2.1. In view of above example, it is clear that any kind of coincidentally commuting is more general than that of compatible mappings.

III. Main Result

Theorem 3.1. Let A and B be occasionally weakly compatible self maps on a complete Menger space $(X, \mathcal{F}, *)$ with t-norm defined as $a * b = \min\{a, b\}$ for all $a, b \in X$ satisfying

(1) $A(X) \subset B(X)$

(2) $\int_0^{1-F_{Ax,Ay}(ct)} \phi(p)dp < \int_0^{1-F_{Bx,By}(t)} \phi(p)dp$ for each x, y $\in$ X, t > 0, c $\in$ [0,1),

where $\phi : R^+ \to R^+$ is a Lebesgue integrable mapping which is summable, non-negative and such that

(3) $\int_0^{\varepsilon} \phi(p)dp > 0$ for each $\varepsilon > 0$.

(4) A and B are any kind of coincidentally commuting mappings.

(5) A(X) or B(X) is a closed subset of X.

Then A & B have a coincidence point. Moreover, A & B have a unique common fixed point.

Proof. Since A and B are any kind of coincidentally commuting maps, therefore there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = u \in X$. Since either A(X) or B(X) is a closed subspace of X, for definiteness we assume that B(X) is a closed subset of X. Further, note that the sequence $\{y_{2n}\}$ which is contained in B(X), so there is a limit in B(X). Call it u such that u = Ba for some a $\in$ X.

Therefore, $\lim_{n\to\infty} Ax_n = u = Ba = \lim_{n\to\infty} Bx_n$. This implies u = Ba $\in$ B(X).

Now we show that u = Aa = Ba. Now from (2), we have

$$\int_0^{1-F_{Aa,Ax_n}(ct)} \phi(p)dp < \int_0^{1-F_{Ba,Bx_n}(t)} \phi(p)dp .$$

Letting n $\to \infty$, and using the Lebesgue Dominated Convergence theorem and c $\in$ [0,1), it follows in view of (3) that Aa = Ba, implies that Aa = Ba = u.

Thus a is the coincidence point of A and B. Since A and B are occasionally weakly compatible, therefore, ABa = BAa = Au = Bu.

Now we show that Au = u. From (2), we have $\int_0^{1-F_{Au,Aa}(ct)} \phi(p)dp < \int_0^{1-F_{Bu,Ba}(t)} \phi(p)dp$

which in turn implies that Au = u. Hence u is common fixed point of A and B.

Uniqueness. Suppose that w($\neq$ u) is also another common fixed point of A and B.

From (2), we have

$$\int_0^{1-F_{u,w}(t)} \phi(p)dp = \int_0^{1-F_{Au,Aw}(t)} \phi(p)dp < \int_0^{1-F_{Bu,Bw}(t/c)} \phi(p)dp = \int_0^{1-F_{u,w}(t/c)} \phi(p)dp$$

where c $\in$ [0, 1). Thus, w = u, therefore uniqueness follows.

References

[1]. Jain, A. and Chaudhary, B., On common fixed point theorems for semi-compatible and occasionally weakly compatible mappings in Menger space, International Journal of Research and Reviews in Applied Sciences, Vol. 14 (3), (2013), 662-670.

[2]. Jain, A. and Singh, B., *Common fixed point theorem in Menger Spaces,* The Aligarh Bull. of Math. 25 (1), (2006), 23-31.

[3]. Kutukcu, S., *A fixed point theorem in Menger spaces,* International Mathematical Forum 1(32), (2006), 1543-1554.

[4]. Menger, K., *Statistical metrics,* Proc. Nat. Acad. Sci. USA. 28(1942), 535 -537.

[5]. Mishra, S.N., *Common fixed points of compatible mappings in PM-spaces*, Math. Japon. 36(2), (1991), 283-289.

[6]. Schweizer, B. and Sklar, A., *Statistical metric spaces*, Pacific J. Math. 10 (1960), 313-334.

[7]. Sehgal, V.M. and Bharucha-Reid, A.T., *Fixed points of contraction maps on probabilistic metric spaces,* Math. System Theory 6(1972), 97- 102.

[8]. Singh, B. and Jain, S., *A fixed point theorem in Menger space through weak compatibility,* J. Math. Anal. Appl., 301 (2005), 439-448.

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