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Nonlocal Boundary Value Problem for Nonlinear Impulsive q_k-Symmetric Integrodifference Equation

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Abstract: *A first order nonlinear impulsive integrodifference equation within the frame of q_k - symmetric quantum calculus is investigated by applying using fixed point theorems.The conditions for existence and uniqueness of solution are obtained.*

Keywords: *q_k - Symmetric integrodifference equation, q_k - symmetric derivatives, q_k - symmetric integrals, Boundary value problem.*

I. Introduction

The $q-$calculus was initiated in twenties of the last century. However,it has gained considerable popularity and importance during the last three decades or so. Their study has not only important theoretical meaning but also wide applications in conformal quantum mechanics, high energy physics, etc. We refer the reader to recent articles[1-7]. Recently, in [8], authors research first order nonlocal boundary value problem for nonlinear impulsive q_k-integrodifference equation.

On this line of thought in this paper, we study the existence and uniqueness of solutions for second order nonlinear q_k - symmetric integrodifference equation with nonlocal boundary condition and impulses:

$$\begin{cases} D_{q_k}u(t) = f(t,u(t)) + {}_{t_k}I_{q_k} g(t,u(t)), & 0<q_k<1, t\in J', \\ \Delta u(t_k) = I_k(u(t_k)), & t_k\in(0,1), k=1,2,\cdots,p, \end{cases} \quad (1)$$

$$u(0^+) = h(u) + u_0, \qquad u_0 \in R, \quad (2)$$

where D_{q_k} , ${}_{t_k}I_{q_k}$ are q_k- symmetric derivatives and q_k- symmetric integrals $(k=0,1,\cdots,p+1)$,respectively. $f,g\in C(J\times R,R)$, $I_k,h\in C(R,R)$, $J=[0,1]\bigcup(1,q_p^{-1}+(1-q_p^{-1})t_p]$, $0=t_0<t_1<t_2<\cdots<t_p<t_{p+1}=1$, $J'=[0,1]\setminus\{t_1,t_2,\cdots,t_p\}$, $\Delta u(t_k)=u(t_k^+)-u(t_k^-)$.

Where $u(t_k^+)$ and $u(t_k^-)$ denote the right and the left limits of $u(t)$ at $t=t_k(t=1,2,\cdots,p)$, respectively.

II. Preliminaries

Let us set $J_0=[0,t_1], J_1=(t_1,t_2],\cdots,J_{p-1}=(t_{p-1},t_p], J_p=(t_p,1], I=(1,q_p^{-1}+(1-q_p^{-1})t_p]$ and introduce the space: $PC(J,R)=\{u:J\to R \mid u\in C((t_k,t_{k+1}]\cup I), k=0,1,\cdots,p$
and $x(t_k^+)$ and $x(t_k^-)$ exist with $x(t_k^-)=x(t_k), k=1,2,\cdots,p\}$. Clearly, it is a Banach space with the norm $\|u\|=\sup_{t\in J}|u(t)|$. Where $J=[0,1]\bigcup(1,q_p^{-1}+(1-q_p^{-1})t_p]$.

Definition A function $u\in PC(J,R)$ with its derivative of second order existing on J is a solution of (1) if it satisfies (2).

For convenience, let us introduce some basic concepts of q_k-symmetric calculus.

For $0<q_k<1$ and $t\in J_k$,we define the q_k-symmetric derivatives of a real valued continuous function f as

$$D_{q_k}f(t) = \frac{f(q_k^{-1}t+(1-q_k^{-1})t_k) - f(q_kt+(1-q_k)t_k)}{(q_k^{-1}-q_k)(t-t_k)}, \quad (3)$$

$$D_{q_k} f(t_k) = \lim_{t\to t_k} D_{q_k} f(t).$$

Higher order $q_k -$ symmetric derivatives are given by

$$D^0_{q_k} f(t) = f(t), D^n_{q_k} f(t) = D_{q_k} D^{n-1}_{q_k} f(t) \; n\in N, t\in J_k. \tag{4}$$

The $q_k -$ symmetric integral of a function f is defined by

$${}_{t_k}I_{q_k} f(t) = \int_{t_k}^t f(s)d_{q_k}s = (1-q_k^2)(t-t_k)\sum_{n=0}^{\infty} q_k^{2n} f(t_k + q_k^{2n+1}(t-t_k)), \quad t\in J_k. \tag{5}$$

Provided the series converges.If $a\in(t_k,t)$ and f is defined on the interval (t_k,t), then

$$\int_a^t f(s)d_{q_k}s = \int_{t_k}^t f(s)d_{q_k}s - \int_{t_k}^a f(s)d_{q_k}s. \tag{6}$$

