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A New Discrete Model for the Simulation of the Dynamics of a Simple Series Circuit

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Abstract: *Non-standard method is an efficient numerical technique that provides a relatively simple way of creating discrete models that can correctly replicate the solution of physical applications that are modeled by linear and nonlinear differential equations. This method, which involves a transformation of the differential and non-linear components of the difference equations using the non standard rules 2 and 3, has become a very powerful member of the finite difference family with applications to diverse physical phenomena. In this paper, we applied the method to simulate the dynamics of a simple series circuit represented by a first order ordinary differential equation. The solution of the scheme has the same monotonic behavior with the analytic solution and also converges to the analytic solutions.*

Keywords: *Non-standard methods, Series circuit, Interpolation functions, Discrete model, Denominator functions*

I. Introduction

Consider a simple RC series circuit driven by a voltage source $v(t)$. Because the resistor and capacitor are connected in series, they must have the same current $i(t)$.

The element constraint for a capacitor C is given by

$$i(t) = C\frac{dv(t)}{dt}$$

The voltage across the resistor, $v_R(t)$ is given by

$$v_R(t) = RC\frac{dv(t)}{dt}$$

Using Kirchhoff's voltage law, the sum of the voltage rises and drops around a loop of a circuit is equal to zero. Therefore, the voltage across the resistor, $v_R(t)$, for the RC series circuit is given by

$$v_R(t) = RC\frac{dv(t)}{dt} + v(t) \qquad (1)$$

Equation (1) is a first-order ordinary differential equation where the unknown function is the capacitor voltage. The state and status of the series circuit can be described by capacitor voltage $v_C(t)$ and inductor current $i_L(t)$. The RC series circuit is a first-order circuit and can be described by a first-order ordinary differential equation.

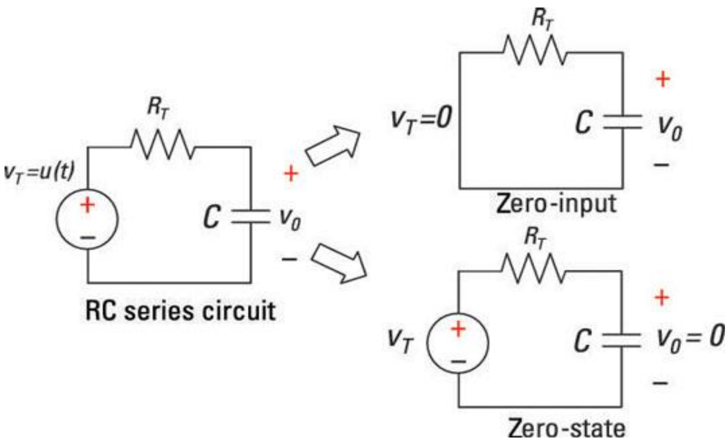

Figure 1: Series Circuit Diagram

Thus, if we have a series circuit containing an inductor L and a resistor R then the dynamics of the flowing current $i(t)$ with respect to the impressed voltage $E(t)$ is also govern by a first order ordinary differential equation of the type

$$L\frac{di}{dt} + Ri = E(t) \tag{2}$$

This relationship leads to a second order differential equation if a single loop series circuit contains an inductance L, a Resistance R and a capacitance C connected in an RLC series circuit. Since the current $i(t)$ is related to the charge on the capacitor $q(t)$ by $i = \frac{dq}{dt}$

the voltage drop across a capacitor with capacitance C is $\frac{1}{C}q$

the differential equation governing the three components Inductor, Resistor and Capacitor with the impressed voltage $E(t)$ can be written as

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t) \tag{3}$$

In this work, we shall consider the first order equation and introduce a second degree interpolating function for the derivation of the discrete model.

