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Fibonacci and Lucas Identities with the Coefficients in Arithmetic Progression

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Abstract: *We define an analogous pair of recurrence relations that yield some Fibonacci, Lucas and generalized Fibonacci identities with the coefficients in arithmetic progression. One relation yields same sign identities and the other alternating signs identities. We also show some new results for negative indexed Fibonacci and Lucas Sequences.*

Keywords: *linear recurrence relation; Fibonacci sequence; Lucas identity; generalized Fibonacci sequence, arithmetic progression.*

I. Introduction

Fibonacci sequence is defined by the linear recurrence relation: $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$ with the initial conditions: $F_0 = 0$ and $F_1 = 1$. Lucas sequence is similar to Fibonacci sequence and defined by the relation: $L_{n+1} = L_n + L_{n-1}$ for $n \geq 1$ with the initial conditions: $L_0 = 2$ and $L_1 = 1$. Generalized Fibonacci sequence is defined by the relation: $G_{n+1} = G_n + G_{n-1}$ for $n \geq 1$ with the initial conditions: $G_0 = x$ and $G_1 = y$ for some x and y. Obviously Fibonacci and Lucas sequences are the special cases of generalized Fibonacci sequence with (0,1) and (2,1) as the values of the pair (x, y) respectively.
A generalized Fibonacci formula is given by the following recurrence relation:

$$\text{For } 0 \leq m \leq n, \quad G_{n+m} = L_m G_n + (-1)^{m+1} G_{n-m}. \tag{1}$$

(1) is known. This is the formula (10a) in 'List of formulae' in the book, Fibonacci and Lucas Numbers, and the Golden Section: Theory and Applications by Steven Vajda; and also the identity 45 in the book, Proofs That Really Count: The Art of Combinatorial Proof by Arthur Benjamin and Jennifer Quinn. Fibonacci and Lucas numbers both satisfy (1). So from (1), we get:

$$\text{For } 0 \leq m \leq n, \quad F_{n+m} = L_m F_n + (-1)^{m+1} F_{n-m}. \tag{1.1}$$

$$\text{For } 0 \leq m \leq n, \quad L_{n+m} = L_m L_n + (-1)^{m+1} L_{n-m}. \tag{1.2}$$

We obtain Fibonacci and Lucas identities with the coefficients in arithmetic progression by using (1.1), (1.2) and two recurrence relations which are defined in the next topic.

II. Pair of Analogous Recurrence Relations And Their Transformations

(a) Recurrence Relation 1

For some numbers: a, b and S_0, we define an n^{th} order recurrence function S_n for $n \geq 1$ by the linear recurrence relation:

$$S_n = \sum_{i=0}^{n-1} (a + ib) S_{n-1-i} + (a + nb). \tag{2}$$

From (2),

$$S_{n+1} = a S_n + (a+b) S_{n-1} + \dots + (a + nb) S_0 + \{a + (n+1)b\}. \tag{2.1}$$

$$= a S_n + a S_{n-1} + \dots + \{a + (n-1)b\} S_0 + (a + nb) + b (S_{n-1} + \dots + S_0 + 1).$$

$$\Rightarrow S_{n+1} = a S_n + S_n + b (S_{n-1} + \dots + S_0 + 1). \tag{22}$$

Similarly

$$S_{n+2} = a S_{n+1} + (a+b) S_n + \dots + (a + (n+1)b) S_0 + (a + (n+2)b). \tag{2.3}$$

$$\Rightarrow S_{n+2} = a S_{n+1} + S_{n+1} + b (S_n + \dots + S_0 + 1). \tag{2.4}$$

From (2.2) and (2.4),

$$b\,(S_{n-1} + \ldots + S_0 + 1) = S_{n+2} - (a+1)\,S_{n+1} - b\,S_n$$

$$= S_{n+1} - (a+1)\,S_n$$

$$\Rightarrow S_{n+2} = (a+2)\,S_{n+1} + \big(b - a - 1\big)\,S_n\,. \qquad (2.5)$$

(2.5) is a second order linear recurrence relation and a reduced version of (2.3) with the initial conditions: $S_1 = aS_0 + (a + b)$ and $S_2 = (a^2 + a + b)\,S_0 + (a^2 + ab + a + 2b)$.

