

Stability Analysis of the Endemic Equilibrium State on the Spread of Malaria Using Bellman and Cooke's Theorem

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Abstract: Malaria is a life threatening blood disease which is caused by parasites transmitted to humans through the bite of the Anopheles mosquito. An infected mosquito bites a human and transmits the parasites which multiply in the host's liver before infecting and destroying red blood cells. In this research work, the model equations were obtained using several known clinical and biological information with the human population subdivided into susceptible, infected, recovered and therapy classes while the vector population is divided into the susceptible and infected classes. The equilibrium states were obtained and the endemic state analysed for stability. The result shows that the non-zero equilibrium state will be stable if $F(0)G'(0) > 0$ and unstable when otherwise.

Keywords: Stability, Characteristic Equation, Malaria, Latency.

I. Introduction

Malaria is a vector-borne disease. Instead of transmitted directly from human to human, its parasites are transferred between human through mosquitoes. The malaria parasite life cycle is divided into two parts, one is within host (human) body and the other within vector (mosquito) body. Human infection starts from a blood meal of an infectious female mosquito. The parasites existing in the infectious mosquito's saliva, called sporozoites at this stage, enter the bloodstream of the human through mosquito bites and migrate to the liver. Within minutes after entering the human body, sporozoites infect hepatocytes and multiply asexually and asymptotically in liver for a period of 5 – 30 days [1] and [2].

Mathematical models can help understand the dynamics of transmission and spread of the infectious disease and thereby, provide guides and suggestions for the control of the disease. According to Hoshen et al [3], the first person to use mathematical model to quantitatively investigate the spread of malaria was Ross. His model was later extended and studied by Mc-Donald which led to the Ross-McDonald Model [4]-[6].

There are different modeling methodologies which include individual-based models [7], habitat-based models [8], integrated models [9] and [10] and others [11]-[16]. The methodology used in this case is predominantly differential equation-based. Using known clinical and biological information on malaria, a set of differential equation is derived based on the human population subdivided into four compartments of susceptible human (S_h), infected human (I_h), recovered human (R_h) and the therapy class (V_h) while the vector population is divided into two compartments of the susceptible vector (S_v) and infected vector I_v . The endemic equilibrium state is examined for stability.

II. Model Equations

The formulation of the model results in the following model equations.

$$\frac{dS_h}{dt} = \wedge_h p + (rR_h + (\theta - \phi)v_h - \mu_h - \beta I_v)S_h \quad (1)$$

$$\frac{dI_h}{dt} = \beta S_h I_v - (\mu_h + \mu_0 + \delta)I_h \quad (2)$$

$$\frac{dR_h}{dt} = \delta I_h - (\mu_h - rS_h)R_h \quad (3)$$

$$\frac{dV_h}{dt} = \wedge_h (1 - p) + (\phi - \theta)S_h v_h \quad (4)$$

$$\frac{dS_v}{dt} = \wedge_v - (\mu_v + eI_h)S_v \quad (5)$$

$$\frac{dI_v}{dt} = eS_v I_h - \mu_v I_v \quad (6)$$

The model consists of six compartments

- S_h = Susceptible human
- I_h = Infected human
- R_h = Recovered human
- S_v = Susceptible vector
- I_v = Infected vector
- V_h = Therapy class

The parameters are

- Λ_v = Birth rate for the vector population
- Λ_h = Birth rate for the human population
- β = Transmission rate from S_h to I_h
- δ = Transmission rate from I_h to R_h
- e = Transmission rate from S_v to I_v
- θ = Transmission rate from V_h to S_h
- ϕ = Transmission rate from S_h to V_h
- μ_o = Natural death in human population
- μ_h = Death resulting from infection in human
- μ_v = Natural death in vector population
- r = Rate of flow from R_h to S_h
- P = Recruitment term.

III. Equilibrium State

At equilibrium state

$$\frac{dS_h}{dt} = \frac{dI_h}{dt} = \frac{dR_h}{dt} = \frac{dV_h}{dt} = \frac{dS_v}{dt} = \frac{dI_v}{dt} = 0 \tag{7}$$

That is

$$\Lambda_h p + (rR_h + (\theta - \phi)v_h - \mu_h - \beta I_v) s_h = 0 \tag{8}$$

$$\beta s_h I_v - (\mu_h + \mu_o + \delta) I_h = 0 \tag{9}$$

$$(\delta I_h - \mu_h - rS_h) R_h = 0 \tag{10}$$

$$\Lambda_h (1 - p) + (\phi - \theta) s_h v_h = 0 \tag{11}$$

$$\Lambda_v - (\mu_v + e I_h) s_v = 0 \tag{12}$$

$$e S_v I_h - \mu_v I_v = 0 \tag{13}$$

From (12)

$$s_v = \frac{\Lambda_v}{\mu_v + e I_h} \tag{14}$$

From (9)

$$I_v = \frac{(\mu_h + \mu_o + \delta) I_h}{\beta s_h} \tag{15}$$

Substituting (14) and (15) into (13) and simplifying gives

$$I_h \left(\frac{e \Lambda_v}{\mu_v + e I_h} - \mu_v \frac{(\mu_h + \mu_o + \delta) I_h}{\beta s_h} \right) = 0 \tag{16}$$