II. Derivation Of The Discrete Model

Let us assume interpolation function of the form

$$y(x) = (a_0 + a_1x + a_2x^2) \tag{4}$$

$$y'(x) = a_1 + 2a_2x \tag{5}$$

$$y''(x) = 2a_2 \tag{6}$$

$$a_2 = \frac{y''(x)}{2} \tag{7}$$

From (5),

$$a_1 = y'(x) - 2a_2x \tag{8}$$

Putting (7) in (8)

$$a_1 = y'(x) - \cancel{2}x\frac{y''(x)}{\cancel{2}}$$

$$\therefore a_1 = y'(x) - xy''(x) \tag{9}$$

Let

$$y(x_n) = a_0 + a_1x_n + a_2x_n^2$$

$$y(x_{n+1}) = a_0 + a_1x_{n+1} + a_2x_{n+1}^2$$

$$y(x_{n+1}) - y(x_n) = \cancel{a_0} + a_1x_{n+1} + a_2x_{n+1}^2 - \cancel{a_0} - a_1x_n - a_2x_n^2$$

$$y(x_{n+1}) = y(x_n) + a_1(x_{n+1} - x_n) + a_2(x_{n+1}^2 - x_n^2) \tag{10}$$

Putting (7) and (9) in (10), we have

$$y(x_{n+1}) = y(x_n) + \{y'(x) - xy''(x)\}(x_{n+1} - x_n) + \frac{y''(x)}{2}(x_{n+1}^2 - x_n^2) \tag{11}$$

but

$$x_n = a + nh, \quad x_{n+1} = a + (n+1)h, \quad x_{n+1} - x_n = h$$

$$x_n^2 = (a + nh)^2 = (a + nh)(a + nh) = a^2 + 2anh + (nh)^2 \tag{12}$$

$$x_{n+1}^2 = (a + nh + h)^2 = a^2 + 2anh + 2ah + (nh)^2 + 2nh^2 + h^2 \tag{13}$$

$$(x_{n+1}^2 - x_n^2) = a^2 + 2anh + 2ah + (nh)^2 + 2nh^2 + h^2 - a^2 - 2anh - (nh)^2 \tag{14}$$

$$(x_{n+1}^2 - x_n^2) = 2ah + 2nh^2 + h^2 \tag{15}$$

putting (15) in (11), we have

$$y(x_{n+1}) = y(x_n) + \{y'(x) - xy''(x)\}h + \frac{y''(x)}{2}(2ah + 2nh^2 + h^2) \tag{16}$$

$$y'(x) = f_n$$

$$y''(x) = f_n'$$

Then, (16) becomes:

$$y(x_{n+1}) = y(x_n) + (f_n - xf_n')h + \frac{f_n'}{2}(2ah + 2nh^2 + h^2) \tag{17}$$

Substituting for x_n in (17) to get

$$y(x_{n+1}) = y(x_n) + hf_n - (a + nh)hf_n' + \frac{f_n'}{2}(2ah + 2nh^2 + h^2) \tag{18}$$

OR

$$y_{n+1} = y_n + hf_n - (a + nh)hf_n' + \frac{(2ah+2nh^2+h^2)}{2} f_n' \tag{19}$$

This is the standard numerical scheme. It is a one step method. The qualitative properties of standard numerical schemes are given in Henrici (1962), Fatunla (1988), Lambert (1991) and Obayomi et al. (2017).

III. Proof Of Convergence

Simplifying equation (19), we obtain

$$y_{n+1} = y_n + hf_n + \left(\frac{(2ah+2nh^2+h^2)}{2} - (a + nh)h\right) f_n' \tag{20}$$

$$y_{n+1} = y_n + hf_n + \frac{h^2}{2} f_n' \tag{21}$$

The incremental function can be written as

$$\phi(x_n, y_n, h) = hf_n + \frac{h^2}{2} f_n'$$

$$\phi(x_n, y_n, h) = Af_n + Bf_n' \tag{22}$$

$$\phi(x_n, y_n, h) - \phi(x_n, y_n^*, h) = A[f(x_n, y_n, h) - f(x_n, y_n^*, h)] + B[f'(x_n, y_n, h) - f'(x_n, y_n^*, h)]$$

$$= A[f(x_n, y_n) - f(x_n, y_n^*)] + B[f'(x_n, y_n) - f'(x_n, y_n^*)]$$

$$= A[\frac{\partial f(x_n, \bar{y})}{\partial y_n}(y_n - y_n^*)] + B[\frac{\partial f'(x_n, \bar{y})}{\partial y_n}(y_n - y_n^*)]$$

