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Skew Chromatic Index of Certain Classes of Graphs

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Abstract : *A skew edge coloring of a graph G is defined to be a set of two edge colorings such that no two edges are assigned the same unordered pair of colors. The skew chromatic index s(G) is the minimum number of colors required for a skew edge coloring of G. In this paper, skew edge coloring of certain classes of graphs are determined. Furthermore, the skew chromatic index of those graphs is obtained in polynomial time.*
Keywords: *NP- complete, skew chromatic index, skew edge coloring.*

I. Introduction

Edge coloring problems are one of the fundamental and important problems of graph theory that have wide range of real time applications. An interconnection network can be represented as an undirected graph where a processor is represented as a node and a communication channel between processors as an edge between corresponding nodes. Star, wheel, fan, crown, hypercube, butterfly, friendship graphs are popular interconnection networks. Wireless networks are extensively studied due to their wide range of applications. In wireless communications, the time slot assignment problem is to assign time slots to communication links with the constraint that if the links are adjacent they must have different time slots in order to avoid interferences. As time slots are critical resources, we need to minimize the number of time slots assigned to links for an interference free schedule. This problem was shown to be *NP*–complete. The time slot assignment problem is generally associated with the problem of proper edge coloring in the graph that represents the network. The vertices of the graph are the network devices and the edges are communication links. Several edge colorings of graphs are developed and applied to particular communication environments or constraints.

Let $G = (V, E)$ be a finite, simple connected undirected graph. An edge coloring of a graph G is an assignment of colors to the edges of G so that no two adjacent edges are assigned the same color [1]. The minimum number of colors required for an edge coloring of G is the edge chromatic number or the chromatic index and is denoted by $\chi'(G)$. The problem of determining the chromatic index of an arbitrary graph is a difficult task. Holyer [2] has proved that edge coloring problem is *NP*–complete. But there are several good approximation algorithms available. Vizing [3] has shown that for any simple graph G, $\chi'(G)$ is either $\Delta(G)$ or $\Delta(G)+1$. In this paper, we consider skew edge coloring problems that are inspired from the study of skew Room squares [4]. The concept of skew chromatic index was introduced by Marsha. F. Foregger and better upper bounds for $s(G)$ was discussed when G is cyclic, cubic or bipartite [5]. All the notations and definitions used in this paper are as in [6]. A skew edge coloring of G is an assignment of an ordered pair of colors (a_i, b_i) to each edge e_i of G such that (i) the a_i 's form an edge coloring of G, (ii) the b_i 's form an edge coloring of G, and (iii) the pairs $\{a_i, b_i\}$ are all distinct. The two edge colorings are referred to as component colorings of the skew edge coloring. The skew chromatic index $s(G)$ is the minimum number of colors required for a skew edge coloring of G. For example, $s(K_3) = 3$ where the first and the second component colorings of K_3 are 1, 2, 3 and 2, 3, 1 respectively. See Fig. 1.

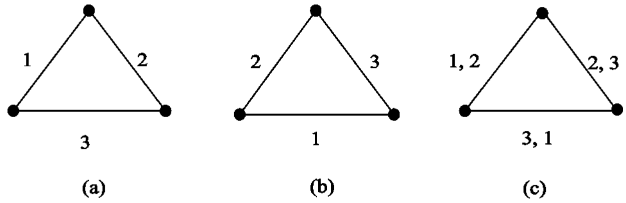

Fig. 1. (a) First component coloring of K_3. (b) Second component coloring of K_3. (c) Skew edge coloring of K_3.

II. Lower Bound On *s*(*G*)

Skew chromatic index, $s(G)$ is defined as the minimum number of colors used in two edge colorings of G such that no two edges are assigned the same unordered pair of colors. Since each component coloring of a skew edge coloring is itself an edge coloring, it is obvious that $s(G) \geq \chi'(G)$. By Vizing's theorem, $\Delta(G) \leq \chi'(G) \leq \Delta(G)+1$, where $\Delta(G)$ is the maximum degree of vertices in G. Hence we have

$s(G) \geq \Delta(G)$. If '*k*' colors are used for skew edge coloring, then there are $\binom{k+1}{2}$ unordered pairs of colors and this number must be at least as large as the number of edges in *G*. Let *k*(*m*) denote the smallest integer '*k*' satisfying $\binom{k+1}{2} \geq m$ where '*m*' denotes the number of edges in *G*. Thus we have $s(G) \geq k\left(|E(G)|\right)$. Hence the best lower bound for *s*(*G*) is $s(G) \geq \max\{\Delta(G), k\left(|E(G)|\right)\}$ as stated in [5]. In this paper, we prove that the bound on skew chromatic index given here is sharp for the certain classes of graphs like star, wheel, fan, palm fan, double wheel, friendship, bistar, helm, flower and sunflower graph.