Observe that

$$D_{q_k}\left({}_{t_k}I_{q_k} f(t)\right) = D_{q_k}\int_{t_k}^t f(s)d_{q_k}s = f(t),$$

$${}_{t_k}I_{q_k}\left(D_{q_k} f(t)\right) = \int_{t_k}^t D_{q_k} f(s)d_{q_k}s = f(t), \tag{7}$$

$${}_aI_{q_k}\left(D_{q_k} f(t)\right) = \int_a^t D_{q_k} f(s)d_{q_k}s = f(t) - f(a), \qquad a\in(t_k,t).$$

For $t\in J_k$,the following reversing order of $q_k -$ integration holds

$$\int_{t_k}^t\int_{t_k}^s f(r)d_{q_k}r d_{q_k}s = \int_{t_k}^{q_k t+(1-q_k)t_k} (t-r) f(r)d_{q_k}r. \tag{8}$$

In fact, $\int_{t_k}^t\int_{t_k}^s f(r)d_{q_k}r d_{q_k}s = (t-t_k)(1-q_k^2)\sum_{n=0}^{\infty} q_k^{2n}\int_{t_k}^{t_k+q_k^{2n+1}(t-t_k)} f(r)d_{q_k}r$

$$= (t-t_k)^2(1-q_k^2)^2\sum_{n=0}^{\infty} q_k^{2n}q_k^{2n+1}\sum_{m=0}^{\infty} q_k^{2m} f(t_k + q_k^{2m+1}(t_k + q_k^{2n+1}(t-t_k)-t_k))$$

$$= (t-t_k)^2(1-q_k^2)^2\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_k^{4n+1+2m} f(t_k + q_k^{2m+2n+2}(t-t_k))$$

$$= (t-t_k)^2(1-q_k^2)^2\sum_{n=0}^{\infty}\sum_{\tau=n}^{\infty} q_k^{2n+1+2\tau} f(t_k + q_k^{2\tau+2}(t-t_k))$$

$$= (t-t_k)^2(1-q_k^2)^2\sum_{\tau=0}^{\infty} q_k^{1+2\tau}(1-q_k^{2+2\tau})\; f(t_k + q_k^{2\tau+2}(t-t_k))$$

$$= \int_{t_k}^{q_k t+(1-q_k)t_k} (t-r) f(r)d_{q_k}r.$$

Note that if $t_k = 0$ and $q_k = q$ in (3) and (5). then $D_{q_k} f = D_q f, {}_{t_k}I_{q_k} f = {}_0I_q f$, where D_q and ${}_0I_q$ are the well-known $q-$ derivative and $q-$ integral of the function $f(t)$ defined by

$$D_{q_k} f(t) = \frac{f(q^{-1}t) - f(q\,t)}{(q^{-1}-q\,)t},$$

$$I_q\; f(t) = \int_0^t f(s)d_q s = (1-q^2)t\sum_{n=0}^{\infty} q^{2n} f(q^{2n+1}t).$$

Lemma 1. For given $y_{q_k}\in C(J,R)$,the function $u\in C(J,R)$ is a solution of the impulsive $q_k -$ symmtric integrodifference equation

$$\begin{aligned} &D_{q_k}u(t) = y_{q_k}(t), \qquad 0 < q_k < 1, t \in J', \\ &\Delta u(t_k) = I_k(u(t_k)), \qquad t_k \in (0,1), k = 1,2,\cdots, p, \\ &u(0^+) = h(u) + u_0, \qquad u_0 \in R. \end{aligned} \tag{9}$$

If and only if u satisfies the $q_k -$ integral equation

$$u(t) = \begin{cases} \int_0^t y_{q_0}(s)d_{q_0}s + h(u) + u_0, t \in J_0 \\ \int_{t_k}^t y_{q_k}(s)d_{q_k}s + \sum_{i=0}^{k-1} y_{q_i}(s)d_{q_i}s + \sum_{i=0}^{k} I_i(u(t_i)) + h(u) + u_0, t \in J_k \\ \int_{t_p}^t y_{q_p}(s)d_{q_p}s + \sum_{i=0}^{p-1} y_{q_i}(s)d_{q_i}s + \sum_{i=0}^{p} I_i(u(t_i)) + h(u) + u_0, t \in J_p \bigcup I. \end{cases} \tag{10}$$

