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Some Qualitative Approach for Bounded Solutions of Some Nonlinear Diffusion Equations with Non-Autonomous Coefficients: Oscillation Criteria

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Abstract:
This work investigates some oscillation criteria for the equation

$$\nabla\cdot\Big\{a(x,u)\Phi_\alpha(\nabla u)\Big\}+c(x,u)\phi_\alpha(u)+f(x,u)=0\quad x\in\Omega\subseteq\mathbb{R}^n.$$

This has been largely done for cases where the coefficients a and c are autonomous (i.e. not depending on the unknown function u). Using some Picone-type formulas we show that if those coefficients are continuous, positive and bounded away from zero and for small $|t|$, $0<\dfrac{f(x,t)}{\phi_\alpha(t)}=O(|t|^\theta)$ with $\theta\geq\alpha>0$, then any bounded and non-trivial solution of the equation is oscillatory.

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I. Introduction

1.1 Preliminaries

Nonlinear elliptic equations arise in many different domains in pure and applied Mathematics. In particular in Nonlinear Diffusion problems,
(slow, fast diffusion flows,...) (see e.g. [1]). The general equation has the form

$$-\nabla\cdot\Big(K_1(x,u,\nabla u)\Big)+b(x,u,\nabla u)=0. \tag{1.1}$$

According to the requirements of the given phenomena and characteristics of the problem, this equation can be put into various forms when the coefficients fulfill some specific conditions. Let $\Omega\subset\mathbb{R}^n$ be an open domain and $W:=\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^n$. In this work, we will be concerned about some models of problems coming from the following general type of problems:

$$\begin{cases}(i)\qquad \nabla\cdot\Big\{A(x,u,\nabla u)\nabla u\Big\}+K(x,u,\nabla u)u+f(x,u)=0;\quad x\in\Omega\\ \text{where for some } m>0,\quad \forall x\in\overline{\Omega}\text{ and } Y\in\mathbb{R}^n\ ,\\ (ii)\quad A\in C^1(x,\,u,\,Y;\,(m,\infty)\,);\quad K\in C(\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^n,\,\mathbb{R});\\ (iii)\quad f\in C(\overline{\Omega}\times\mathbb{R},\,\mathbb{R}).\end{cases} \tag{1.2}$$

In the sequel a solution of (1.2)(i) will be any $u\in\Re(\Omega)$ which satisfies weakly the equation (i) where

$$\Re(\Omega):=\{w\in C^1(\overline{\Omega})\bigcap C^2(\Omega)\,;\,A(x,w,\nabla w)\nabla w\in C^1(\overline{\Omega}\,\}. \tag{1.3}$$

Among other, one of the important qualitative aspects of the solutions (when they are bounded) of those problems is the oscillation criteria when that solution is obtained (or extended) in an unbounded domain Ω, say. The oscillatory criteria have been mainly investigated in the literature for the cases where the coefficients ($A(..)$ and $K(..)$) are autonomous (i.e. independent of the unknown functions $u,\ \nabla u,..$) (see [4, 6, 5, 11, 12] and references therein).

A function u is said to be oscillatory in Ω if $\forall R > 0$, it has zeros in
$\Omega_R := \{x \in \Omega;\ |x| > R\}$ and strongly oscillatory if its support has non-void open, bounded and connected components in any Ω_R.

1.2 Some recalls on equations with autonomous coefficients

It is about proiblems whose equations are of the type

$$\nabla \cdot \left\{ A(x)\Phi_\alpha(\nabla u) \right\} + C(x)\Phi_\alpha(u) + F(x,u,\nabla u) = 0; \quad x \in \Omega_R \qquad \textbf{(P0)}$$

where $\alpha > 0$ and $\Phi_\alpha(S) := |S|^{\alpha-1}S$. For this type of equations, in [6] the theorems 1.5 and 2.4 for one-dimensional cases and 3.1, 3.2 and 3.3 for multidimensional ones offer interesting results when the coefficients A and C are strictly positive functions and some boundedness conditions set on F.