L1 = $\mathrm{SUP}_{(x_n, y_n) \in D} \ \frac{\partial f(x_n, \bar{y})}{\partial y_n}$ and

L2 = $\mathrm{SUP}_{(x_n, y_n) \in D} \ \frac{\partial f'(x_n, \bar{y})}{\partial y_n}$

then

$$\phi(x_n, y_n, h) - \phi(x_n, y_n^*, h) = A[L1(y_n - y_n^*)] + B[L2(y_n - y_n^*)]$$

Let M = |A.L1+B.L2|

$\phi(x_n, y_n, h) - \phi(x_n, y_n^*, h) \le M|y_n - y_n^*|$ which is the condition for convergence

IV. Consistency Of The Schemes

$$y_{n+1} = y_n + hf_n + \frac{h^2}{2} f_n'$$

$$y_{n+1} = y_n + \{A\}f_n + \{B\}f_n'$$

When $h = 0 \quad A = 0 \text{ and } B = 0$

$\Rightarrow y_{n+1} = y_n$ and the incremental function is identically zero when $h = 0$

$\Rightarrow \phi(x_n, y_n, 0) \equiv 0$

V. Stability Of The Schemes

Consider the equation

$$y_{n+1} = y_n + \{A\}f_n(x_n, y_n) + \{B\}f_n'(x_n, y_n)$$

Let $p_{n+1} = p_n + \{A\}f_n(x_n, P_n) + \{B\}f_n'(x_n, P_n)$ (23)

$$y_{n+1} - p_{n+1} = y_n - p_n + \{A\}[f_n(x_n, y_n) - f_n(x_n, P_n)] + \{B\}[f_n'(x_n, y_n) - f_n'(x_n, P_n)]$$

$$= y_n - p_n + A[\frac{\partial f(x_n, p_n)}{\partial p_n}(y_n - p_n)] + B[\frac{\partial f'(x_n, p_n)}{\partial p_n}(y_n - p_n)]$$

L1 = $\mathrm{SUP}_{(x_n, y_n) \in D} \ \frac{\partial f(x_n, p_n)}{\partial p_n}$ and

L2 = $SUP_{(x_n, y_n)\in D} \frac{\partial f'(x_n, p_n)}{\partial p_n}$

$y_{n+1} - p_{n+1} = y_n - p_n + A.L1(y_n - p_n) + B.L2(y_n - p_n)$

$|y_{n+1} - p_{n+1}| = |y_n - p_n| + [A.L1 + B.L2]|(y_n - p_n)|$

Let N = $|1 + [A.L1 + B.L2]|$

$|y_{n+1} - p_{n+1}| \leq N\, |y_n - p_n|$

Let $y_0 = y(x_0) = \xi$ and $p_0 = p(x_0) = \xi^*$ then

$|y_{n+1} - p_{n+1}| \leq K\ |\xi - \xi^*|$

VI. Application To Series Circuits

Let the inductance of an LC series circuit be given by

$$L(t) = \begin{cases} 1 - \frac{1}{10}t & 0 \leq t < 10 \\ 0 & t \geq 10 \end{cases} \tag{24}$$

if the resistance $R = 0.2$ ohms, the impressed voltage $E(t) = 4$ and the current $i(t) = 0$ at $t = 0$, then the dynamics of governing the state of the series circuit is given by

$$L\frac{di}{dt} + Ri = E(t)$$

which implies

$$\left(1 - \frac{t}{10}\right)\frac{di}{dt} + 0.2i = 4, i(0) = 0 \quad 0 \leq t < 10 \tag{25}$$

Applying the developed scheme, we have

$$i_{n+1} = i_n + hf_n + \frac{h^2}{2}f'_n$$

$$i' = \frac{40 - 2i}{(10 - t)} = f_n$$

$$i'' = \frac{2i - 40}{(10 - t)^2} = f'_n$$

$$i_{n+1} = i_n + h\left\{\frac{40 - 2i}{(10 - t)}\right\} + \frac{h^2}{2}\left\{\frac{2i - 40}{(10 - t)^2}\right\} \text{ .... NEWh} \tag{26}$$

From (26) above, two hybrid schemes will be obtained by changing h to $\psi = \sin(\propto h)$ and $\psi = \frac{(e^{\lambda h} - 1)}{\lambda}$ and renaming them NEWsin, NEWexp respectively.