Proof. Let u be a solution of $q_k -$ symmetric difference equation (9) . For $t \in J_0$,applying the operator ${}_0I_{q_0}$ on both sides of $D_{q_0}u(t) = y_{q_0}(t),$ we have

$$u(t) = u(0^+) + \int_0^t y_{q_0}(s)d_{q_0}s,$$

Similarly, for $t \in J_1$, applying the operator ${}_{t_1^+}I_{q_1}$ on both sides of $D_{q_1}u(t) = y_{q_1}(t),$ then

$$u(t) = u(t_1^+) + \int_{t_1}^t y_{q_1}(s)d_{q_1}s.$$

In view of $\Delta u(t_1) = u(t_1^+) - u(t_1^-) = I_1(u(t_1))$, it holds

$$u(t) = u(0^+) + \int_{t_1}^t y_{q_1}(s)d_{q_1}s + I_1(u(t_1)), t \in J_1.$$

Repeating the above process,we can get

$$u(t) = u(0^+) + \int_{t_k}^t y_{q_k}(s)d_{q_1}s + \sum_{i=0}^{\infty}\int_{t_i}^{t_{i+1}} y_{q_i}(s)d_{q_i}s + \sum_{i=1}^{k} I_i(u(t_i)), t \in J_k.$$

Using the boundary value conditions given in (9), it follows

$$u(t) = \int_{t_k}^t y_{q_k}(s)d_{q_k}s + \sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}} y_{q_i}(s)d_{q_i}s + \sum_{i=1}^{k} I_i(u(t_i)) + h(u) + u_0, t \in J_k.$$

For $t \in J_p \bigcup I$,we have

$$u(t) = \int_{t_p}^t y_{q_p}(s)d_{q_p}s + \sum_{i=0}^{p-1}\int_{t_i}^{t_{i+1}} y_{q_i}(s)d_{q_i}s + \sum_{i=1}^{p} I_i(u(t_i)) + h(u) + u_0.$$

Thus we can get (10). Conversely, assume that u satisfies the impulsive $q_k -$ integral equation (9), applying D_{q_k} on both sides of (10) and substituting $t = 0$ in (10), then (9) holds. This completes the proof.

Remark 1. In (10)， t allow belongs to I . Since $(q_p^{-1}t + (1-q_p^{-1})t_p)q_p^{2n+1} + (1-q_p^{2n+1})t_p < t, n \in \mathrm{N}$

and $\int_{t_p}^{q_p^{-1}t+(1-q_p^{-1})t_p} y_{q_p}(s)d_{q_p}s = (1-q_p^2)q_p^{-1}(t-t_p)\sum_{n=0}^{\infty} q_p^{2n} y_{q_p}(q_p^{-1}t + (1-q_p^{-1})t_p)q_p^{2n+1} + (1-q_p^{2n+1})t_p)$,

We see that $u(q_p^{-1}t + (1-q_p^{-1})t_p),$ for $t \in I$ have definition.

III. Main Results

Letting $y_{q_k}(t) = f(t,u(t)) + {}_{t_k}I_{q_k} g(t,u(t))$, in view of Lemma 1, we introduce an operator $PC \to PC$ as

$$(Qu)(t) = \int_{t_k}^{t} [f(s,u(s)) + \int_{t_k}^{s} g(r,u(r)d_{q_k}r]d_{q_k}s$$
$$+\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}} [f(s,u(s)) + \int_{t_i}^{s} g(r,u(r)d_{q_k}r]d_{q_i}s + \sum_{i=1}^{k} I_i(u(t_i)) + h(u) + u_0.$$

(11)

By (8), we obtain

$$(Qu)(t) = \int_{t_k}^{t} f(s,u(s))d_{q_k}s + \int_{t_k}^{q_k t+(1-q_k)t_k} (t-s)g(s,u(s)d_{q_k}s$$
$$+\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}} [f(s,u(s))d_{q_i}s + \sum_{i=0}^{k-1}\int_{t_i}^{q_i t_{i+1}+(1-q_i)t_i} (t_{i+1}-s)g(s,u(s))d_{q_i}s + \sum_{i=1}^{k} I_i(u(t_i)) + h(u) + u_0.$$

(12)

Then, the impulsive q_k - symmetric integrodifference equation (1) (2) has a solution if and only if the operator equation $u = Qu$ has a fixed point .

In order to prove the existence of solutions for (1) (2), we need the following known result (J. X. Sun, 2008).

Lemma 2. Let E be a Banach space. Assume that $T : E \to E$ is a completely continuous operator and the set $V = \{x \in E \,|\, x = \mu Tx, 0 < \mu < 1\}$ is bounded. Then T has a fixed point in E .