In the hypotheses the coefficients A and C are required to keep (each) the same sign mainly for technical reasons; the results we establish rely on the fact that the corresponding "half-linear equations "

$$\nabla \cdot \left\{ A(x)\Phi_\alpha(\nabla u) \right\} + C(x)\Phi_\alpha(u) = 0$$

are odd in the sense that if u solves the equation, so will $-u$. Those results were obtained via usage of some Picone-type formulae and comparison principles.

Results for such problems with $\alpha = 1$, namely

$\nabla \cdot \left\{ a(u)\nabla u \right\} + C(x)u + f(x,u) = 0$ have been investigated in [8].

II. Models Problems And Main Resultys

In the sequel we define the following:

$\forall \gamma > 0,\quad s \in \mathbb{R}$ and $S \in \mathbb{R}^n \quad \phi_\gamma(s) := |s|^{\gamma-1}s$ and $\Phi_\gamma(S) := |S|^{\gamma-1}S$ and after easy and elaborate calculations, they have the forrlowing properties:

$$\begin{cases} s\phi_\gamma(s) = |s|^{\gamma+1}; \quad s\phi'_\gamma(s) = \gamma\phi_\gamma(s) \quad \textit{and} \quad \phi_\gamma(st) = \phi_\gamma(s)\phi_\gamma(t); \\ S\Phi_\gamma(S) = |S|^{\gamma+1} \quad \textit{and} \quad \Phi_\gamma(TS) = \Phi_\gamma(T)\Phi_\gamma(S); \\ \text{and for a function } s \text{ and } S = \nabla s, \quad s\nabla[\phi_\mu(s)] = \mu S\phi_\mu(s). \end{cases} \tag{2.1}$$

For the type of equation we will be dealing with, say

$$\nabla \cdot \left\{ a(x,u)\Phi_\gamma(\nabla u) \right\} + c(x,u)\phi_\gamma(u) + f(x,u) = 0 \quad x \in \Omega,$$

we first lay down the following hypotheses on the coefficients:

$$\textbf{(H)} \left[\begin{array}{l} \text{For some } \gamma,\ m > 0 \\ \textbf{(a)}: \quad \text{the coefficients } a \in C^1(\overline{\Omega} \times \mathbb{R},\ (m,\ \infty)\ ) \text{ and} \\ \qquad c \in C(\overline{\Omega} \times \mathbb{R},\ (m,\ \infty)\ ); \\ \textbf{(b)}: \quad a \text{ and } c \text{ are even in their secong argument} \\ \qquad \text{i.e. } \forall (x,t) \in \Omega \times \mathbb{R}, \quad a(x,t) = a(x,-t) \text{ and } c(x,t) = c(x,-t). \\ \textbf{(f)}: \quad \text{(i)} \quad f \in C(\overline{\Omega} \times \mathbb{R},\ \mathbb{R}) \text{ satisfies} \quad \dfrac{f(x,t)}{\phi_\alpha(t)} > 0 \quad \forall (x,t) \in \overline{\Omega} \times \mathbb{R} \setminus \{0\} \text{ and} \\ \qquad (ii) \quad \exists \theta > \alpha;\ |f(x,t)| = O(|t|^\theta) \text{ for small } |t|. \end{array} \right.$$

The conditions on f mainly the (ii) ensures that for smaff $|u|$, $\phi_\alpha(u)$ remains the leading term ensuring that the solution cannot have a compact support. (see e.g. [1])

2.1 Problems without damping terms

The first model problem is the following:
under thew hypotheses in (H), we consider
for $\Omega \subset \mathbb{R}^n$; $\alpha, m > 0$; $\phi := \phi_\alpha$ and $\Phi := \Phi_\alpha$ the equation

$$\nabla \cdot \left\{ a(x,u)\Phi(\nabla u) \right\} + c(x,u)\phi(u) + f(x,u) = 0 \quad x \in \Omega. \tag{2.2}$$

It is important to note that from (f) above , the function f is a restoring function in t (i.e. $\forall (x,s) \in \Omega \times \mathbb{R} \setminus \{0\}$, $sf(x,s) > 0$), if u is a solution of the equation (2.2), so would be $-u$; thus we can say that u is non-oscillatory if u is eventually strictly positive or eventually strictly negative, because a and c are set positive.