(b) Recurrence Relation 2

For some numbers: a, b and T_0, we define an n^{th} order recurrence function T_n, for $n \geq 1$ by the linear recurrence relation:

$$T_n = \sum_{i=0}^{n-1} (-1)^i\,(a + ib)\,T_{n-1-i} + (-1)^n\,(a + nb)\,. \qquad (3)$$

We obtain (2.1), …, (2.5) from (2). In like manner we obtain below (3.1), …, (3.5) from (3):

$$T_{n+1} = \sum_{i=0}^{n} (-1)^i\,(a + ib)\,T_{n-i} + (-1)^{n+1}\{a + (n+1)b\}. \qquad (3.1)$$

$$T_{n+1} = a\,T_n - T_n + b\,\{-T_{n-1} + \ldots + (-1)^n\,T_0 + (-1)^{n+1}\}\,. \qquad (3.2)$$

$$T_{n+2} = \sum_{i=0}^{n+1} (-1)^i\,(a + ib)\,T_{n+1-i} + (-1)^{n+2}\{a + (n+2)b\}. \qquad (3.3)$$

$$T_{n+2} = a\,T_{n+1} - T_{n+1} + b\,\{-T_n + \ldots + (-1)^{n+1}\,T_0 + (-1)^{n+2}\}\,. \qquad (3.4)$$

$$T_{n+2} = (a-2)\,T_{n+1} + \big(a - b - 1\big)T_n\,. \qquad (3.5)$$

(3.5) is a second order linear recurrence relation and the reduced version of (3.3) with the initial conditions: $T_1 = aT_0 - (a + b)$; and $T_2 = (a^2 - a - b)T_0 - (a^2 + ab - a - 2b)$.

(c) Transformation of Second Order Linear Recurrence Relation to *n*-th Order Linear Recurrence Relations of Two Kinds

It follows from (2), (2.5), (3) and (3.5) that second order linear relation:

$$U_{n+2} = x\,U_{n+1} + y\,U_n \qquad (4)$$

has n^{th} order versions of two kinds. (4) is (2.5) with the conditions: $x = a + 2$ and $y = b - a - 1$; and is (3.5) with the conditions: $x = a - 2$ and $y = a - b - 1$. In other words (i) putting $a = x - 2$ and $b = x + y - 1$ in (2), we find the same sign n-th order version of (4); and (ii) putting $a = x + 2$ and $b = x - y + 1$ in (2), we find the alternating signs n-th order version of (4).

III. Usual Or Positive Indexed Fibonacci And Lucas Identities With The Coefficients In Arithmetic Progression

(a) Fibonacci and Lucas Identities from Recurrence Relation 1

(2.5) is analogous with (1).

(i) Putting $a = b = L_m - 2$ in (2.5), we get:

$$S_{n+2} = L_m\,S_{n+1} - S_n\,. \qquad (5.1)$$

(ii) Putting $a = L_m - 2$ and $b = L_m$ in (2.5), we get:

$$S_{n+2} = L_m\,S_{n+1} + S_n\,. \qquad (5.2)$$

The values of the pair (a, b) and L_m for $m \in (1, 2, 3, 4)$ are listed in Table 1.

Table 1

m	L_m	$(a,\ b)$
1	1	(–1, 1)
2	3	(1, 1)
3	4	(2, 4)
4	7	(5, 5)

(5.1) and (5.2) depend on S_0 also. By trial, it is found that (5.1) and (5.2) represent some particular forms of (1.1) and (1.2) for some values of the pair (m, S_0). Consequently, Recurrence relation 1 (the relation for S_n) or any higher order relation (the relation for S_{n+k} for $k \geq 1$) can yield the Fibonacci and Lucas identities with the coefficients in arithmetic progression. But the results are limited. The values of S_0 for $m \in (1, 2, 3, 4)$, which can generate the desired identities, are listed in Table 2.