Suppose $I_h \neq 0$

$$\frac{e \Lambda_v}{\mu_v + e I_h} = \mu_v \frac{(\mu_h + \mu_o + \delta) I_h}{\beta s_h} \tag{17}$$

Simplifying equation (17) gives

$$I_h = \frac{\beta e \Lambda_v s_h - \mu_v^2 (\mu_h + \mu_o + \delta)}{\mu_v e (\mu_h + \mu_o + \delta)} \tag{18}$$

From equation (10)

Suppose $R_h \neq 0$

$$\delta I_h - \mu_h - rS_h = 0 \tag{19}$$

Put equation (18) into (19)

$$\delta \left(\frac{\beta e \wedge_v s_h - \mu^2 v (\mu_h + \mu_0 + \delta)}{\mu_v e (\mu_h + \mu_0 + \delta)} \right) - \mu_h - rS_h = 0 \tag{20}$$

Simplifying (20) gives

$$s_h = \frac{(\mu_v + e\mu_h)\mu_v e (\mu_h + \mu_0 + \delta)}{e^2 (\delta\beta \wedge_v - \mu_v (\mu_h + \mu_0 + \delta))} \tag{21}$$

Substituting (21) into (18) and simplifying

$$I_h = \frac{\beta e^2 \wedge_v \mu_v (\mu_v + e\mu_h) (\mu_h + \mu_0 + \delta)}{\mu_v e^3 (\delta\beta \wedge_v - \mu_v (\mu_h + \mu_0 + \delta)) (\mu_h + \mu_0 + \delta)} - \frac{\mu^2 v (\mu_h + \mu_0 + \delta)}{\mu_v e (\mu_h + \mu_0 + \delta)} \tag{22}$$

Similarly, substituting (22) into (14)

$$s_v = \frac{\wedge_v}{\mu_v + e \left(\frac{\beta e^2 \wedge_v \mu_v (\mu_v + e\mu_h) (\mu_h + \mu_0 + \delta)}{\mu_v e^3 (\delta\beta \wedge_v - \mu_v (\mu_h + \mu_0 + \delta)) (\mu_h + \mu_0 + \delta)} - \frac{\mu^2 v (\mu_h + \mu_0 + \delta)}{\mu_v e (\mu_h + \mu_0 + \delta)} \right)} \tag{23}$$

Similarly, substituting (21) and (23) into (15)

$$I_v = \frac{(\mu_h + \mu_0 + \delta) \frac{\beta e^2 \wedge_v \mu_v (\mu_v + e\mu_h) (\mu_h + \mu_0 + \delta)}{\mu_v e^3 (\delta\beta \wedge_v - \mu_v (\mu_h + \mu_0 + \delta)) (\mu_h + \mu_0 + \delta)} - \frac{\mu^2 v (\mu_h + \mu_0 + \delta)}{\mu_v e (\mu_h + \mu_0 + \delta)}}{\beta \left(\frac{(\mu_v + e\mu_h)\mu_v e (\mu_h + \mu_0 + \delta)}{e^2 (\delta\beta \wedge_v - \mu_v (\mu_h + \mu_0 + \delta))} \right)} \tag{24}$$

Now from equation (11)

$$\wedge_h (1 - p) + (\phi - \theta) s_h v_h = 0 \tag{25}$$

Simplifying (25) and making v_h the subject

$$v_h = \frac{\wedge_h (1 - p)}{(\theta - \phi) s_h} \tag{26}$$

Putting (21) into (26)

$$v_h = \frac{\wedge_h (1 - p)}{(\theta - \phi) \frac{(\mu_v + e\mu_h)\mu_v e (\mu_h + \mu_0 + \delta)}{e^2 (\delta\beta \wedge_v - \mu_v (\mu_h + \mu_0 + \delta))}} \tag{27}$$

Also from equation (8)

$$\wedge_h p + (rR_h + (\theta - \phi)v_h - \mu_h - \beta I_v) s = 0 \tag{28}$$

Simplifying (28) and making R_h the subject

$$R_h = \frac{\mu_h + \beta I_v}{r} - \frac{(\theta - \phi)v_h}{r} - \frac{\wedge_h p}{rs_h} \tag{29}$$