With Ω extended to the whole space the following result will be obtained:

Theorem 2.1. *Assume that* a, c, f *satisfy the hypotheses in (H).*

Then any non-trivial and bounded solution u of (2.2) in the whole $\mathbb{R}^n$ *is oscillatory in the sense that it has zeros in any exterior domain* Ω_R *of* $\mathbb{R}^n$.

Next we have this slight different result for one-dimensional equations:

Theorem 2.2. *Given the functions*

$$\textbf{(}\boldsymbol{H}\textbf{1)}: \quad \frac{D}{B}, \ D \in C(\mathbb{R}, \mathbb{R}^+) \text{ and the increasing } B \in C^1(\mathbb{R}, \mathbb{R}^+),$$

any bounded and non-trivial solution of

$$\left\{ B(u)\phi_\alpha(u') \right\}' + D(u)\phi_\alpha(u) = 0 \quad t \in \Omega_R; \quad u(R) > 0$$

is strongly oscillatory.

Remark 2.2
As we will see in the proof, the same conclusion holds for the equation

$$\left\{ B(u)\phi_\alpha(u') \right\}' + D(u)\phi_\beta(u) = 0 \quad t \in \Omega_R; \quad u(R) > 0$$

provided that $\beta > 0$. (see also [6, 7])

2.2 Problems with damping terms

In terms of oscillation character, when the coefficient a satisfies (H)(a), the equation (2.2) and

$$\nabla \cdot \left\{ \Phi(\nabla u) \right\} + \frac{\nabla a(x,u)}{a(x,u)} \Phi(\nabla u) + \frac{c(x,u)}{a(x,u)} \phi(u) + \frac{f(x,u)}{a(x,u)} = 0 \tag{2.3}$$

are equivalent in the sense that whenever one oscillates so does the other. For the equation (2.3), $\frac{\nabla a(x,u)}{a(x,u)} \Phi(\nabla u)$ denotes its damping term. Thus for the model of this type of problem we consider in Ω the problem

$$\begin{cases} (i) \quad \nabla \cdot \left\{ \Phi(\nabla u) \right\} + B(x,u) \cdot \Phi(\nabla u) + C(x,u)\phi(u) + F(x,u) = 0 \\ \text{where} \\ (ii) \quad \exists b \in C^1(\Omega \times \mathbb{R}, \mathbb{R}) \text{ such that } \forall u \in C^1(\mathbb{R}^n, R) \\ \qquad \nabla b(x,u) = B(x,u) := \nabla_x b(x,t)|_{t=u} + \frac{\partial}{\partial t} b(x,t)|_{t=u} \nabla_x u \\ (iii) \quad \text{and} \quad C \text{ and } F \text{ are as } c \text{ and } f \text{ in (H).} \end{cases} \tag{2.4}$$

We will establish the following result:

Theorem 2.3. *Under the hypotheses in (2.4)(iii), assume that*

a) *ether* $b(x,t) \equiv b(t)$ *and* $\forall t \in \mathbb{R}$ $b'(t) > 0$ *or*

b) $x \mapsto c(x,t)$ *is increasing and unbounded in* $|x|$ *while for some continuous non-negative function* $\rho \in C(\mathbb{R})$, $|\nabla_x b(x,t)| < \rho(t)$ $\forall x \in \Omega$.

Then any non-trivial and bounded solution of the equation is oscillatory.

III. Basic Picone-Type Formulae

Given two (supposedly) solutions u and v in Ω respectively of

$$\begin{cases} (a) \quad \nabla \cdot \Big\{ A(x,u,\nabla u)\nabla u \Big\} + K(x,u,\nabla u)u + f(x,u) = 0 \quad \text{and} \\ (b) \quad \nabla \cdot \Big\{ A_1(x,v,\nabla v)\nabla v \Big\} + K_1(x,v,\nabla v)v + f_1(x,v) = 0, \end{cases} \tag{3.1}$$