Table 2

Values of m	**Values of S_0**
1	–4, –3, –2, –1, 0, 1
2	–3, –2, –1, 0, 1, 2
3	–5, –3, –2, –1, 0, 1, 2, 3, 5
4	–6, –3, 0, 3, 6, 9

In Row 1, the values of the pair (m, S_0) are: (1, –4), (1, –3), (1, –2), (1, –1), (1, 0), and (1,1,) where the common value of m is 1. All these six pairs can yield the identities of the desired pattern. Similarly in Row 2, six values of (m, S_0): (2, –3), (2, –2), (2, –1), (2, 0), (2, 1), and (2,2) can yield six identities; and so on. It is found that some values of the pair: (m, S_0) yield the same identity of desired pattern.

Example: We show the result when (m, S_0) is (2, –2). For $m = 2$, $L_m = L_2 = 3$ and $a = b = L_m - 2 = 1$; and then for $S_0 = -2$,

(2) yields: $S_1 = 0 = F_0$; and $S_2 = -1 = -F_2$;

(5.2) yields: $S_3 = L_2 S_2 - S_1 = -L_2 F_2 + F_0 = -F_4$;

$S_4 = L_2 S_3 - S_2 = -L_2 F_4 + F_2 = -F_6$;

… …

In general,

$$S_{n+2} = L_2 S_{n+1} - S_n = -L_2 F_{2n} + F_{2n-2} = -F_{2n+2} \text{ . [By (1.1)]}$$

Then from (2.3), we get the following Fibonacci identity with the coefficients in arithmetic Progression.

$$-F_{2n+2} = -F_{2n} - 2F_{2n-2} - \ldots - nF_0 - (n+1).$$

$$\Rightarrow F_{2n} = \sum_{i=0}^{n-1} i\, F_{2(n-i)} + n\,.$$

We can obtain also the above identity when the value of (m, S_0) is (2, 1).

The values of (m, S_0) and the Fibonacci and Lucas identities with respect to Recurrence Relation 1 are listed in Table 3 below.

Table 3

Sl. No.	*m*	S_0	Fibonacci Identities
1.	1	0, –1, –2, –3	$F_n = \sum_{i=1}^{n} (i-2) F_{n-i} + n.$ (6.1)
2.	2	1, –2	$F_{2n} = \sum_{i=0}^{n-1} i F_{2(n-i)} + n.$ (6.2)
3.	2	0, –1	$F_{2n+1} = \sum_{i=1}^{n} i F_{2(n-i)+1} + 1.$ (6.3)
4.	3	–3	$F_{3n} = 2\left\{\sum_{i=1}^{n} (2i-1) F_{3(n-i)} + n\right\}.$ (6.4)
5.	3	3, –1	$F_{3n+1} = 2\left\{\sum_{i=1}^{n} (2i-1) F_{3(n-i)+1}\right\} + 1.$ (6.5)
6.	3	2, –2	$F_{3n+2} = 2\sum_{i=1}^{n} (2i-1) F_{3(n-i)+2} + 2n + 1.$ (6.6)
7.			$F_{4n} = 5\sum_{i=1}^{n} i F_{4(n-i)} + 3n.$ (6.7)
8.	4	3	$F_{4n+1} = 5\sum_{i=1}^{n} i F_{4(n-i)+1} - n + 1.$ (6.8)
9.	4	–3, 6	$F_{4n+2} = 5\sum_{i=1}^{n} i F_{4(n-i)+2} + 2n + 1.$ (6.9)
10.	4	0	$F_{4n+3} = 5\sum_{i=1}^{n} i F_{4(n-i)+3} + n + 2.$ (6.10)
			Lucas Identities
11.	1	1, –4	$L_n = \sum_{i=1}^{n} (i-2) L_{n-i} + n + 2.$ (6.11)
12			$L_{2n} = \sum_{i=0}^{n-1} i L_{2(n-i)} + n + 2.$ (6.12)
13.	2	–3, 2	$L_{2n+1} = \sum_{i=1}^{n} i L_{2(n-i)+1} + 2n + 1.$ (6.13)
14.	3	5	$L_{3n} = 2\left\{\sum_{i=1}^{n} (2i-1) L_{3(n-i)} - n + 1\right\}.$ (6.14)
15.	3	–5	$L_{3n+1} = 2\sum_{i=1}^{n} (2i-1) L_{3(n-i)+1} + 4n + 1.$ (6.15)
16.	3	0, 1	$L_{3n+2} = 2\sum_{i=1}^{n} (2i-1) L_{3(n-i)+2} + 2n + 3.$ (6.16)
17.			$L_{4n} = 5\sum_{i=1}^{n} i L_{4(n-i)} - 5n + 2$ (6.17)
18.	4	9	$L_{4n+1} = 5\sum_{i=1}^{n} i L_{4(n-i)+1} + 5n + 1.$ (6.18)
19.			$L_{4n+2} = 5\sum_{i=1}^{n} i L_{4(n-i)+2} + 3.$ (6.19)
20.	4	–6	$L_{4n+3} = 5\sum_{i=1}^{n} i L_{4(n-i)+3} + 5n + 4.$ (6.20)