III. Skew Edge Coloring Of Certain Classes Of Graphs

In this section, skew edge coloring and hence the skew chromatic index of star graph, wheel graph, fan graph, palm graph, double wheel graph, friendship graph, bistar graph, helm graph, flower graph and sunflower graph is obtained.

Definition 3.1. A star graph [7] $K_{1,n}$ is a complete bipartite graph consisting of one internal node and *n* leaves. It is a tree obtained by adding *n* pendant edges to the central vertex [8].

Star networks are one of the common network topologies that are modeled after the star graph which plays an important role in distributed computing.

***Theorem* 3.1.** For a star graph $K_{1,n}$, $n > 3$, the skew chromatic index $s(K_{1,n}) = n$.

Proof. Let $v_1, v_2, ..., v_n$ be the pendant vertices of $K_{1,n}$ and v_0 be the central vertex of $K_{1,n}$ adjacent to $v_i, 1 \leq i \leq n$, where $\deg(v_0) = n$. The *n* edges v_0v_i are assigned the pairs of colors of the form $(i, i), 1 \leq i \leq n.$ The above method of coloring guarantees a skew edge coloring of $K_{1,n}$ as the *n* edges receive *n* different pairs of colors that form skew edge coloring of $K_{1,n}$. See Fig. 2.

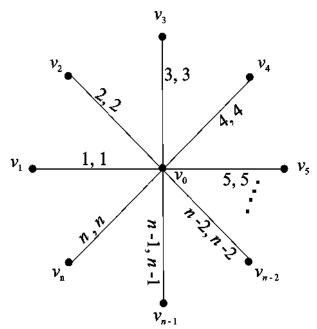

Fig. 2. Skew edge coloring of star graph $K_{1,n}$.

Definition 3.2. A wheel graph [9] W_n, $n \geq 3$ is a graph that has a central vertex connected to all *n* vertices in a cycle C_n. The central vertex is also known as hub. The edges corresponding to cycle are called the rim edges and the edges joining the central vertex and the vertices of the cycle are called spokes. This type of network plays an important role in circuit layout and interconnection network designs.

***Theorem* 3.2.** For a wheel graph W_n, $n \geq 3$, the skew chromatic index $s(W_n) = n$.

Proof. By the definition of wheel graph, W_n is obtained by joining the vertices of the cycle C_n to the central vertex v_0. Let $V = \{ v_0, v_1, v_2, ..., v_n \}$ be the vertex set of W_n. It is a planar undirected graph with *n* + 1 vertices and 2*n* edges. The spokes of the wheel graph v_0v_i are assigned the pairs of colors of the form $(i, i), 1 \leq i \leq n.$ The $n - 3$ rim edges of the wheel graph namely $v_iv_{i+1}, 1 \leq i \leq n-3$ are assigned the colors (*i* + 2, *i* + 3). The remaining three edges are assigned the colors (*n*, 1), (1, 2) and (2, 3) respectively. The above method of coloring guarantees a skew edge coloring of W_n as it can be easily seen that all a_i's form an edge coloring, all b_i's form an edge coloring and the pairs $\{a_i, b_i\}$ are all distinct. Hence $s(W_n) = n$. See Fig. 3.

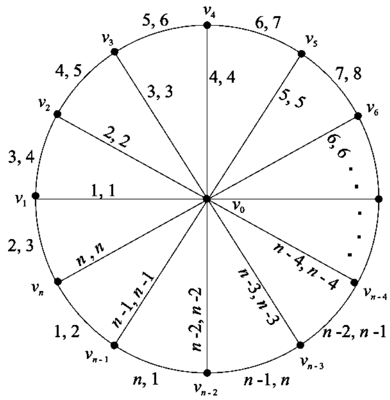

Fig. 3. Skew edge coloring of wheel graph W_n.

Definition 3.3: A fan graph F_n is a graph $P_n + K_1$ which is constructed from a wheel graph W_n by deleting one edge in C_n [10, 11].