wherever $v \neq 0$,

$$\begin{aligned} &\nabla \cdot \Big\{ uA(x,u,\nabla u)\nabla u - \frac{u^2}{v}A_1(x,v,\nabla v)\nabla v \Big\} = \Big[A - A_1 \Big] |\nabla u|^2 \\ &+ A_1 Z(u,v) + u^2 \Big[K_1 - K \Big] + u^2 \Big\{ \frac{f_1}{v} - \frac{f}{u} \Big\} \end{aligned} \tag{3.2}$$

where $\forall w, z \in C^1$, the nonnegative 2-form Z is defined by

$$Z(w,z) := |\nabla w|^2 - 2\frac{w}{z}\nabla w \nabla z + |\frac{w}{z}\nabla z|^2 = \Big(\nabla w - \frac{w}{z}\nabla z \Big)^2. \tag{3.3}$$

This type of formula is the main tool for establishing the results. In fact (3.2) is valuable only in a domain where v is non zero. Thus if say $v \neq 0$ in a domain D and $u|_{\partial D} = 0$ then the integration of (3.2) over D would give 0 at the left.

So if it happens that the right hand of (3.2) is strictly positive (or negative) in that D, we have a contradiction implying that v has zero inside that D.

IV. Equations Without Damping Terms

Given a bounded region $\Omega \subset \mathbb{R}^n$, an α, $m > 0$, and

in the sequel, unless indicated otherwise $\phi\ (\Phi) := \phi_\alpha\ (\Phi_\alpha)$,

we consider the equation (2.2) under the hypotheses (H).

AS seen in (3.2)-(3.3), if v, w are solutions for

$\nabla \cdot \{a_1(x,v)\Phi(\nabla v)\} + c_1(x,v)\phi(v) + f_1(x,v) = 0$ and

$\nabla \cdot \{a(x,u)\Phi(\nabla u)\} + c(x,u)\phi(u) + f(x,u) = 0$ respectively,

formally wherever $v \neq 0$

$$\begin{cases} \nabla \cdot \Big\{ ua(x,u)\Phi(\nabla u) - u\phi(\frac{u}{v})a_1(x,v)\Phi(\nabla v) \Big\} = \\ \Big[a(x,u) - a_1(x,v) \Big] |\nabla u|^{\alpha+1} + a_1(x,v)\zeta_\alpha(u,v) + \\ u^{\alpha+1} \Big[c_1(x,v) - c(x,u) \Big] + u^{\alpha+1} \Big\{ \frac{f_1(x,v)}{\phi(v)} - \frac{f(x,u)}{\phi(u)} \Big\} \end{cases} \tag{4.1}$$

where $\forall z, y \in C^1(\Omega)$ the non-negative two-form ζ_γ, $(\gamma > 0)$ is defined by

$$\zeta_\gamma(z,y) := |\nabla z|^{\gamma+1} - (\gamma+1)|\frac{z}{y}\nabla y|^{\gamma-1}\frac{z}{y}\nabla y \cdot \nabla z + \gamma|\frac{z}{y}\nabla y|^{\gamma+1} \tag{4.2}$$

which is positive and is zero only if $\exists \mu \in \mathbb{R}; \quad z = \mu y$ ([11, 12]).

These types of equations are found in various Reaction Diffusion problems like flows through porous media, Plasma Physics ([11, 9]).

Unless indicated otherwise, in the sequel the coefficients $a_i(x,.)$ in the principal parts of the equations and $c_i(x,.)$ of the half-linear part will fullfil the hypotheses in (H).

4.1 Proof of Theorem 2.1

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Proof. Let u be such a bounded solution and $b := |u|_\infty$. It is known that for $A(x) := \max_{|t|\leq b}\{a(x,t)\}$

$$\nabla \cdot \Big\{A(x)\Phi(\nabla w)\Big\} + m\phi(w) = 0$$

is oscillatory given the hypotheses on $a(.,.)$ (see [6]).