The identities are listed in Table 3 in the order of the expressions for Fx and then for Lx such that the values of x are the positive integers; and then the positive integers of the types: $2n$, $2n + 1$, $3n$, $3n + 1$, $3n + 2$, $4n$, $4n + 1$, $4n + 2$ and $4n + 3$ in the successive rows. The values of (m, S_0) are not found for (6.7), (6.12), (6.19) and (6.20). Applying the rule: $F_{4n} = F_{4n+2} - F_{4n+1}$, we can easily obtain (6.7) from (6.8) and (6.9). In like manner (6.12), (6.17) and (6.19) are the easy consequences of (6.13), (6.18) and (6.20).

(b) Fibonacci and Lucas Identities from Recurrence Relation 2

Only five alternating signs identities (four Fibonacci and one Lucas) are found by trial. These are listed in Table 4.

Table 4

Sl. No.	m	$(a,\ b)$	T_0	Fibonacci and Lucas Identities
1.	1	(3, 1)	1	$F_{n+2} = \sum_{i=1}^{n} (-1)^{i-1} (i+2) F_{n-i+2} + (-1)^n$ (7.1)
2.	2	(5, 5)	3	$F_{2n+1} = 5\sum_{i=1}^{n} (-1)^{i-1}\, i\, F_{2(n-i)+1} + (-1)^n (2n+1)$ (7.2)
3.	3	(6, 4)	2	$F_{3n+1} = 2\sum_{i=1}^{n} (-1)^{i-1} (2i+1) F_{3(n-i)+1} + (-1)^n (2n+1)$ (7.3)
4.	4	(9, 9)	1	$F_{4n+2} = 9\sum_{i=1}^{n} (-1)^{i-1}\, i\, F_{4(n-i)+2} + (-1)^n$ (7.4)
5.	1	(3, 1)	0	$L_{n+2} = \sum_{i=1}^{n} (-1)^{i-1} (i+2) L_{n-i+2} + (-1)^n (2n+3)$ (7.5)

(c) Generalized Fibonacci Identities from Recurrence Relation 1 and Recurrence Relation 2

(i) From (2.5), we get: $S_{n+2} = S_{n+1} + S_n$ for $a = -1$ and $b = 1$. On the other hand, the rule for generalized Fibonacci sequence is: $G_{n+1} = G_n + G_{n-1}$. Let $S_1 = G_0$ and $S_2 = G_1$. From (2), (2.1), … (2.5), we then get: $G_1 = 1 - G_0$ and $S_{n+2} = G_{n+1}$; and finally we get:

For $G_1 = 1 - G_0$ and $G_{n+1} = G_n + G_{n-1}$,

$$G_{n+1} = \sum_{i=0}^{n-1} (i-1)\, G_{n-i} + G_1 + n \tag{8.1}$$

(8.1) is a generalized Fibonacci identity with the coefficients in arithmetic progression.

(ii) From (3.5), we get: $T_{n+2} = T_{n+1} + T_n$ for $a = 3$ and $b = 1$. Let $T_1 = G_0$, $T_2 = G_1$ and $T_0 = x$. From (3), (3.1),, …, (3..5), we then get:

For $G_0 = 3x - 4$, $G_1 = 5x - 7$ and $G_{n+1} = G_n + G_{n-1}$,

$$G_{n+1} = \sum_{i=0}^{n} (-1)^i (i+3)\, G_{n-i} + (-1)^{n+1}\{(n-1)x + 4x - 5\} \tag{8.2}$$

(8.2) is another generalized Fibonacci identity with the coefficients in arithmetic progression. (8.1) and (8.2) are the same and alternating signs generalized Fibonacci identities respectively.