VII. Derivation Of The Non-Standard Schemes

We will apply rules 2 and 3 in Mickens (1992) and Mickens (1994) and their extensions in Angueluv and Lubuma (2000), Angueluv and Lubuma (2003) and Obayomi (2005) to each of the components of the equations as shown below

Rule 2

Denominator function for the discrete derivatives must be expressed in terms of more complicated function of the step-sizes than those conventionally used. This rule allows the introduction of complex analytic function of h that satisfy certain conditions in the denominator, Mickens (1992) and Mickens (1994).

Rule 3

The non-linear terms must in general be modeled (approximated) non-locally on the computational grid or lattice in many different ways, Mickens (1992) and Mickens (1994).

Application of the combination of these two rules will give us the following transformations for

$$\frac{dy}{dx} \equiv \frac{(y_{k+1} - y_k)}{\psi} \quad \text{where } \psi(h) \to h + 0(h^2) \text{ as } h \to 0 \tag{27}$$

$$\frac{dy}{dx} \equiv \frac{(y_{k+1} - \beta y_{k-1})}{2\psi} \quad \text{where } \psi(h) \to h + 0(h^2),\ \beta(h) \to 1 \text{ as } h \to 0 \tag{28}$$

The following non-local approximations for

$$y_{k+1} \equiv a y_{k+1} + b y_k \quad a + b = 1 \tag{29}$$

Sample renormalisation functions to be employed are

$$\psi = \sin(\propto h),\ \propto \in \mathrm{R} \qquad \rightarrow h + 0(h^2) \quad \text{as } h \rightarrow 0 \tag{30}$$

$$\psi = \frac{(e^{\lambda h} - 1)}{\lambda},\ \lambda \in \mathrm{R}, \qquad \rightarrow h + 0(h^2) \quad \text{as } h \rightarrow 0 \tag{31}$$

$$\beta = cos(\propto h),\ \propto \in \mathrm{R} \qquad \rightarrow 1 \quad \text{as } h \rightarrow 0$$

VIII. One step Scheme

Applying non-local approximation and transformation equations in (25)

$$\left(1 - \frac{t}{10}\right)\frac{di}{dt} + 0.2i = 4,\ i(0) = 0 \quad 0 \leq t < 10$$

$$i' = \frac{40 - 2i}{(10 - t)}$$

we have the following

$$\left[\frac{i_{k+1} + i_k}{\psi}\right] = \frac{40 - 2\,(a\, i_{k+1} + b\, i_k)}{10 - t_k}$$

$$i_{k+1} = \frac{[10 - t - 2b\psi]y_k + 40\psi}{[10 - t + 2a\psi]}, \tag{32}$$

The three Non-standard scheme is obtained by using $\psi = h$ NSh, $\psi = \sin(\delta h)$ and $\psi = \frac{(e^{\lambda h} - 1)}{\lambda}$ which will be named CNSh, CNSsin and CNSe respectively.

IX. GRAPHICAL REPRESENTATION OF THE NUMERICAL RESULTS

The algorithm of these schemes has been coded into a software program. The sample results for $h = 0.001$ and $h = 0.01$ are given in 3D graphs below