Theorem 3. Assume the following .

(H_1) There exist nonnegative bounded function $M_i(t)$ $(i = 1,2,3,4)$ such that

$$|f(t,u)| \le M_1(t) + M_2(t)|u|, \quad |g(t,u)| \le M_3(t) + M_4(t)|u|, \quad \text{for any } t \in J, u \in R,$$

denote $\sup_{t\in J}|M_i(t)| = M_i$, $i = 1,2,3,4$.

(H_2) There exists positive constants $\bar{L}$, L' such that $|I_k(u)| \le \bar{L}$, $h(u) \le L'$ for any $u \in R$, $k = 1,2,\cdots,p$.

Then the problem (1) (2) has at least one solution provided.

$$\tau = \sup_{t\in J}\{(q_p^{-1} + (1-q_p^{-1})t_p)M_2(t) + M_4(t)\sum_{i=0}^{p}(t_{i+1}-t_i)^2 + M_4(t)q_p^{-2}(1-t_p)^2\} < 1.$$

Proof. Firstly, similar to the proof of Theorem 3 in [8], we prove the operator Q is completely continuous. Next we define the set $W_1 = \{u \in PC(J,R) \,|\, u = \lambda Qu, 0 < \lambda < 1\}$. We show W_1 is bounded. Let $u \in W_1$, then $u = \lambda Qu$, $0 < \lambda < 1$. For any $t \in J$ by conditions (H_1) and (H_2), we have

$$|u(t)| = \lambda|(Qu)(t)|$$
$$\le \int_{t_k}^{t} |f(s,u(s))|d_{q_k}s + \int_{t_k}^{q_k t+(1-q_k)t_k} (t-s)|g(s,u(s)|d_{q_k}s$$
$$+\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}} [|f(s,u(s))|d_{q_i}s + \sum_{i=0}^{k-1}\int_{t_i}^{q_i t_{i+1}+(1-q_i)t_i} (t_{i+1}-s)|g(s,u(s))|d_{q_i}s + \sum_{i=1}^{k}|I_i(u(t_i))| + |h(u)| + |u_0|$$

$$\le (M_1 + M_2 \| u \|)(t - t_k) + (M_3 + M_4 \| u \|)q_k (t - t_k)^2$$

$$+ \sum_{i=0}^{k-1} [(M_1 + M_2 \| u \|)(t_{i+1} - t_i) + q_i (t_{i+1} - t_i)^2 (M_3 + M_4 \| u \|)] + \sum_{i=1}^{k} \overline{L} + L' + | u_0 |$$

$$\le (M_1 + M_2 \| u \|)(q_p^{-1} + (1 - q_p^{-1})t_p) + (M_3 + M_4 \| u \|)\sum_{i=0}^{p} (t_{i+1} - t_i)^2$$

$$+ (M_3 + M_4 \| u \|)q_p^{-2} (1 - t_p)^2 + \sum_{i=0}^{k-1} [(M_1 + M_2 \| u \|)(t_{i+1} - t_i)$$

$$+ q_i (t_{i+1} - t_i)^2 (M_3 + M_4 \| u \|)] + \sum_{i=1}^{k} \overline{L} + L' + | u_0 |$$

$$\le M_1 (q_p^{-1} + (1 - q_p^{-1})t_p) + M_3 \sum_{i=0}^{p} (t_{i+1} - t_i)^2 + M_3 q_p^{-2} (1 - t_p)^2 + m\overline{L} + | u_0 |$$

$$+ \| u \| (M_2 (q_p^{-1} + (1 - q_p^{-1})t_p) + M_4 \sum_{i=0}^{p} (t_{i+1} - t_i)^2 + M_4 q_p^{-2} (1 - t_p)^2.$$

$$\| u \| \le \frac{1}{1 - \tau} [M_1 (q_p^{-1} + (1 - q_p^{-1})t_p) + M_3 \sum_{i=0}^{p} (t_{i+1} - t_i)^2 + M_3 q_p^{-2} (1 - t_p)^2 + m\overline{L} + | u_0 | := \text{constant}$$

So, the set W_1 is bounded. Thus, Lemma 2 ensures the impulsive q_k- symmetric integrodifference equation (1) (2) has at least one solution.

Corollary 4. Assume the following.

(H_3) There exist nonnegative constants L_i, $i = 1,2,3,4$ such that

$$|f(t,u)| \le L_1, \quad |g(t,u)| \le L_2, \quad |I_k(u)| \le L_3, \quad |h(u)| \le L_4$$

for any $t \in J$, $u \in R$, $k = 1,2,\cdots, p$. Then problem (1) (2) has at least one solution.