Putting (21), (24) and (27) into (29)

$$R_h = \frac{(\mu_h + \mu_0 + \delta) \frac{\beta e^2 \wedge_v \mu_v (\mu_v + e\mu_h)(\mu_h + \mu_0 + \delta)}{\mu_v e^3 (\delta\beta \wedge_v - \mu_v (\mu_h + \mu_0 + \delta))(\mu_h + \mu_0 + \delta)} - \frac{\mu^2 v (\mu_h + \mu_0 + \delta)}{\mu_v e (\mu_h + \mu_0 + \delta)}}{\beta \left(\frac{(\mu_v + e\mu_h) \mu_v e (\mu_h + \mu_0 + \delta)}{e^2 (\delta\beta \wedge_v - \mu_v (\mu_h + \mu_0 + \delta))} \right)} - \frac{(\theta - \phi) \frac{\wedge_h (1-p)}{(\theta - \phi) \frac{(\mu_v + e\mu_h) \mu_v e (\mu_h + \mu_0 + \delta)}{e^2 (\delta\beta \wedge_v - \mu_v (\mu_h + \mu_0 + \delta))}}}{r} - \frac{\wedge_h p}{r \left(\frac{(\mu_v + e\mu_h) \mu_v e (\mu_h + \mu_0 + \delta)}{e^2 (\delta\beta \wedge_v - \mu_v (\mu_h + \mu_0 + \delta))} \right)} \quad (30)$$

IV. Stability Analysis of the Endemic Equilibrium

The characteristic equation is obtained by

$$A = \begin{vmatrix} \alpha_1 - \lambda & 0 & rs_h & (\theta - \phi)s_h & 0 & -\beta s_h \\ \beta I_v & -\alpha_2 - \lambda & 0 & 0 & 0 & \beta s_h \\ -rR_h & \delta R_h & \alpha_3 - \lambda & 0 & 0 & 0 \\ (\theta - \phi)v_h & 0 & 0 & \alpha_4 - \lambda & 0 & 0 \\ 0 & -es_v & 0 & 0 & \alpha_5 - \lambda & 0 \\ 0 & es_v & 0 & 0 & eI_h & -\mu_v - \lambda \end{vmatrix} = 0 \quad (31)$$

Which gives

$$\begin{aligned} & (-r\lambda^3 + (r\alpha_5 + \delta\beta I_v - r(\mu + \alpha_2))\lambda^2 + (res_v\beta s_h + (-\beta\delta I_v + r(\mu + \alpha_2))\alpha_5 + \\ & (-r\alpha_2 + \beta\delta I_v)\mu)\lambda - re\beta s_v(eI_h + \alpha_5)s_h - \alpha_5(-r\alpha_2 + \delta\beta I_v)\mu(-\alpha_4 + \lambda)R_h(rs_h) \\ & + (-\alpha_3 + \lambda)(\lambda^5 + (\mu - \alpha_4 - \alpha_1 + \alpha_2 - \alpha_5)\lambda^4 + ((-\beta es_v - v_h(-\theta + \phi)^2)s_h \\ & + (-\mu + \alpha_1 + \alpha_4 - \alpha_2)\alpha_5 + (\alpha_2 - \alpha_4 - \alpha_1)\mu + (\alpha_1 - \alpha_2)\alpha_4 - \alpha_1\alpha_2)\lambda^3 \\ & + (((\beta es_v + v_h(-\theta + \phi)^2)\alpha_5 + \beta^2 I_v es_v + e(eI_h + \alpha_1 + \alpha_4)\beta s_v - v_h(-\theta + \phi)^2(\mu + \alpha_2))s_h \\ & + ((-\alpha_2 + \alpha_4 + \alpha_1)\mu + (-\alpha_1 + \alpha_2)\alpha_4 + \alpha_1\alpha_2)\alpha_5 + ((\alpha_1 - \alpha_2)\alpha_4 - \alpha_1\alpha_2)\mu + \alpha_1\alpha_1\alpha_4)\lambda^2 \\ & + (\beta es_v v_h(\theta + \phi)^2 s^2 h + ((-\beta^2 I_v es_v - es_v(\alpha_1 + \alpha_4)\beta + v_h(-\theta + \phi)^2(\mu + \alpha_2))\alpha_5 \\ & - es_v(-ev_h + I_v\alpha_4)\beta^2 - e((eI_h + \alpha_1)\alpha_4 + eI_h\alpha_1)\beta s_v - \mu v_h\alpha_2(-\theta + \phi)^2)s_h \\ & + (((-\alpha_1 + \alpha_2)\alpha_4 + \alpha_1\alpha_2)\mu - \alpha_1\alpha_2\alpha_4)\alpha_5 + \alpha_1\alpha_2\alpha_4\mu)\lambda - (\theta + \phi)^2\beta es_v(eI_h + \alpha_5)v_h s_h^2 \\ & + ((\beta^2 I_v es_v\alpha_4 + \alpha_1 es_v\beta\alpha_4 + v_h\mu\alpha_2(-\theta + \phi)^2)\alpha_5 + e^2\alpha_4\beta s_v(-\beta v_h + \alpha_1))s_h - \alpha_1\alpha_2\alpha_4\alpha_5\mu) \end{aligned} = 0 \quad (32)$$

