IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 13, Issue 1 Ver. I (Jan. - Feb. 2017), PP 72-77
www.iosrjournals.org

Achieve asymptotic stability using Lyapunov's second method

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Abstract: *This paper discusses asymptotic stability for autonomous systems by means of the direct method of Liapunov.Lyapunov stability theory of nonlinear systems is addressed .The paper focuses on the conditions needed in order to guarantee asymptotic stability by Lyapunov'ssecond method in nonlinear dynamic autonomous systems of continuous time and illustrated by examples.*
Keywords*:Lyapunov function, Lyapunov's second method, asymptotic stability, autonomous nonlinear differential system.*

I. Introduction

The most useful and general approach for studying the stability of nonlinear systems is the theory introduced in the late 19th century by the Rusian Mathematician Alexander MikhailovichLyapunov [1,2]. Lyapunov stability is a fundamental topic in mathematics and engineering, it is a general and useful approach to analyze the stability of nonlinear systems. Lyapunov stability concepts include two approaches: Lyapunov indirect (first) method and Lyapunov direct(second) method. For Lyapunov indirect method the idea of system linearization around a given point is used and one can achieve **local stability** with small stability regions. On the other hand the Lyapunov direct method is the most important tool for design and analysis of nonlinear systems. This method can be applieddirectly to a nonlinear system without the need to linearization and achieves **global stability**. The fundamental concept of the Lyapunov direct method is that if the total energy of a system is continuously dissipating, then the system will eventually reach an equilibrium point and remain at that point. Hence, the Lyapunov direct method consists of two steps. Firstly, a suitable scalar function v(x) is chosen and this function is referred as Lyapunov function [3,4]. Secondly, we have to evaluate its firstordertime derivative along the trajectory of the system. If the derivative of a Lyapunov function is decreasing along the system trajectory as time increase, then the system energy is dissipating and the system will finally settle down[5].

In this paper the tools of Lyapunov stability theory will be considered . Lyapunov's second method is presented to achieve asymptotically stable of nonlinear systems . some examples illustrate the procedure for studying the asymptotic stability of nonlinear system.The paper is organized as follows. In sec. 2 A brief review of Lyapunov stability theory is presented . In sec.3 Lyapunov's methods (direct and indirect) methods is studied . Lyapunov stability theory will desscuss to achive the main subject of the paper . Examples to illustrate The above concept is presented in Sec .4. Concluding remarks are given in Sec. 5 .

II. A Brief Review of LyapunovStability Theory

Consider the autonomous systems
$\dot{x} = f(x)$ (1)

$f: D \to R^n$, D =open connected subset R^n, f locally Lipschitz , the system (1) has an equilibrium point $\check{x} \in D$ i,e., $f(\check{x}) = 0$. For convenience , we state all definitions and theorems for case when the equilibrium point is at the origin $x = 0$.

Definition1: The equilibrium point $x = 0$ of (1) is
1- Stable , if for each $\varepsilon > 0$, there is $\delta = \delta(\varepsilon) > 0$ such that

$\|x(0)\| < \delta \to \|x(t)\| < \varepsilon$ for all $t \geq 0$. (2)
2- Asymptotically stable, if it is stable and δ can be chosen such that

$\|x(0)\| < \delta \to \lim_{t\to\infty} x(t) = 0$. (3)
3- Unstable, if not stable.

Definition 2 : Let $V: D \to R$ be a continuously differentiable function defined in a domain $D \subset R^n$ that contains the origin , the derivative of V along the trajectories of (1) , denoted by $\dot{V}(x)$, is given by

$$\dot{V}(x) = \frac{d}{dt}V(x) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}\frac{d}{dt}x_i$$

$$= \left[\frac{\partial V}{\partial x_1}\frac{\partial V}{\partial x_2} \dots \frac{\partial V}{\partial x_n}\right]\dot{x} = \frac{\partial V}{\partial x}f(x). \quad (4)$$

If $\dot{V}(x)$ is negative , V will decrease along the trajectory of (1) passing through x.