***Theorem* 3.3.** For a fan graph F_n, $n > 3$, the skew chromatic index $s(F_n) = n$.

Proof. By the definition, F_n is obtained by deleting one edge in the cycle C_n of the wheel graph W_n. It is a planar undirected graph with $n + 1$ vertices and $2n – 1$ edges. Let $V = \{ v_0, v_1, v_2, ..., v_n \}$ be the vertex set of F_n and let the vertex at the centre be v_0 (apex vertex). Since n edges are incident to the apex vertex v_0, the edges v_0v_i are colored using the pairs of the form $(i, i), 1 \le i \le n$. The edges v_iv_{i+1} are assigned the colors $(i+2, i+3), 1 \le i \le n-3$. The remaining two edges namely (v_{n-2}, v_{n-1}) and (v_{n-1}, v_n) are assigned the colors $(n, 1)$ and $(1, 2)$ respectively.

The above method of coloring guarantees a skew edge coloring of the fan graph. Hence $s(F_n) = n$. See Fig. 4.

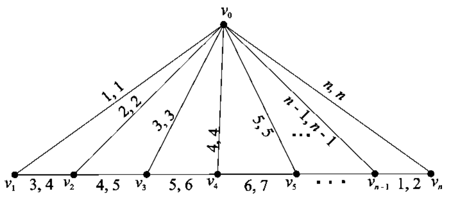

Fig. 4. Skew edge coloring of fan graph F_n.

Definition 3.4. A fan graph F_n with a pendant edge attached with the apex vertex is called a fan with a handle or palm fan [12, 13] and is denoted by F_n^*.

***Theorem* 3.4.** For a palm fan F_n^*, $n > 3$, the skew chromatic index $s(F_n^*) = n+1$.

Proof. By the definition, F_n^* is obtained by joining a pendant edge to the apex vertex v_0 of the fan graph F_n. Let $V = \{u_0, v_0, v_1, \ldots, v_n\}$ be the vertex set of F_n^*, where u_0 is the pendant vertex, v_0 is the apex vertex and $deg(v_0) = n + 1$. Since $n + 1$ edges are incident with the apex vertex, at least $n + 1$ colors are required for its proper edge coloring. Hence assign the colors of the form (i, i) to the edges $v_0v_i, 1 \le i \le n$. Then assign the color $(n + 1, n + 1)$ to the edge u_0v_0. The edges of the path namely v_iv_{i+1} are assigned the colors $(i+2, i+3), 1 \le i \le n-3$. The remaining two edges namely (v_{n-2}, v_{n-1}) and (v_{n-1}, v_n) are assigned the colors $(n, 1)$ and $(1, 2)$ respectively. The above method of coloring guarantees a skew edge coloring of the palm fan graph. Hence $s(F_n^*) = n + 1$. See Fig. 5.

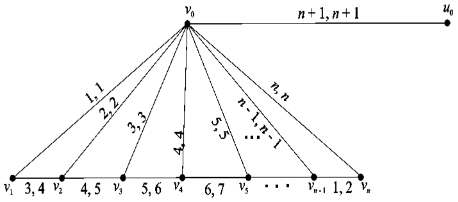

Fig. 5. Skew edge coloring of palm fan graph F_n^*.

Definition 3.5. A double-wheel graph [14] DW_n is a graph $2C_n + K_1$ consisting of two cycles of size n, the vertices of the two cycles being connected to a central vertex.

Theorem 3.5. For a double wheel graph DW_n, $n > 3$, the skew chromatic index $s(DW_n) = 2n$.

Proof. By the definition of double wheel graph, DW_n is obtained by joining the vertices of the two n-cycles to the central vertex v_0. Let $V = \{ v_0, v_1, v_2, ..., v_n, v_{n+1}, v_{n+2}, ..., v_{2n} \}$ be the vertex set of DW_n. It is a planar undirected graph with $2n + 1$ vertices and $4n$ edges. The spokes of the inner wheel v_0v_i are colored using the pairs of the form $(i, i), 1 \le i \le n$. The rim edges of the inner wheel namely v_iv_{i+1} are assigned the colors $(i+2, i+3), 1 \le i \le n-3$. The remaining three edges of the inner rim are assigned the colors $(n, 1)$, $(1, 2)$ and $(2, 3)$ respectively. The spokes of the outer wheel v_0v_i are colored using the pairs of the form $(i, i), n+1 \le i \le 2n$. The rim edges of the outer wheel namely v_iv_{i+1} are assigned the colors $(i+2, i+3), n+1 \le i \le 2n-3$. The remaining three edges of the outer rim are assigned the colors $(2n, n + 1)$, $(n + 1, n + 2)$ and $(n + 2, n + 3)$ respectively. The above method of coloring guarantees a skew edge coloring of DW_n since it can be easily seen that all a_i's form an edge coloring, all b_i's form an edge coloring and the pairs $\{a_i, b_i\}$ are all distinct. Hence $s(DW_n) = 2n$. See Fig. 6.