Assume that $u > 0$ in some $\Omega_R := \{x \in \mathbb{R}^n; \quad |x| > R\}$. With this w, (4.1) reads in Ω_R

$$\begin{aligned}\nabla \cdot \Big\{wA(x)\Phi(\nabla w) - w\phi(\frac{w}{u})a(x,u)\Phi(\nabla u)\Big\} &= \Big[A(x) - a(x,u)\Big]|\nabla w|^{\alpha+1} \\ &+ a(x,u)\zeta_\alpha(w,u) + w^{\alpha+1}\Big[c(x,u) - m\Big] + w^{\alpha+1}\frac{f(x,u)}{\phi(u)}\end{aligned} \tag{4.3}$$

whose integration over any nodal set $D(w) \subset \Omega_R$ would give

$0 = \int_{D(w)}\Big\{\Big[A(x) - a(x,u)\Big]|\nabla w|^{\alpha+1} +$
$a(x,u)\zeta_\alpha(w,u) + w^{\alpha+1}\Big[c(x,u) - m\Big] + w^{\alpha+1}\frac{f(x,u)}{\phi(u)}\Big\}dx > 0$

which is absurd if $u > 0$ in Ω_R. Therefore $u > 0$ in Ω_R cannot hold for any $R > 0$. □

4.2 Proof of Theorem 2.2

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Proof. Althnough this result can be an application of Theorem 2.1, we provide here an analytical proof.

With

$$\text{increasing } B \in C^1(\mathbb{R}, \mathbb{R}^+) \quad \text{and} \quad \frac{D}{B}, D \in C(\mathbb{R}, \mathbb{R}^+),$$

if for some $R > 0$ a bounded and non-trivial solution u of

$\Big\{B(u)\phi_\alpha(u')\Big\}' + D(u)\phi_\alpha(u) = 0 \quad t \in \Omega_R; \quad u(R) > 0$

is strictly positive and bounded then from the equation

$u'(t)B'(u)\phi_\alpha(u') + B(u)[\phi_\alpha(u')]' = B'(u)|u'|^{\alpha+1} + B(u)[\phi_\alpha(u')]'$
$= -D(u)\phi_\alpha(u) \leq 0$ in Ω_R.

This implies that $[\phi_\alpha(u')]' = u''\phi'_\alpha(u') = \alpha|u'|^{\alpha-1}u'' \leq 0$. So, $u'' \leq 0$ in Ω_R and being bounded, $u' \geq 0$ and decreases to zero there.

Thus if $u(t) > 0 \quad \forall t > R$, it increases to its upper bound in Ω_R. Let $M := \sup_{\Omega_R} u(t)$ and $\delta := \min_{[u(T),M]}\frac{D(u)}{B(u)}$. From the equation

$\phi_\alpha(u'(t)) \leq \phi_\alpha(u'(T)) - \delta M(t-T), \quad \forall t > T$ which is eventually strctly negative, conflicting with the fact that $u' \geq 0$ in Ω_T. Therefore no bounded and non-trivial solution of the equation can be eventually positive.

V. Equations With Damping Terms

Let

$$\begin{cases} b \in C^1(\mathbb{R}^n \times \mathbb{R},\ (0,\ \infty)); \quad B(x,t) := \nabla b(x,t); \\ c(x,t) \ \text{ and } \ f(x,t) \ \text{ are as those in (H)}. \end{cases} \tag{5.1}$$

Let for some strictly positive $C \in C(\mathbb{R},\ (0,\ \infty)$ u and v be respectively solutions of

$$\begin{cases} (i) \quad \nabla \cdot \Big\{ \Phi(\nabla u) \Big\} + B(x,u) \cdot \Phi(\nabla u) + c(x,u)\phi(u) + f(x,u) = 0; \\ (ii) \quad \nabla \cdot \Big\{ \Phi(\nabla v) \Big\} + m\phi(v) = 0. \end{cases} \tag{5.2}$$

Assume that $u \neq 0$ in say, $D \subset \mathbb{R}^n$. Then in D easy calculations show that with
$\Gamma(v,u) := v\Phi(\nabla v) - v\Phi(\frac{v}{u}\nabla u)$,
the corresponding Picone-type formulae are:

$$\begin{cases} \nabla \cdot \Big\{ b(x,u)\Gamma(v,u) \Big\} = b(x,u) \Big\{ \zeta_\alpha(v,u) + \\ \Big[c(x,u) - m + \dfrac{f(x,u)}{\phi(u)} \Big] |v|^{\alpha+1} + vB(x,u) \cdot \Phi(\frac{v}{u}\nabla u) \Big\} \\ + B(x,u) \cdot \Gamma(v,u). \end{cases} \tag{5.3}$$

5.1 Proof of Theorem 2.3

Proof. .