Remark 1: Periodic Oscillations of the Sequences from Recurrence Relation 1 and Recurrence Relation 2

From (2) and (2.5), we find the sequence: (S_1, S_2, S_3, ...) for $a = b = -1$; and from (3) and (3.5), the sequence: (T_1, T_2, T_3, ...) for $a = b = 3$ such that both the sequences have the typical periodic oscillations with the periodic cycles of six elements. The periodic cycle of the first sequence is composed of six elements: $-S_0 - 2$, $-S_0 - 1$, 1, $S_0 + 2$, $S_0 + 1$, –1; and the periodic cycle of the second sequence is composed of six elements: $3T_0 - 6$, $3T_0 - 9$, –3, $6 - 3T_0$, $9 - 3T_0$ and 3 in succession. The last three elements are the negative values of the first three and vice versa in both the periodic cycles of six elements. Obviously the periodic cycle of the first sequence is independent of the values of S_0 and the second one is independent of the values of T_0. It is found by trial that the sequences: (S_1, S_2, S_3, …) and (T_1, T_2, T_3, …) can be convergent or divergent depending on the values of the pair (a, b).

IV. Negative Indexed Fibonacci And Lucas Identities With The Coefficients In Arithmetic Progression

Negative indexed Fibonacci sequence or extended Fibonacci sequence is : … F_{-6}, F_{-5}, F_{-4}, F_{-3}, F_{-2}, F_{-1} defined by the relation: $F_{-n+1} = F_{-n} + F_{-n-1}$ for $n \geq 2$ with the initial conditions: $F_{-1} = 1$ and $F_{-2} = -1$. This sequence is also called 'Negafibonacci sequence'. We can obtain Fibonacci sequence with extension: $Fs(E)$ by the association of (i) the extended or the left hand sequence: $Fs(L)$ and (ii) the usual or the right hand sequence: $Fs(R)$. That is, $Fs(E) = Fs(L) + Fs(R)$. Replacing n by $-n$ in the rule for $Fs(R)$, we get the rule for $Fs(L)$. Fibonacci numbers under $Fs(L)$ and $Fs(R)$ satisfy the relation: $F_{-n} = (-1)^{n+1} F_n$.. The special case is: $F_{-0} = F_0 = 0$ for $n = 0$, which is under $Fs(R)$. F_{-0} is not a usual symbol. $Fs(E)$ is shown in Table 5.

Table 5

Fibonacci Sequence with Extension: *Fs*(*E*)												
..	…	F_{-17}	F_{-16}	F_{-15}	F_{-14}	F_{-13}	F_{-12}	F_{-11}	F_{-10}	F_{-9}	F_{-8}	F_{-7}
...	…	1597	–987	610	–377	233	–144	89	–55	34	–21	13
F_{-6}	F_{-5}	F_{-4}	F_{-3}	F_{-2}	F_{-1}	$\boldsymbol{F_0}$	F_1	F_2	F_3	F_4	F_5	F_6
–8	5	–3	2	–1	1	**0**	1	1	2	3	5	8
F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}	F_{14}	F_{15}	F_{16}	F_{17}	…	…
13	21	34	55	89	144	233	377	610	987	1597	…	…

Negative indexed Lucas sequence or extended Lucas sequence is : … L_{-6}, L_{-5}, L_{-4}, L_{-3}, L_{-2}, L_{-1} defined by the relation: $L_{-n+1} = L_{-n} + L_{-n-1}$ for $n \geq 2$ with the initial conditions: $L_{-1} = -1$ and $L_{-2} = 3$. Then Lucas sequence with extension: $Ls(E)$ is the association of (i) the extended or the left hand sequence: $Ls(L)$ and (ii) the usual or the right hand sequence: $Ls(R)$. Lucas numbers under $Ls(L)$ and $Ls(R)$ satisfy the relation: $L_n = (-1)^{n+1} L_{-n}$. $Ls(E)$ is shown in Table 6.