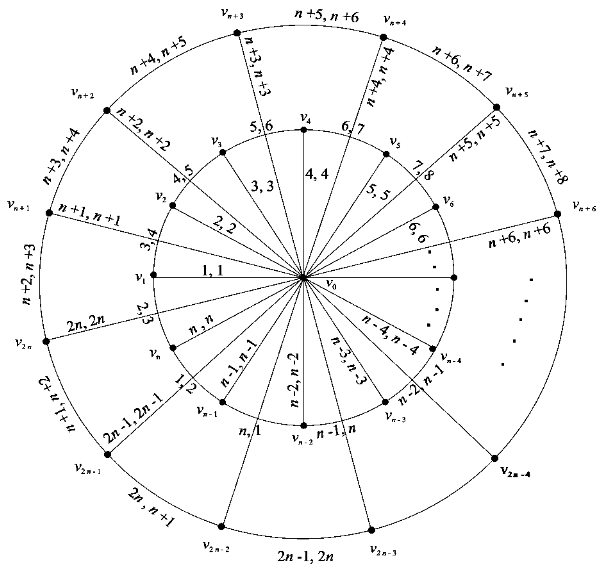

Fig. 6. Skew edge coloring of double wheel graph DW_n.

Definition 3.6. A friendship graph [15] Fd_n is constructed by joining n copies of the cycle graph C_3 with a common vertex. It is a planar undirected graph with $2n + 1$ vertices and $3n$ edges.

A graph is called a friendship graph if every pair of its vertices has exactly one common neighbour. This condition is called the friendship condition [16].

Theorem 3.6. For a friendship graph Fd_n, $n \ge 2$, the skew chromatic index $s(Fd_n) = 2n$.

Proof. Let $V = \{ v_0, v_1, v_2, ..., v_n, v_{n+1}, v_{n+2}, ..., v_{2n} \}$ be the vertex set. Since $2n$ edges are incident to the central vertex v_0, the edges v_0v_i are colored using the pairs of the form $(i, i), 1 \le i \le n$. Next the edges v_iv_{i+1} are assigned the colors of the form $(i+2, i+3)$, $i = 1, 3, 5, 7, ..., 2n-3$. Finally the last edge namely $v_{2n-1}v_{2n}$ is assigned the color $(1, 2)$. This forms the skew edge coloring of the friendship graph. Hence $s(Fd_n) = 2n$. See Fig. 7.

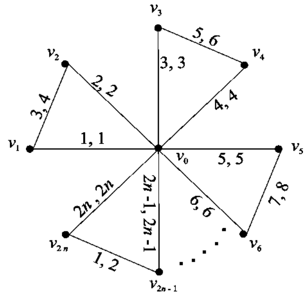

Fig. 7. Skew edge coloring of friendship graph Fd_n.

Definition 3.7. A bistar graph [17] $B_{n,n}$ is a graph obtained by joining the central vertices of two copies of $K_{1,n}$ by an edge.

Theorem 3.7. For a bistar graph $B_{n,n}$, $n > 4$, the skew chromatic index $s(B_{n,n}) = n + 1$.

Proof. Let $V = \{v_0, v_1, v_2, ..., v_n, v_0^{'}, v_1^{'}, v_2^{'}, ..., v_n^{'}\}$ be the vertex set of $B_{n,n}$. By the definition of a bistar graph n edges are incident to the central vertex v_0, so assign the n distinct pairs of colors of the form (i, i) to the edges $v_0v_i, 1 \le i \le n$. Then assign the colors to the edges $v_0^{'}v_i^{'}$ using the pairs of the form $(i, i+1), 1 \le i \le n-1$. The remaining two edges namely $v_0^{'}v_n^{'}$ and $v_0v_0^{'}$ are assigned the colors (n, 1) and (n + 1, n + 1) respectively.

The above method of coloring guarantees a skew edge coloring of $B_{n,n}$ as all the 2n + 1 edges receive distinct pairs of colors that form skew edge coloring of $B_{n,n}$. See Fig. 8.