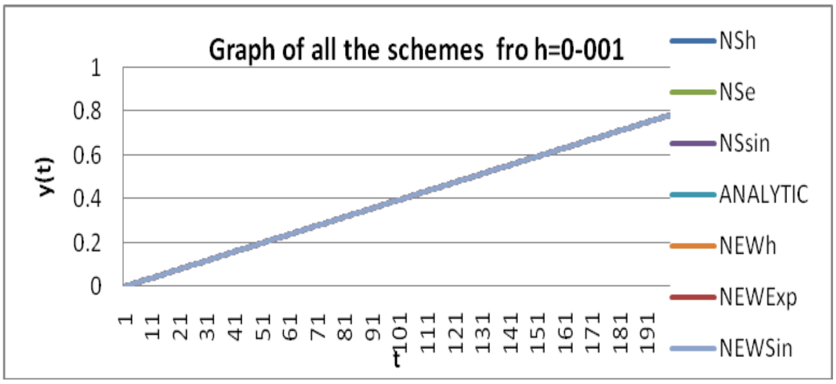

Fig 2: Graph of Schemes and analytic solution for the series circuit

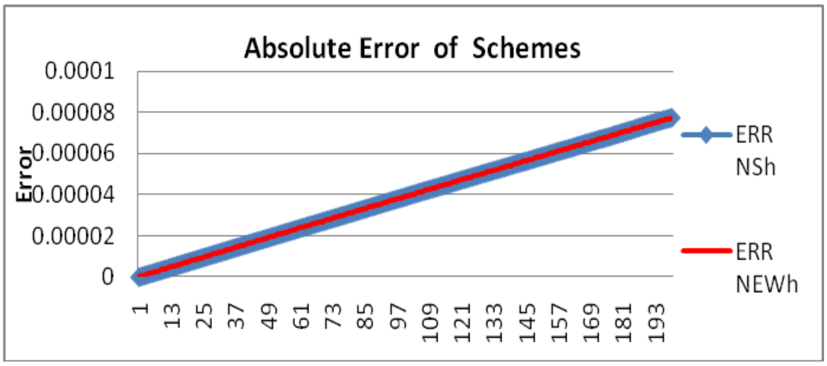

Fig 3a: Absolute Error of Deviation of the Non-standard schemes from the analytic solution using h =0.001

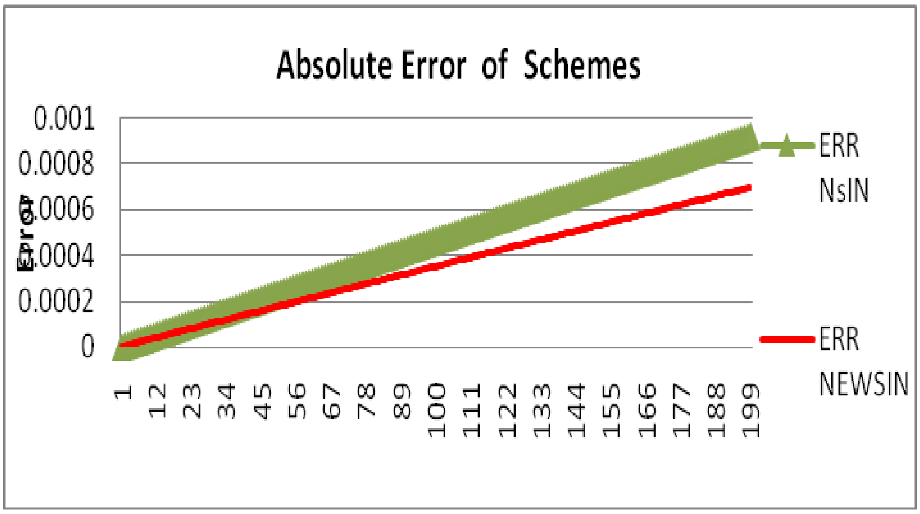

Fig 3b: Absolute Error of Deviation of the Non-standard schemes from the analytic solution using h =0.001