We chose as before $m := \min_{|t| \leq b} \{c(x,t)\}$.

Assume that there is such a solution u for (2.4)(i) such that $|u|_\infty = b$.

$$\nabla \cdot \Big\{ \Phi(\nabla v) \Big\} + m\phi(v) = 0$$

is oscillatory and as in (5.3)(a), a version of Picone-type formula for v and u reads

$$\begin{aligned} &\nabla \cdot \Big\{ b(u)\Gamma(v,u) \Big\} = B(u) \cdot \Gamma(v,u) + b(u) \Big\{ \zeta_\alpha(v,u) + \\ &\Big[c(x,\beta) - m + \frac{f(x,u)}{\phi(u)} \Big] |v|^{\alpha+1} + vB(x,u) \cdot \Phi(\frac{v}{u}\nabla u) \Big\}. \end{aligned} \tag{5.4}$$

a) The equation (5.2)(i) is unchanged if b is replaced by $b(u) + \lambda$, $\forall \lambda \in \mathbb{R}$. If we assume that $u > 0$ in some Ω_R, then for any such λ, the integration of (5.4) over any nodal set $D(v^+) \subset \Omega_R$ gives

$$\begin{aligned} 0 = &\int_{D(v^+)} (b(u) + \lambda) \Big\{ \zeta_\alpha(v,u) + \Big[c(x,\beta) - m + \frac{f(x,u)}{\phi(u)} \Big] |v|^{\alpha+1} \\ &+ vB(x,u) \cdot \Phi(\frac{v}{u}\nabla u) \Big\} dx + \int_{D(v^+)} B(x,u) \cdot \Gamma(v,u) dx. \end{aligned}$$

For this to hold for any arbitrary λ, each integrand has to be zero in $D(v^+)$ and in particular

$$\zeta_\alpha(v,u) + \Big[c(x,\beta) - m + \frac{f(x,u)}{\phi(u)} \Big] |v|^{\alpha+1} + vB(x,u) \cdot \Phi(\frac{v}{u}\nabla u) \equiv 0. \tag{5.5}$$

But $vB(x,u)\cdot\Phi(\frac{v}{u}\nabla u) = vb'(u)|\nabla u|^{\alpha-1}|\nabla u|^{\alpha+1} \geq 0$ and each term in the equation above is non-negative. Therefore the assumption that such a bounded non-trivial solution u can remain strictly positive eventually is not possible.

b) In this case, the segment $\left[c(x,\beta) - m + \frac{f(x,u)}{\phi(u)}\right]|v|^{\alpha+1} + vB(x,u)\cdot\Phi(\frac{v}{u}\nabla u)$ remains strictly positive eventually as $c(x,t)$ is unbounded and

$|vB(x,u)\cdot\Phi(\frac{v}{u}\nabla u)| \leq const.\ |v|^{\alpha+1}$.

Therefore that segment is eventually positive.

VI. Some Applications

Oscillation criteria for a porous medium equation with source (see [3])

Such an equation (in steady form) can be

$$\begin{cases} \nabla\cdot\left\{\Phi_\alpha(\nabla V(x))\right\} + \phi_p(u) = 0 \quad x\in\mathbb{R}^n;\ n>1; \\ \qquad \beta,\ \alpha,\ p>0; \quad \text{with } V(x) := \phi_\beta(u) = |u(x)|^{\beta-1}u(x)\ . \end{cases} \tag{6.1}$$