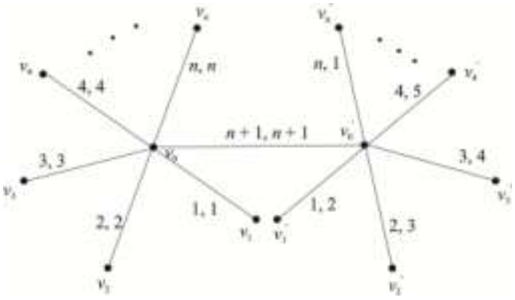

Fig. 8. Skew edge coloring of bistar graph $B_{n,n}$.

Definition 3.8. A helm graph [18, 19] H_n is a graph obtained from a wheel graph W_n by attaching a pendant edge to each vertex of the cycle C_n. It has 2n + 1 vertices and 3n edges.

Theorem 3.8. For a helm graph H_n, $n > 4$, the skew chromatic index $s(H_n) = n$.

Proof. Let v_0 be the central vertex and $v_1, v_2, ..., v_n$ be the vertices of the cycle C_n that are adjacent to the central vertex and $v_{n+1}, v_{n+2}, ..., v_{2n}$ be the pendant vertices of the cycle C_n. Since all the vertices $v_i, 1 \le i \le n$ are incident with the central vertex v_0, the n edges v_0v_i are colored using the n distinct pairs of colors of the form $(i, i), 1 \le i \le n$. The $n-3$ edges v_iv_{i+1} of the cycle C_n are colored using the pairs of the form $(i+2, i+3), 1 \le i \le n-3$. The remaining three edges of the cycle C_n are assigned the colors (n, 1), (1, 2) and (2, 3) respectively. The pendant edges v_iv_{n+i} are colored using the pairs $(i-1, i+1),\ 2 \le i \le n-1$. The remaining two pendant edges namely v_nv_{2n} and v_1v_{n+1} are assigned the colors $(n-1, 1)$ and $(n, 2)$ respectively. The above method of coloring guarantees a skew edge coloring of H_n as all the 3n edges receive distinct pairs of colors that form skew edge coloring of H_n. Hence $s(H_n) = n$. See Fig. 8.

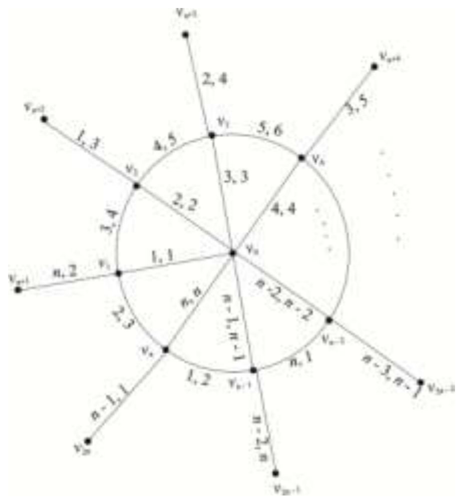

Fig. 8. Skew edge coloring of helm graph H_n.

Definition 3.9. A flower graph [20] Fl_n is a graph obtained from a helm graph H_n by joining each pendant vertex to the central vertex of the helm graph.

Theorem 3.9. For a flower graph Fl_n, $n > 4$, the skew chromatic index $s(Fl_n) = 2n$.
Proof. Let v_0 be the central vertex and $v_1, v_2, ..., v_n$ be the vertices of the cycle C_n that are adjacent to the central vertex and $v_{n+1}, v_{n+2}, ..., v_{2n}$ be the pendant vertices of the cycle C_n. Since all the vertices $v_i, 1 \le i \le n$ are incident with the central vertex v_0, the n edges v_0v_i are colored using the n distinct pairs of colors of the form $(i, i), 1 \le i \le n$. The $n-3$ edges of the cycle v_iv_{i+1} are colored using the pairs of the form $(i+2, i+3), 1 \le i \le n-3$. The remaining three edges of the cycle C_n are assigned the colors $(n, 1)$, $(1, 2)$ and $(2, 3)$ respectively. The pendant edges v_iv_{n+i} are colored using the pairs $(i-1, i+1), 2 \le i \le n-1$. The remaining two pendant edges namely v_nv_{2n} and v_1v_{n+1} are assigned the colors $(n-1, 1)$ and $(n, 2)$ respectively. The additional n edges joining the pendant vertices with the central vertex, namely $v_0v_i, n+1 \le i \le 2n$ are assigned the pairs of colors of the form $(i, i), n+1 \le i \le 2n$.
The above method of coloring guarantees a skew edge coloring of Fl_n as all the $2n$ edges incident with central vertex v_0 receive $2n$ distinct pairs of colors that form skew edge coloring of Fl_n. Hence $s(Fl_n) = 2n$ for $n > 4$. See the following Fig. 9.