Where $\alpha_1 = rR_h + (\theta - \phi)v_h - \mu_h - \beta I_v$

$$\alpha_2 = -(\mu_h + \mu_0 + \delta)$$

$$\alpha_3 = \delta I_h - \mu_h - rS_h$$

$$\alpha_4 = ((\theta - \phi)s_h)$$

$$\alpha_5 = (\mu_v + eI_h)$$

Bellman and Cooke's theorem is used to establish the stability or otherwise of the model.

Theorem

Let $H(Z) = P(z, e^z)$ where $p(z, w)$ is a polynomial with principal term.

Suppose $H(iy)$, $y \in R$, is separated into its real and imaginary parts,

$$H(z) = F(y) + iG(y) \quad (33)$$

If all zeros of $H(z)$ have negative real parts, then the zeros of $F(y)$ and $G(y)$ are real, simple and alternate and

$$G'(0)F(0) - G(0)F'(0) > 0 \text{ for all } y \in R. \tag{34}$$

Conversely, all zeros of $H(y)$ will be in the left-half plane provided that either of the following conditions is satisfied:

- (i) All the zeros of $F(y)$ and $G(y)$ are real, simple, and alternate and the inequality (34) is satisfied for at least one y .
- (ii) All the zeros of $F(y)$ are real and, for each zero, the relation (34) is satisfied
- (iii) All the zeros of $G(y)$ are real and, for each zero, the relation (33) is satisfied.

From (32) let

$H(\lambda)$

$$\begin{aligned} & - (r\lambda^3 + (r\alpha_5 + \delta\beta_v - r(\mu + \alpha_2))\lambda^2 + (res_v\beta s_h + (-\beta\delta_v + r(\mu + \alpha_2))\alpha_5 \\ & + (-r\alpha_2 + \beta\delta_v)\mu)\lambda - re\beta s_v(eI_h + \alpha_5)s_h - \alpha_5(-r\alpha_2 + \delta\beta_v)\mu)(-\alpha_4 + \lambda)R_h(rs_h) \\ & + (-\alpha_3 + \lambda)(\lambda^5 + (\mu - \alpha_4 - \alpha_1 + \alpha_2 - \alpha_5)\lambda^4 + ((-\beta es_v - v_h(-\theta + \phi)^2)s_h + \\ & (-\mu + \alpha_1 + \alpha_4 - \alpha_2)\alpha_5 + (\alpha_2 - \alpha_4 - \alpha_1)\mu + (\alpha_1 - \alpha_2)\alpha_4 - \alpha_1\alpha_2)\lambda^3 \\ = & + (((\beta es_v + v_h(-\theta + \phi)^2)\alpha_5 + \beta^2 I_v es_v + e(eI_h + \alpha_1 + \alpha_4)\beta s_v - v_h(-\theta + \phi)^2(\mu + \alpha_2))s_h \\ & + ((-\alpha_2 + \alpha_4 + \alpha_1)\mu + (-\alpha_1 + \alpha_2)\alpha_4 + \alpha_1\alpha_2)\alpha_5 + ((\alpha_1 - \alpha_2)\alpha_4 - \alpha_1\alpha_2)\mu + \alpha_1\alpha_1\alpha_4)\lambda^2 \\ & + (\beta es_v v_h(\theta + \phi)^2 s^2 h + ((-\beta^2 I_v es_v - es_v(\alpha_1 + \alpha_4))\beta + v_h(-\theta + \phi)^2(\mu + \alpha_2))\alpha_5 \\ & - es_v(-ev_h + I_v\alpha_4)\beta^2 - e((eI_h + \alpha_1)\alpha_4 + eI_h\alpha_1)\beta s_v - \mu v_h\alpha_2(-\theta + \phi)^2)s_h + \\ & (((-\alpha_1 + \alpha_2)\alpha_4 + \alpha_1\alpha_2)\mu - \alpha_1\alpha_2\alpha_4)\alpha_5 + \alpha_1\alpha_2\alpha_4\mu)\lambda - (\theta + \phi)^2\beta es_v(eI_h + \alpha_5)v_h s_h^2 + \\ & ((\beta^2 I_v es_v\alpha_4 + \alpha_1 es_v\beta\alpha_4 + v_h\mu\alpha_2(-\theta + \phi)^2)\alpha_5 + e^2\alpha_4\beta s_v(-\beta v_h + \alpha_1))s_h - \alpha_1\alpha_2\alpha_4\alpha_5\mu) \end{aligned} \tag{35}$$