As $\nabla\cdot\{|u(x)|^{\beta-1}u(x)\} := \nabla\cdot\{(u(x)^2)^{[\beta-1]/2}u(x)\}$, $\nabla\cdot\Phi_\alpha(\nabla V(x)) = \beta^\alpha|u(x)|^{\alpha(\beta-1)}\Phi_\alpha(\nabla u)$ and the equation in (6.1) becomes

$$\begin{cases} (i) \quad \nabla\cdot\left\{|u(x)|^{\alpha(\beta-1)}\Phi_\alpha(\nabla u)\right\} + \beta^{-\alpha}\phi_p(u) = 0; \quad x\in\mathbb{R}^n; \\ \text{and wherever } u\neq 0, \text{ it becomes for } \quad F(u,\nabla u) := \dfrac{\alpha(\beta-1)u}{|u|^2}|\nabla u|^{\alpha+1} \\ (ii) \quad \nabla\cdot\left\{\Phi_\alpha(\nabla u)\right\} + F(u,\nabla u) + \beta^{-\alpha}\phi_p(u) = 0; \quad x\in\mathbb{R}^n. \end{cases} \tag{6.2}$$

One-dimensional problem

In one-dimension, for $V(t) = |u(t)|^{\beta-1}u(t)$, after some calculations

$$\left\{|u|^{(\beta-1)\alpha}\,\phi_\alpha(u')\right\}' + \phi_p(u) = 0,\ t>0. \tag{6.3}$$

Wherever $u\neq 0$, this equation is equivalent to

$$\begin{cases} \left\{\phi_\alpha(u')\right\}' + \alpha(\beta-1)\dfrac{u}{|u|^2}|u'|^{\alpha+1} + \beta^{-\alpha}\,\phi_q(u) = 0,\ t>0. \\ \text{where } q := p + (1-\beta)\alpha = p + \alpha - \beta\alpha. \end{cases} \tag{6.4}$$

Theorem 6.1. *Let*

$\alpha > 0,\ \beta > 1$ *and* $p + \alpha > \alpha\beta$.

Then any bounded and non-trivial solution of (6.4) is oscillatory i.e. (6.3) is oscillatory.

Proof. Assume that there is a bounded and non-trivial solution $u\in C^2([0,\infty))$ such that $u(r) > 0$ in some $\Omega_R,\ R>0$.

When $p+\alpha > \alpha\beta$, $q > 0$. In Ω_R the middle term of (6.4) is positive and the result is an application of Theorem 2.4 of [6].

Multidimensional problem

From Theorem 3.3 of [6] we have

Theorem 6.2. *If $q = \alpha$ or $p = \alpha\beta$ then (6.2)(ii) is oscillatory as* $uF(u, \nabla u) > 0 \qquad \forall u \neq 0.$

6.1 Radially symmetric solutions

If there is a radially symmetric solution with $V(r) := \phi_\beta(u(|x|)) = |u|^{\beta-1}u;\ r := |x|$ of the equation, then easy calculations give

$$\left\{r^{n-1}|u(r)|^{\alpha(\beta-1)}\phi_\alpha(u')\right\}' + \frac{r^{n-1}}{\beta^\alpha}\phi_p(u) = 0, \quad r > 0$$

and for ease writing we consider

$$\left\{r^{n-1}|u(r)|^{\alpha(\beta-1)}\phi_\alpha(u')\right\}' + r^{n-1}\phi_p(u) = 0, \quad r > 0. \tag{6.5}$$

If we assume that $u > 0$ in say, Ω_R then there this equation in u becomes

$$\begin{cases} (i) \quad \left\{r^{n-1}\phi_\alpha(u')\right\}' + \dfrac{\alpha(\beta-1)r^{n-1}}{u}|u'|^{\alpha+1} \\ \qquad + r^{n-1}|u|^{(1-\beta)\alpha}\phi_p(u) = 0 \quad \text{or} \\ (ii) \quad \left\{r^{n-1}\phi_\alpha(u')\right\}' + \dfrac{\alpha(\beta-1)r^{n-1}}{u}|u'|^{\alpha+1} + r^{n-1}\phi_q(u) = 0 \end{cases} \tag{6.6}$$

with

$q = p + \alpha - \alpha\beta$. Because $u > 0$ in $\Omega_T \Longrightarrow \quad \dfrac{\alpha(\beta-1)r^{n-1}}{u}|u'|^{\alpha+1} \geq 0$ there, (6.6) is oscillatory if

$$\left\{r^{n-1}\phi_\alpha(u')\right\}' + r^{n-1}\phi_q(u) = 0; \quad u(0) = 0;\ u'(R) = b > 0 \tag{6.7}$$

is oscillatory.(see [6])

We recall also that similarly (6.7) is oscillatory in $\Omega_R;\ R > 0$ if $\left\{\phi_\alpha(u')\right\}' + \dfrac{n-1}{r}\phi_\alpha(u') + \phi_q(u) = 0; \quad r > R$ is.