Theorem 5. Assume the following.

(H_4) There exist nonnegative bounded functions $M(t)$ and $N(t)$ such that

$$|f(t,u) - f(t,v)| \le M(t)|u - v|, \quad |g(t,u) - g(t,v)| \le N(t)|u - v|, \quad \text{for } t \in J, u, v \in R.$$

(H_5) There exist positive constants K, G such that

$$|I_k(u) - I_k(v)| \le K|u - v|, \quad |h(u) - h(v)| \le G|u - v|, \quad \text{for } u, v \in R \text{ and } k = 1,2,\cdots, p.$$

(H_6) $K_1 = \sup_{t \in J} \{M(t)t + mK + G + N(t)\sum_{i=0}^{p} (t_{i+1} - t_i)^2 + N(t)q_p^{-2} (1 - t_p)^2\} < 1.$

Then problem (1) (2) has a unique solution.

Proof. Clearly Q is a continuous operator. Denote $\sup_{t \in J} |M(t)| = M$, $\sup_{t \in J} |N(t)| = N$. For $u, v \in PC(J, R)$, by (H_4) and (H_5), we have

$$|(Qu)(t) - (Qv)(t)|$$

$$\leq \int_{t_k}^{t} | f(s,u(s)) - f(s,v(s)) | d_{q_k} s + \int_{t_k}^{q_k t+(1-q_k)t_k} (t-s) | g(s,u(s) \text{-} g(s,v(s) | d_{q_k} s$$

$$+ \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} [| f(s,u(s)) - f(s,v(s)) | d_{q_i} s + \sum_{i=0}^{k-1} \int_{t_i}^{q_i t_{i+1}+(1-q_i)t_i} (t_{i+1}-s) | g(s,u(s)) \text{-} g(s,v(s) | d_{q_i} s$$

$$+ \sum_{i=1}^{k} | I_i(u(t_i)) \text{-} I_i(v(t_i)) | + | h(u) - h(v) |$$

$$\leq \| u - v \| (\int_{t_k}^{t} M(s) d_{q_k} s + \int_{t_k}^{q_k t+(1-q_k)t_k} (t-s) N(s) d_{q_k} s$$

$$+ \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} M(s) d_{q_i} s + \sum_{i=0}^{k-1} \int_{t_i}^{q_i t_{i+1}+(1-q_i)t_i} (t_{i+1}-s) N(s) d_{q_i} s + \sum_{i=1}^{k} \mathrm{K} + G$$

$$\leq K_1 \|u - v\|.$$

As $K_1 < 1$ by (H_6). Therefore, Q is a contractive map. Thus, the conclusion of the Theorem 5 follows by Banach contraction mapping principle.

IV. Example

Consider the following second order nonlinear $q_k -$ symmetric integrodifference equation with impulses

$$D_{1/2+k} u(t) = 8 + 3\sqrt{t} + \ln(1 + 2t^3 + \frac{t^2}{25^3} | u(t) |)$$

$$+ \int_{1/2+k}^{t} (10s + \frac{s^3}{16^4} \sin(u(s)) d_{1/2+k} s, t \in (0,1), t \neq \frac{1}{1+k},$$

$$\Delta u(\frac{1}{1+k}) = \cos(u(\frac{1}{1+k})), k = 1,2,3,4,$$

$$u(0) = 5 + e^{-u^2}, \quad k = 1,2,3,4.$$

Obviously, $q_k = \frac{1}{2+k}$, $t_k = \frac{1}{1+k}$, $k = 1,2,3,4.$

$f(t,u) = 8 + 3\sqrt{t} + \ln(1 + 2t^3 + \frac{t^2}{25^3} | u |), \quad g(t,u) = 10t + \frac{t^3}{16^4} \sin u, \quad I_k(u) = \cos u$, and $u(0) = 5 + e^{-u^2}$.

By a simple calculation,we can get $| f(t,u) | \leq 1 + 3\sqrt{t} + 2t^3 + \frac{t^2}{25^3} | u |, \quad | g(t,u) | \leq t + \frac{t^3}{16^4} | u |,$ $|I_k(u)| \leq 1$, $| h(u) | \leq 1$. Take $M_1(t) = 1 + 3\sqrt{t} + 2t^3, M_2(t) = \frac{t^2}{25^3}, M_3(t) = t, M_4(t) = \frac{t^3}{16^4}, \overline{L} = L' = 1$ .Then all conditions of Theorem 3, the above nonlinear impulsive $q_k -$ symmetric integrodifference has at least one solution.

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