Simplifying in powers of (λ)

$H(\lambda) =$

$$\begin{aligned} & \lambda^6 + (-\alpha_3 + \mu - \alpha_4 - \alpha_1 + \alpha_2 - \alpha_5)\lambda^5 + (rR_h(rs_h) \\ & + (-es_v\beta - v_h(-\theta + \phi)^2)s_h + (-\mu + \alpha_1 + \alpha_4 - \alpha_2)\alpha_5 \\ & + (\alpha_2 - \alpha_4 - \alpha_1)\mu + (\alpha_1 - \alpha_2)\alpha_4 - \alpha_1\alpha_2 - \alpha_3(\mu - \alpha_4 - \alpha_1 + \alpha_2 - \alpha_5))\lambda^4 + \\ & (((\beta es_v + v_h(-\theta + \phi)^2)\alpha_5 + \beta^2 I_v es_v + e(eI_h + \alpha_1 + \alpha_4)\beta s_v \\ & - v_h(-\theta + \phi)^2(\mu + \alpha_2))s_h + ((-\alpha_2 + \alpha_4 + \alpha_1)\mu + (-\alpha_1 + \alpha_2)\alpha_4 + \alpha_1\alpha_2)\alpha_5 + \\ & ((\alpha_1 - \alpha_2)\alpha_4 - \alpha_1\alpha_2)\mu + \alpha_1\alpha_2 - \alpha_3((-\beta es_v - v_h(-\theta + \phi)^2)s_h \\ & + (-\mu + \alpha_1 + \alpha_4 - \alpha_2)\alpha_5 + (\alpha_2 - \alpha_4 - \alpha_1)\mu + (\alpha_1 - \alpha_2)\alpha_4 - \alpha_1\alpha_2) \\ & - (r\alpha_5 + \delta\beta_v - r(\mu + \alpha_2) + r\alpha_4)R_h(rs_h))\lambda^3 + (es_v\beta v_h(-\theta + \phi)^2)s_h^2 \\ & + ((-\beta^2 I_v es_v - es_v(\alpha_1 + \alpha_4))\beta + v_h(-\theta + \phi)^2(\mu + \alpha_2))\alpha_5 - es_v(-I_h\alpha_4)\beta^2 \\ & - e((eI_h + \alpha_1)\alpha_4 + eI_h\alpha_1)\beta s_v - v_h\mu\alpha_1(-\theta + \phi)^2)s_h((-\alpha_1 + \alpha_2)\alpha_4 \\ & + \alpha_1\alpha_2)\mu - \alpha_1\alpha_2\alpha_4)\alpha_5 + \alpha_1\alpha_2\alpha_4\mu - \alpha_3(((es_v\beta + v_h(-\theta + \phi)^2)\alpha_5 \\ & + \beta^2 I_v es_v + e(eI_h + \alpha_1 + \alpha_4)\beta s_v - v_h(-\theta + \phi)^2)(\mu + \alpha_2))s_h \\ & + ((-\alpha_2 + \alpha_4 + \alpha_1)\mu + (-\alpha_1 + \alpha_2)\alpha_4 + \alpha_1\alpha_2)\alpha_5 + ((\alpha_1 - \alpha_2)\alpha_4 \\ & - \alpha_1\alpha_2)\mu + \alpha_1\alpha_2\alpha_4) - (res_v\beta s_h + (-\delta\beta_v + r(\mu + \alpha_2))\alpha_5 + \\ & (-r\alpha_2 + \delta\beta_v - r(\mu + \alpha_2))\alpha_4)R_h(rs_h))\lambda^2 + (-\alpha_3(es_v\beta v_h(-\theta + \phi)^2)s_h^2 \\ & + ((-\beta^2 I_v es_v - es_v(\alpha_1 + \alpha_4))\beta + v_h(-\theta + \phi)^2(\mu + \alpha_2))\alpha_5 \\ & - es_v(-eI_h\alpha_4)\beta^2 - e((eI_h + \alpha_1)\alpha_4 + eI_h\alpha_1)\beta s_v - v_h\mu\alpha_2(-\theta + \phi)^2)s_h \\ & (((-\alpha_1 + \alpha_2)\alpha_4 + \alpha_1\alpha_2)\mu - \alpha_1\alpha_2\alpha_4)\alpha_5 + \alpha_1\alpha_2\alpha_4\mu) - (-\theta + \phi)^2\alpha_5 + \\ & e^2\alpha_4\beta s_v(-\beta v_h + \alpha_1))s_h - \alpha_1\alpha_2\alpha_4\alpha_5\mu - (-\alpha_5(-r\alpha_2 + \delta\beta_v)\mu)\alpha_4)R_h(rs_h))\lambda \\ & + (-\alpha_5(-r\alpha_2 + \delta\beta_v)\mu - re\beta s_v(eI_h + \alpha_5)s_h)\alpha_4 R_h(rs_h) \\ & - \alpha_3(-\theta + \phi)^2 e\beta s_v eI_h\alpha_5 v_h s_h^2 + ((\beta^2 I_v es_v\alpha_4 + \alpha_1 es_v\beta\alpha_4 \\ & + v_h\mu\alpha_2(-\theta + \phi)^2)\alpha_5 + e^2\alpha_4\beta s_v(-\beta v_h + \alpha_1))s_h - \alpha_1\alpha_2\alpha_4\alpha_5\mu \end{aligned} \tag{36}$$