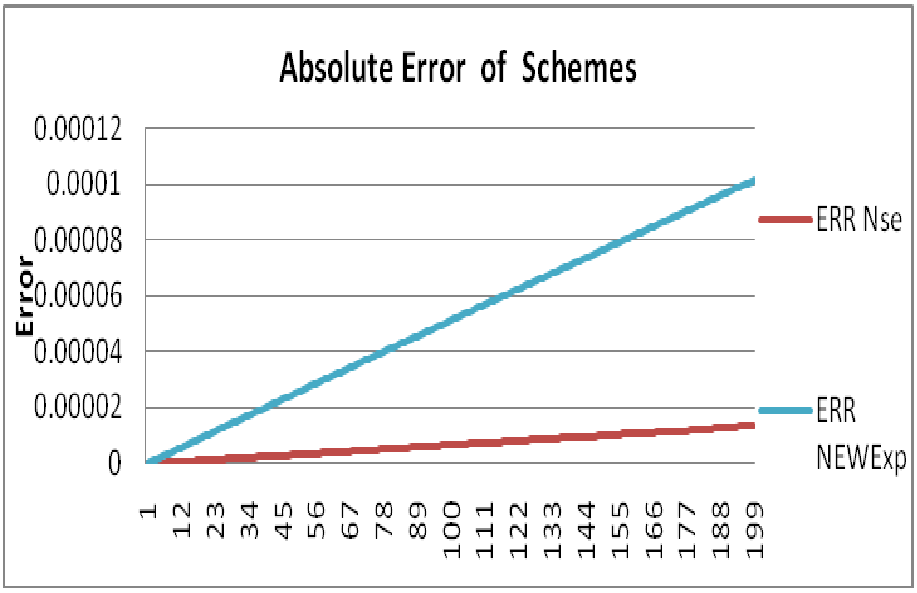

Fig 3c: Absolute Error of Deviation of the Nonstandard schemes from the analytic solution using h =0.001