As the term $\dfrac{n-1}{r}\phi_\alpha(u')$ has the form $[\log(r^{n-1})]'\phi_\alpha(u')$ (damping term), (6.7) will be oscillatory if (as $q = \alpha + (p - \alpha\beta)$)

$$\left\{\phi_\alpha(u')\right\}' + |u|^{p-\alpha\beta}\,\phi_\alpha(u) = 0; \quad r > R \quad \text{is.} \tag{6.8}$$

From [2] if $u \in C^2(\Omega)$ is a non negative solution then a Pohozaev formula for u reads

$$\begin{cases} R^n\left\{\dfrac{\alpha}{\alpha+1}|u'(R)|^{\alpha+1} + \dfrac{u(R)^{q+1}}{q+1} + \dfrac{n-(\alpha+1)}{(\alpha+1)R}\phi_\alpha(u'(R))u(R)\right\} \\ = \left[\dfrac{n(\alpha-q)}{(q+1)(\alpha+1)} + 1\right]\displaystyle\int_0^R r^{n-1}u(r)^{q+1}dr, \qquad \forall R > 0. \end{cases} \tag{6.9}$$

Define $\Gamma(\alpha,\beta,p) := \left[\dfrac{n(\alpha-q)}{(q+1)(\alpha+1)} + 1\right] = \dfrac{n(\alpha\beta-p)}{(\alpha+1)(p+\alpha-\alpha\beta+1)} + 1.$

As $u(0) = 0$ and $u'(0) > 0$, $u(r) > 0$ in some $(0, \mu)$.

If $\exists R_1 := \min\{R > 0 \mid \quad u'(R) = 0\}$, then

$$\frac{R_1^n u(R_1)^{q+1}}{q+1} = \Gamma(\alpha,\beta,p)\int_0^{R_1} r^{n-1}u(r)^{q+1}dr$$

and obviously such an R_1 would not exist if $\Gamma(\alpha,\beta,p)<0$. Thus any chance of finding an oscillatory solution for (6.7) requires that $\Gamma(\alpha,\beta,p)>0$ i.e.

$$\begin{cases} \text{if (6.7) is oscillatory then} \\ \Gamma(\alpha,\beta,p):=\left[\dfrac{n(\alpha-q)}{(q+1)(\alpha+1)}+1\right]=\dfrac{n(\alpha\beta-p)}{(\alpha+1)(p+\alpha-\alpha\beta+1)}+1>0. \end{cases} \tag{6.10}$$

Notice that if $q\le\alpha$ then this necessary condtition holds (as in Theorem 6.2 above).

We finally have the following result for the radially symmetric equation (6.5), as applications of results in [6]:

Theorem 6.3. *Given* $\alpha>0$ *and* $\beta>1$, *if ether*

a) $p=\alpha\beta$ *or*

b) $0\le p-\alpha\beta<\dfrac{(p+\alpha+1-\alpha\beta)(\alpha+1)}{n}$ *or*

c) $0<p<\alpha\beta$,

then any non-trivial and bounded solution for (6.5) is strongly oscillatory.

Proof. First each of the conditions a)-c) implies that the necessary condition (6.10) holds. The theorem 1.5 of [6] applies to (6.7) if a) holds:

the Theorem 1.3 of [6] applies to (6.8) if b) holds and the theorem 1.4 of [6] applies to (6.8) if c) holds. □

Dedications

To my late brother Fopa Kayem Rigobert and grand-son Kuate Ceddrick;

Que nos ancêtres vous guident vers la sérénité éternelle.

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