Let $\lambda = ip$

$H(ip) =$

$$\begin{aligned}
 & (ip)^6 + (-\alpha_3 + \mu - \alpha_4 - \alpha_1 + \alpha_2 - \alpha_5)(ip)^5 + (rR_h(rs_h)) \\
 & + (-es_v\beta - v_h(-\theta + \phi)^2)s_h + (-\mu + \alpha_1 + \alpha_4 - \alpha_2)\alpha_5 \\
 & + (\alpha_2 - \alpha_4 - \alpha_1)\mu + (\alpha_1 - \alpha_2)\alpha_4 - \alpha_1\alpha_2 - \alpha_3(\mu - \alpha_4 - \alpha_1 + \alpha_2 - \alpha_5)(ip)^4 \\
 & + (((\beta es_v + v_h(-\theta + \phi)^2)\alpha_5 + \beta^2 I_v es_v + e(eI_h + \alpha_1 + \alpha_4)\beta s_v - v_h(-\theta + \phi)^2)(\mu + \alpha_2))s_h \\
 & + ((-\alpha_2 + \alpha_4 + \alpha_1)\mu + (-\alpha_1 + \alpha_2)\alpha_4 + \alpha_1\alpha_2)\alpha_5 + ((\alpha_1 - \alpha_2)\alpha_4 - \alpha_1\alpha_2)\mu + \alpha_1\alpha_2 - \\
 & \alpha_3((-\beta es_v - v_h(-\theta + \phi)^2)s_h + (-\mu + \alpha_1 + \alpha_4 - \alpha_2)\alpha_5 + (\alpha_2 - \alpha_4 - \alpha_1)\mu \\
 & + (\alpha_1 - \alpha_2)\alpha_4 - \alpha_1\alpha_2) - (r\alpha_5 + \delta\beta I_v - r(\mu + \alpha_2) + r\alpha_4)R_h(rs_h))(ip)^3 \\
 & + (es_v\beta v_h(-\theta + \phi)^2)s_h^2 + ((-\beta^2 I_v es_v - es_v(\alpha_1 + \alpha_4)\beta + v_h(-\theta + \phi)^2)(\mu + \alpha_2))\alpha_5 \\
 & - es_v(-I_h\alpha_4)\beta^2 - e((eI_h + \alpha_1)\alpha_4 + eI_h\alpha_1)\beta s_v - v_h\mu\alpha_1(-\theta + \phi)^2)s_h(((\alpha_1 - \alpha_2)\alpha_4 + \alpha_1\alpha_2)\mu \\
 & - \alpha_1\alpha_2\alpha_4)\alpha_5 + \alpha_1\alpha_2\alpha_4\mu - \alpha_3(((es_v\beta + v_h(-\theta + \phi)^2)\alpha_5 + \beta^2 I_v es_v + e(eI_h + \alpha_1 + \alpha_4)\beta s_v \\
 & - v_h(-\theta + \phi)^2)(\mu + \alpha_2))s_h + ((-\alpha_2 + \alpha_4 + \alpha_1)\mu + (-\alpha_1 + \alpha_2) \\
 & \alpha_4 + \alpha_1\alpha_2)\alpha_5 + ((\alpha_1 - \alpha_2)\alpha_4 - \alpha_1\alpha_2)\mu + \alpha_1\alpha_2\alpha_4) - (res_v\beta s_h + (-\delta\beta I_v + r(\mu + \alpha_2))\alpha_5 \\
 & + (-r\alpha_2 + \delta\beta I_v - r(\mu + \alpha_2))\alpha_4)R_h(rs_h))(ip)^2 + (-\alpha_3(es_v\beta v_h(-\theta + \phi)^2)s_h^2 \\
 & + ((-\beta^2 I_v es_v - es_v(\alpha_1 + \alpha_4)\beta + v_h(-\theta + \phi)^2)(\mu + \alpha_2))\alpha_5 - es_v(-eI_h\alpha_4)\beta^2 \\
 & - e((eI_h + \alpha_1)\alpha_4 + eI_h\alpha_1)\beta s_v - v_h\mu\alpha_1(-\theta + \phi)^2)s_h(((\alpha_1 - \alpha_2)\alpha_4 + \alpha_1\alpha_2)\mu \\
 & - \alpha_1\alpha_2\alpha_4)\alpha_5 + \alpha_1\alpha_2\alpha_4\mu) - (-\theta + \phi)^2)\alpha_5 + e^2\alpha_4\beta s_v(-\beta I_v\alpha_1))s_h - \alpha_1\alpha_2\alpha_4\alpha_5\mu \\
 & - (-\alpha_5(-r\alpha_2 + \delta\beta I_v)\mu)\alpha_4)R_h(rs_h))(ip) + (-\alpha_5(-r\alpha_2 + \delta\beta I_v)\mu - re\beta s_v(eI_h + \alpha_5)s_h)\alpha_4R_h(rs_h) \quad (37) \\
 & - \alpha_3(-\theta + \phi)^2 e\beta s_v eI_h\alpha_5 v_h s_h^2 + ((\beta^2 I_v es_v\alpha_4 + \alpha_1 es_v\beta\alpha_4 + v_h\mu\alpha_2(-\theta + \phi)^2)\alpha_5 \\
 & + e^2\alpha_4\beta s_v(-\beta I_v\alpha_1))s_h - \alpha_1\alpha_2\alpha_4\alpha_5\mu
 \end{aligned}$$