A function $V(x)$ is

1- Positive definite if
a- $V(0) = 0$ and
b- $V(x) > 0$ for $x \neq 0$.

2- Positive semidefinite if
a- $V(0) = 0$ and
b- $V(x) \geq 0$ for $x \neq 0$.

3- Negative definite if
a- $-V(x)$ is positive definite

4- Negative semidefinite if
a- $-V(x)$ is positive semidefinite

Lyapunov's stability theorem states that the origin is **stable** if , in a domain D that contains the origin, there is a continuously differentiable positive definite function $V(x)$ so that $\dot{V}(x)$ is negative semidefinite, and it is **asymptotically stable** if $\dot{V}(x)$ is negative definite, when the condition for stability is satisfied, the function $V(x)$ is called a Lyapunov function [3].

III. LyapunovDirect Method

Lyapunov's direct method is a mathematical extension of the fundamental physical observation that an energy dissipative system must eventually settle down to an equilibrium point. It states that if there is an energy-like function $V$for a system, that is strictly decreasing along every trajectory of the system, then the trajectories are asymptotically attracted to an equilibrium. The function V is then said to be a Lyapunov function for the system [9].

To prove the equilibrium is asymptotically stable we have to seek a scalar function of the states and this function is positive definite in region around the equilibrium point :$V(x) > 0$, except $V(x) = 0$. The existence of a Lyapunov function is sufficient to prove stability in the region . If $\dot{V}(x)$ is negative definite , the equilibrium is asymptotically stable[9].

Theorem 1[10]: Let$x = 0$ be an equilibrium point for (1) where$f: D \to R^n$is a locally Lipchitz and$D \subset R^n$a domain that contains the origin. Let $V: D \to R$be a continuously differentiable, positivedefinite function in D such that

$V(0) = 0$ and $V(x) > 0, \forall x \in D\backslash\{0\}$,

Then $x = 0$ is a **stable** equilibrium point, if

$\dot{V}(x) \leq 0, \forall x \in D.$ (5)

Moreover, if
$\dot{V}(x) < 0$, $\forall x \in D\backslash\{0\}$, (6)

Then $x = 0$ is an **asymptotically stable** equilibrium point.

In both cases above V is called a Lyapunov function. Moreover, if the conditions hold for all$x \in R^n$and
$\|x\| \to \infty$implies that $V(x) \to \infty$, (7)

then $x = 0$ is **globally stable** in (5) and **globallyasymptotically stable** in (6).

Proof[10]: Suppose $\epsilon > 0$, choose $r \in (0, \varepsilon]$ such that $B_r = \{x \in R^n, \|x\| \leq r\} \subset D$. Let $\alpha = \min_{\|x\|=r} V(x)$. Choose $\beta = (0, \alpha)$ and define $\Omega_\beta = \{x \in B_r, V(x) \leq \beta\}$. It holds that if $(0) \in \Omega_\beta \to x(t) \in \Omega_\beta \forall t$ because

$$\dot{V}(x(t)) \leq 0 \to V(x(t)) \leq V(x(0)) \leq \beta$$

Further $\exists\ \delta > 0$ such that $\|x\| < \delta \to V(x) < \beta$. Therefore, we have that

$$\beta_\delta \subset \Omega_\beta \subset \beta_r$$

And furthermore

$$x(0) \in \beta_\delta \to x(0) \in \Omega_\beta \to x(t) \in \Omega_\beta \to x(t) \in \beta_r$$

Finally, it follows that

$$\|x(0)\| < \delta \to \|x(t)\| < r \leq \varepsilon, \forall t > 0.$$

This means that the equilibrium point is **stable** at the point $x = 0$.
In order to show asymptotic stability, we need to show that $x(t) \to 0$ as $t \to \infty$. In this case, it turns out that it is sufficient to show that $V(x(t)) \to 0$ as $t \to \infty$. Since V is monotonically decreasing and bounded from below by 0, then

$V(x) \to c \geq 0$,as $t \to \infty$

Finally, it can be further shown by contradiction that the limit c is actually equal to 0.