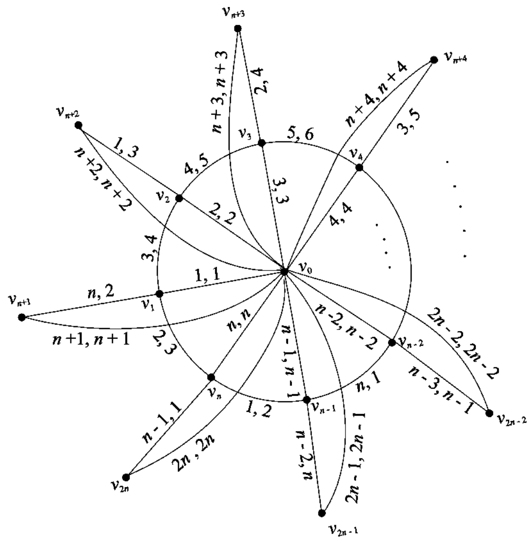

Fig. 9. Skew edge coloring of flower graph Fl_n.

Definition 3.10. A sunflower graph [20] SFl_n is a graph obtained from a flower graph of wheels W_n by adding n pendant edges to the central vertex. It has $3n + 1$ vertices and $5n$ edges.

Theorem 3.10. For a sunflower graph SFl_n, $n > 4$, the skew chromatic index $s(SFl_n) = 3n$.
Proof. Let v_0 be the central vertex and $v_1, v_2, ..., v_n$ be the vertices of the cycle C_n that are adjacent to the central vertex and $v_{n+1}, v_{n+2}, ..., v_{2n}$ be the pendant vertices of the cycle C_n. Since all the vertices $v_i, 1 \le i \le n$ are incident with the central vertex v_0, the n edges v_0v_i are colored using the n distinct pairs of colors of the form $(i, i), 1 \le i \le n$. The $n-3$ edges of the cycle C_n namely $v_iv_{i+1}, 1 \le i \le n-3$ are colored using the pairs of the form $(i+2, i+3), 1 \le i \le n-3$. The remaining three edges of the cycle C_n are assigned the colors $(n, 1)$, $(1, 2)$ and $(2, 3)$ respectively. The pendant edges v_iv_{n+i} are colored using the pairs $(i-1, i+1), 2 \le i \le n-1$. The remaining two pendant edges namely v_nv_{2n} and v_1v_{n+1} are assigned the colors $(n-1, 1)$ and $(n, 2)$ respectively. The additional n edges joined with the central vertex namely v_0v_i are assigned the pairs of colors of the form $(i, i), 2n+1 \le i \le 3n$.
The above method of coloring guarantees a skew edge coloring of SFl_n. Hence $s(SFl_n) = 3n$. See Fig. 10.

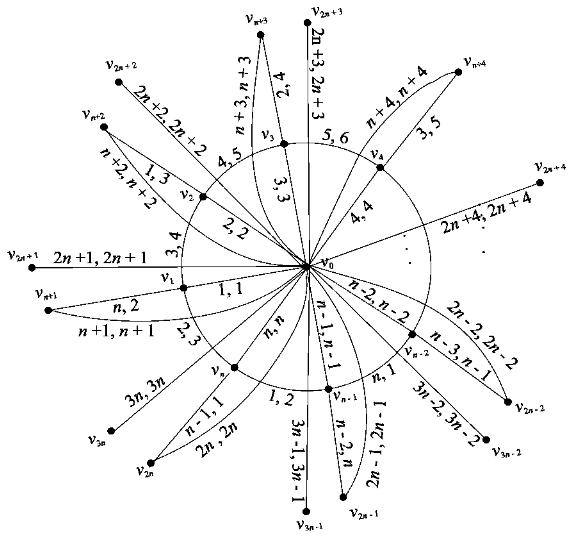

Fig. 10. Skew edge coloring of sunflower graph SFl_n.

IV. Conclusion

In this paper, we have obtained an optimal solution for the skew chromatic index of star, wheel, fan, palm fan, double wheel, friendship, bistar, helm, flower and sunflower graph and their skew chromatic index is $s(G) = \Delta(G)$. It would be interesting to identify the skew chromatic index for various interconnection networks.

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