Resolving into real and imaginary

$$H(ip) = F(p) + iG(p) \quad (38)$$

$$\begin{aligned}
 & F(p) = (-p)^6 + (rR_h(rs_h)) + (-es_v\beta - v_h(-\theta + \phi)^2)s_h + \\
 & (-\mu + \alpha_1 + \alpha_4 - \alpha_2)\alpha_5 + (\alpha_2 - \alpha_4 - \alpha_1)\mu + (\alpha_1 - \alpha_2)\alpha_4 \\
 & - \alpha_1\alpha_2 - \alpha_3(\mu - \alpha_4 - \alpha_1 + \alpha_2 - \alpha_5)(p)^4 + (es_v\beta v_h(-\theta + \phi)^2)s_h^2 \\
 & + ((-\beta^2 I_v es_v - es_v(\alpha_1 + \alpha_4)\beta + v_h(-\theta + \phi)^2)(\mu + \alpha_2))\alpha_5 - es_v(-I_h\alpha_4)\beta^2 \\
 & - e((eI_h + \alpha_1)\alpha_4 + eI_h\alpha_1)\beta s_v - v_h\mu\alpha_1(-\theta + \phi)^2)s_h(((\alpha_1 - \alpha_2)\alpha_4 + \alpha_1\alpha_2)\mu \\
 & - \alpha_1\alpha_2\alpha_4)\alpha_5 + \alpha_1\alpha_2\alpha_4\mu - \alpha_3(((es_v\beta + v_h(-\theta + \phi)^2)\alpha_5 + \beta^2 I_v es_v \\
 & + e(eI_h + \alpha_1 + \alpha_4)\beta s_v - v_h(-\theta + \phi)^2)(\mu + \alpha_2))s_h + ((-\alpha_2 + \alpha_4 + \alpha_1)\mu + (-\alpha_1 + \alpha_2) \\
 & \alpha_4 + \alpha_1\alpha_2)\alpha_5 + ((\alpha_1 - \alpha_2)\alpha_4 - \alpha_1\alpha_2)\mu + \alpha_1\alpha_2\alpha_4) - (res_v\beta s_h + (-\delta\beta I_v + r(\mu + \alpha_2))\alpha_5 + \\
 & (-r\alpha_2 + \delta\beta I_v - r(\mu + \alpha_2))\alpha_4)R_h(rs_h))(-p)^2 + (-\alpha_5(-r\alpha_2 + \delta\beta I_v)\mu - re\beta s_v(eI_h + \alpha_5)s_h)\alpha_4R_h \\
 & (rs_h) - \alpha_3(-\theta + \phi)^2 e\beta s_v eI_h\alpha_5 v_h s_h^2 + ((\beta^2 I_v es_v\alpha_4 + \alpha_1 es_v\beta\alpha_4 + v_h\mu\alpha_2(-\theta + \phi)^2)\alpha_5 \\
 & + e^2\alpha_4\beta s_v(-\beta I_v\alpha_1))s_h - \alpha_1\alpha_2\alpha_4\alpha_5\mu
 \end{aligned} \quad (39)$$