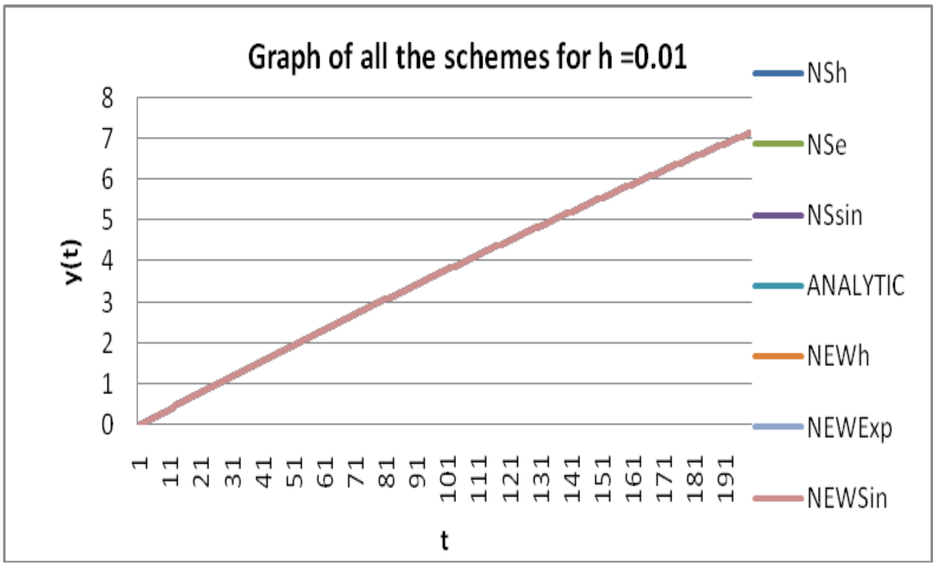

Fig 4: Graph of Schemes and analytic solution for the series circuit

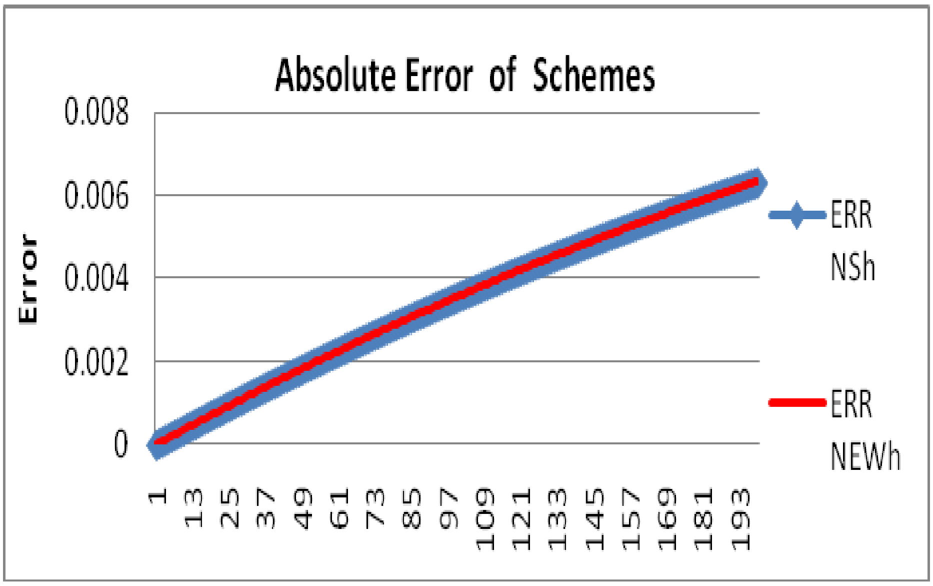

Fig 5a: Absolute Error of Deviation of the Nonstandard schemes from the analytic solution using h =0.01

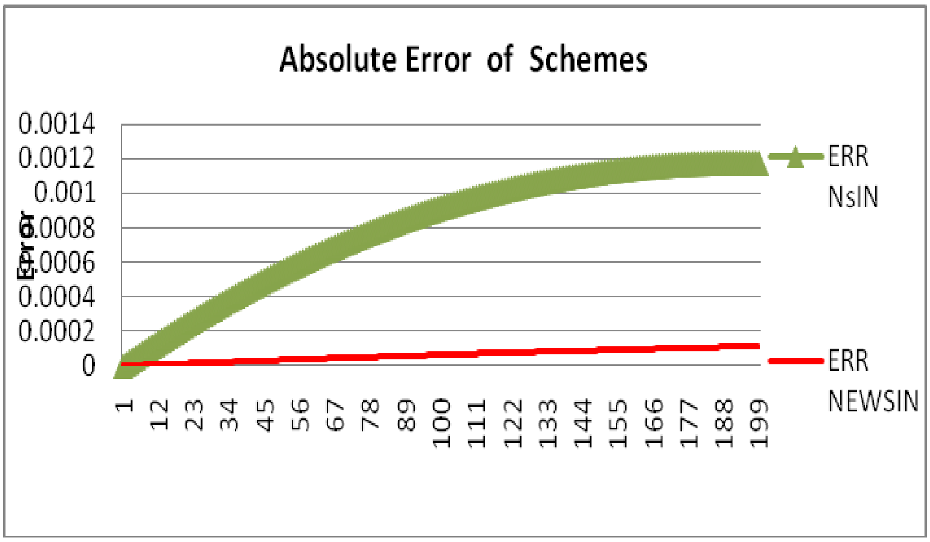

Fig 5b: Absolute Error of Deviation of the Nonstandard schemes from the analytic solution using h =0.01