Table 6

Lucas Sequence with Extension: *Ls*(*E*)												
..	…	L_{-17}	L_{-16}	L_{-16}	L_{-14}	L_{-13}	L_{-12}	L_{-11}	L_{-10}	L_{-9}	L_{-8}	L_{-7}
...	…	–1974	1354	– 610	843	–521	322	–199	123	–76	47	–29
L_{-6}	L_{-5}	L_{-4}	L_{-3}	L_{-2}	L_{-1}	$\boldsymbol{L_0}$	L_1	L_2	L_3	L_4	L_5	L_6
18	–11	7	–4	3	–1	**2**	1	3	4	7	11	18
L_7	L_8	L_9	L_{10}	L_{11}	L_{12}	F_{13}	L_{14}	L_{15}	L_{16}	L_{17}	…	…
29	47	76	123	199	322	521	843	1364	2207	3571	…	…

$Fs(E)$ differs from $Ls(E)$ remarkably by the following relations:

(i) The numbers under $Fs(E)$ satisfy the relations: $F_{2n} = -F_{-2n}$ and $F_{2n-1} = F_{-2n+1}$.

(ii) The numbers under $Ls(E)$ satisfy the relations: $L_{2n} = L_{-2n}$ and $L_{2n-1} = -L_{-2n+1}$.

Some negative indexed Fibonacci and Luks identities are shown below.

$$F_{-n-2} = \sum_{i=2}^{n+1} (-1)^{i+1} (i-2) F_{-n-1+i} + (-1)^{n+1}(n+1) \qquad (9.1)$$

$$= -\sum_{i=1}^{n} (i+2) F_{-n-2+i} - 1 . \qquad (9.2)$$

$$F_{-2n} = \sum_{i=1}^{n-1} i F_{-2(n-i)} - n . \qquad (9.3)$$

$$F_{-2n+1} = \sum_{i=1}^{n-1} i\, F_{-2(n-i)+1} + 1. \tag{9.4}$$

$$F_{-2n-1} = 5\sum_{i=1}^{n} (-1)^{i-1}\, i\, F_{-2(n-i)-1} + (-1)^n (2n+1). \tag{9.5}$$

$$F_{-3n-1} = -2\sum_{i=1}^{n} (2i+1)\, F_{-3(n-i)-1} + (-1)^{2n} (2n+1). \tag{9.6}$$

$$F_{-3n-2} = 2\sum_{i=1}^{n} (-1)^{i}(2i-1)\, F_{-3(n-i)-2} + (-1)^{n+1} (2n+1). \tag{9.7}$$

$$F_{-4n+2} = 5\sum_{i=1}^{n-1} i\, F_{-4(n-i)+2} - 2n + 1. \tag{9.8}$$

$$F_{-4n-2} = 9\sum_{i=1}^{n} (-1)^{i-1}\, i\, F_{-4(n-i)-2} + (-1)^{n-1}. \tag{9.9}$$

$$L_{-n-2} = \sum_{i=2}^{n+1} (-1)^{i+1} (i-2)\, L_{-n-1+i} + (-1)^n (n+3) \tag{9.10}$$

$$= -\sum_{i=1}^{n} (i+2)\, L_{-n+i-2} + 2n + 3. \tag{9.11}$$

$$L_{-2n+1} = \sum_{i=0}^{n-1} i\, L_{-2(n-i)+1} - 2n + 1. \tag{9.12}$$

It is curious that the numbers under $Fs(E)$ satisfy (1.1) and the numbers under $Ls(E)$ satisfy (1.2) for the wide ranges of the values of n and m such that all $n, m \in$ z with $|n| \geq |m|$.

Examples: Let $|n| = 5$ and $|m| = 3$. Then the four values of the pair (n, m) are: (5, 3), (5, –3), (–5, 3) and (–5, –3) which yield four Fibonacci relations from (1.1) and four Lucas relations from (1.2).

(i) From (1.1), the four relations are:

$$L_3 F_5 = F_8 - F_2,$$

$$L_{-3} F_5 = F_2 - F_8,$$

$$L_3 F_{-5} = F_{-2} - F_{-8},$$

$$L_{-3} F_{-5} = F_{-8} - F_{-2},$$

(ii) From (1.2), the four relations are:

$$L_3 L_5 = L_8 - L_2.$$

$$L_{-3} L_5 = L_2 - L_8.$$

$$L_3 L_{-5} = L_{-2} - L_{-8}.$$

$$L_{-3} L_{-5} = L_{-8} - L_{-2}.$$

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