$$\begin{aligned}
 G(p) = & (-\alpha_3 + \mu - \alpha_4 - \alpha_1 + \alpha_2 - \alpha_5)(p)^5 + (((\beta es_v + v_h(-\theta + \phi)^2)\alpha_5 + \beta^2 I_v es_v + \\
 & e(eI_h + \alpha_1 + \alpha_4)\beta s_v - v_h(-\theta + \phi)^2)(\mu + \alpha_2))s_h + ((-\alpha_2 + \alpha_4 + \alpha_1)\mu + (-\alpha_1 + \alpha_2)\alpha_4 + \alpha_1\alpha_2)\alpha_5 + \\
 & ((\alpha_1 - \alpha_2)\alpha_4 - \alpha_1\alpha_2)\mu + \alpha_1\alpha_2 - \alpha_3((-\beta es_v - v_h(-\theta + \phi)^2)s_h + (-\mu + \alpha_1 + \alpha_4 - \alpha_2)\alpha_5 + \\
 & (\alpha_2 - \alpha_4 - \alpha_1)\mu + (\alpha_1 - \alpha_2)\alpha_4 - \alpha_1\alpha_2) - (r\alpha_5 + \delta\beta I_v - r(\mu + \alpha_2) + r\alpha_4)R_h(rs_h))(-p)^3 \\
 & + (-\alpha_3(es_v\beta v_h(-\theta + \phi)^2 s_h^2 + ((-\beta^2 I_v es_v - es_v(\alpha_1 + \alpha_4)\beta + v_h(-\theta + \phi)^2)(\mu + \alpha_2))\alpha_5 \\
 & - es_v(-eI_h\alpha_4)\beta^2 - e((eI_h + \alpha_1)\alpha_4 + eI_h\alpha_1)\beta s_v - v_h\mu\alpha_2(-\theta + \phi)^2)s_h(((-\alpha_1 + \alpha_2)\alpha_4 + \alpha_1\alpha_2)\mu \\
 & - \alpha_1\alpha_2\alpha_4)\alpha_5 + \alpha_1\alpha_2\alpha_4\mu) - (-\theta + \phi)^2)\alpha_5 + e^2\alpha_4\beta s_v(-\beta I_v\alpha_1))s_h - \alpha_1\alpha_2\alpha_4\alpha_5\mu - \\
 & (-\alpha_5(-r\alpha_2 + \delta\beta I_v)\mu)\alpha_4)R_h(rs_h))(p)
 \end{aligned} \tag{40}$$

Differentiating (39) and (40) with respect to p and setting $p = 0$

$$F'(0) = 0 \tag{41}$$

$$\begin{aligned}
 G'(0) = & (-\alpha_3(es_v\beta v_h(-\theta + \phi)^2 s_h^2 + ((-\beta^2 I_v es_v - es_v(\alpha_1 + \alpha_4)\beta \\
 & + v_h(-\theta + \phi)^2)(\mu + \alpha_2))\alpha_5 - es_v(-eI_h\alpha_4)\beta^2 - e((eI_h + \alpha_1)\alpha_4 \\
 & + eI_h\alpha_1)\beta s_v - v_h\mu\alpha_2(-\theta + \phi)^2)s_h(((-\alpha_1 + \alpha_2)\alpha_4 + \alpha_1\alpha_2)\mu \\
 & - \alpha_1\alpha_2\alpha_4)\alpha_5 + \alpha_1\alpha_2\alpha_4\mu) - (-\theta + \phi)^2)\alpha_5 + e^2\alpha_4\beta s_v(-\beta I_v\alpha_1))s_h \\
 & - \alpha_1\alpha_2\alpha_4\alpha_5\mu(-\alpha_5(-r\alpha_2 + \delta\beta I_v)\mu)\alpha_4)R_h(rs_h))
 \end{aligned} \tag{42}$$

Setting $p = 0$ into (39) and (40) gives

$$\begin{aligned}
 F(0) = & (-\alpha_5(-r\alpha_2 + \delta\beta I_v)\mu - re\beta s_v(eI_h + \alpha_5)s_h)\alpha_4 R_h(rs_h) \\
 & - \alpha_3(-\theta + \phi)^2 e\beta s_v eI_h\alpha_5 v_h s_h^2 + ((\beta^2 I_v es_v\alpha_4 + \alpha_1 es_v\beta\alpha_4 + \\
 & v_h\mu\alpha_2(-\theta + \phi)^2)\alpha_5 + e^2\alpha_4\beta s_v(-\beta I_v\alpha_1))s_h - \alpha_1\alpha_2\alpha_4\alpha_5\mu
 \end{aligned} \tag{43}$$

$$G(0) = 0 \tag{44}$$

$$\text{Hence } F(0)G'(0) > 0 \tag{45}$$

$$\text{Let } S_* = F(0)G'(0) \tag{46}$$

Therefore, the non-zero state will be stable when $S_* > 0$ and unstable if otherwise.

V. Conclusion

This paper proposes a mathematical model on the spread of malaria using a system of ordinary differential equations with six compartments. The equilibrium states were obtained and the endemic state analysed for stability. It is observed from the mathematical analysis that the non-zero equilibrium state will be stable when $F(0)G'(0) > 0$ and unstable otherwise.

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