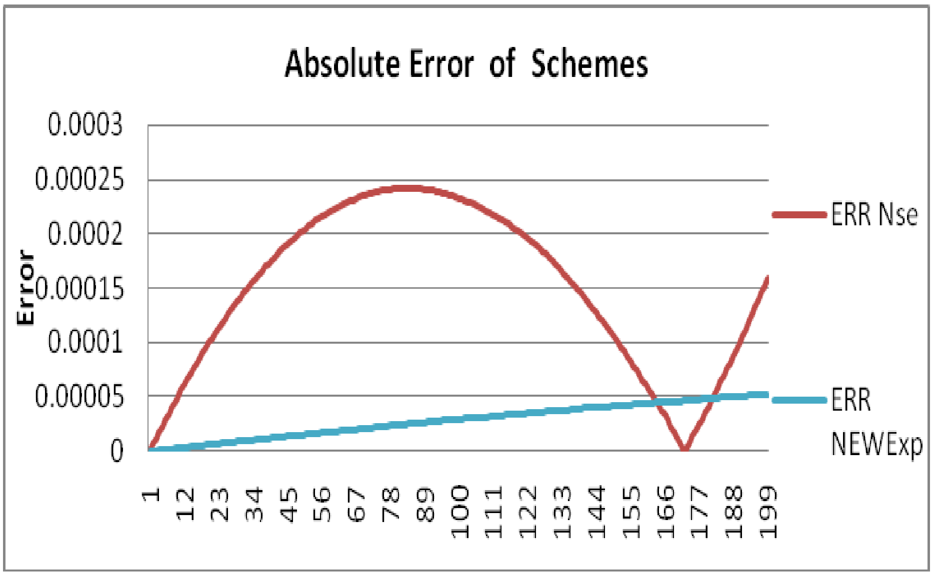

Fig 5c: Absolute Error of Deviation of the Nonstandard schemes from the analytic solution using h =0.01

X. Error Analysis

The absolute error of deviation for each scheme is depicted in Figures (3a,b,c and,5a,b,c). For 100 iterations respectively the absolute error has a bound of $0.2h^2$ and this is consistent for any step-size.

The Non-standard scheme proposed by Mickens (1994) suggested the transformation of the Euler Method $y_{n+1} = y_n + hf(x_n y_n)$ by replacing h with a function $\psi(h)$ which satisfies $\psi(h) \rightarrow h + 0(h^2) as\ h \rightarrow 0$

With a carefully selected function $\psi(h)$, the truncation error is of order (h^2) and the global truncation error is of order h. The proponent of this method was much more interested in obtaining solution curves that behaves closely like the curve of the original equation. The major achievement of this method is that, if the rules are satisfied then the schemes derived will possess some desirable qualitative properties like preservation of the hyperbolic and non-hyperbolic fixed points of the original equation and monotonicity of solutions.

XI. Summary And Conclusion

The solution curves for the schemes correspond to the curve of the analytic solution and have the same monotonic behaviour. The three schemes considered produces solution curves that converges to the analytic solution. The performance of the schemes shows that both the Non-standard and the hybrid schemes are adequate for the simulation of a simple series circuit model. One of the advantages of Non-standard method is that, if the rules are followed, it creates schemes that are less susceptible to numerical instability. In particular, it does not produce extraneous solution and it produces solution that carry along the dynamics of the original equation as it is in the cases above. The major significance of this work is that this new class of schemes can be used to simulate the behaviour of any simple series circuit.

References

[1]. Anguelov R. and Lubuma J.M.S. (2000). On Mathematical Foundation of the Nonstandard Finite Differential Method, in Fiedler B, Grooger K, Sprekels .J (Eds), Proceeding of the International Conference on Differential Equations, EQUADIFF' 99 Word Scientific, Singapore, 1401-1403.

[2]. Anguelov R. and Lubuma J.M.S. (2003). Nonstandard Finite Difference Method by Nonlocal Approximatio Mathematics and Computers in Simulation 6, 465-475.

[3]. Fatunla S.O. (1988). Numerical Methods for Initial Value Problems in Ordinary Differential Equations, Academic press Inc., London.

[4]. Henrici, P. (1962). Discrete Variable Methods in Ordinary Differential Equations, John Willey & Sons, New York.

[5]. Lambert J.D. (1991). Numerical Methods for Ordinary Differential Systems-The Initial Value Problems. John Willey & Sons, New York. Mickens R.E. (1992). Finite Difference Schemes having the correct Linear Stability Properties for all Finite Step-sizes. Dynamics Systems and Applications 1, 329-340.

[6]. Mickens R.E. (1994). Non-standard Finite Difference Models of Differential Equations.

[7]. World Scientific, Singapore.

[8]. Obayomi A.A. (2015). Non-standard Numerical Approaches for Solutions of Some Second Order Ordinary Differential Equations, British Journal of Math. & Computer Sc.

[9]. Obayomi A.A., Oke M.O., Olaosebikan T.E. and Raji R.A. (2017). An Hybrid of Numerically Stable Simulation Model for the Solution of some Initial Value Problems in Ordinary Differential Equations, Global Journal of Mathematics, 9(1), 574 – 580.

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