To prove globally stable and globally asymptotically stable

Proof[10]: Given the point $x \in R^n$, let $c = V(x)$. Condition (7) implies that for any > 0 , there is $r > 0$ such that $V(x) > c$ whenever $\|x\| > r$. Thus $\Omega_\beta \subset \beta_r$, which implies that Ω_β bounded.

IV. Application And Illustrative Examples

Example 1 [10]:Consider the following system:

$$\begin{aligned}\dot{x}_1 &= x_1(x_1^2 + x_2^2 - \beta^2) + x_2 \\ \dot{x}_2 &= -x_1 + x_2(x_1^2 + x_2^2 - \beta^2)\end{aligned} \quad (8)$$

Now we will choose Lyapunov function $V(x) = 1/2(x_1^2 + x_2^2)$ we have

$$\dot{V}(x) = \nabla . f(x)$$
$$= [x_1, x_2][x_1(x_1^2 + x_2^2 - \beta^2) + x_2^2, -x_1 x_2(x_1^2 + x_2^2 - \beta^2)]^T$$
$$= x_1^2(x_1^2 + x_2^2 - \beta^2) + x_2^2(x_1^2 + x_2^2 - \beta^2)$$
$$= (x_1^2 + x_2^2)x_1^2 + x_2^2 - \beta^2)$$

Since, $V(x) \succ 0$ and $\dot{V}(x) \leq 0$, provided that $(x_1^2 + x_2^2) \prec \beta^2$, it follows that the origin is an *asymptotically stable* equilibrium point .

Example 2 [9]:Consider the simple pendulum (pendulum with friction), (Figure 3.1), k is a coefficient of friction, l denotes the length of the rod and m denotes the mass of the bob. Let θ denote the angle subtended by the rod and the vertical axis through the pivot point. The gravitational force equal to mg, where g is the acceleration due to gravity. Using Newton's second law of motion and take the state variables as $x_1 = \theta$ and $x_2 = \dot{\theta}$. Therefore the state equations are

$$\begin{aligned}\dot{x}_1 &= x_2 = f_1(x)\\ \dot{x}_2 &= -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2 = f_2(x)\end{aligned} \quad (9)$$

where $x=0$ is an equilibrium point , to study the stability of the equilibrium at the origin we propose a Lyapunov function candidate $V(x)$. In this case we use total energy $E(x)$ which is a positive function as the sum of its potential and kinetic energies and $E(0)=0$, we get

$E = K + P$ (Kinetic plus potential energy)

$$= \frac{1}{2}m(wl)^2 + mgh$$

Where

$$w = \dot{\theta} = x_2$$

$$h = l(1-\cos\theta) = l(1-\cos x_1)$$

Finally,

$$E = \frac{1}{2}ml^2x_2^2 + mgl(1-\cos x_1)$$

We know define $V(x) = \text{E}$ (positive definite) as

$$V(x) = \frac{1}{2}ml^2x_2^2 + mgl(1-\cos x_1),$$

and the energy is

$$E(x) = V(x) = \int_0^{x_1} a\sin y + \frac{1}{2}x_2^2 = a(1-\cos x_1) + \frac{1}{2}x_2^2$$

Thus the derivative of $V(x)$ is

$$\dot{V}(x) = [mgl\sin x_1, ml^2x_2][x_2, -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2]^T = -kl^2x_2^2$$

As we saw $\dot{V}(x)$ is *negative semi-definite* (not negative definite) because at $\dot{V}(x)=0$ we find $x_2 = 0$ regardless of the value of x_1 (thus $\dot{V}(x)=0$ along the x_1 axis), this means that the origin is *stable* by (theorem 1), not *asymptotic stability*.

Here Lyapunov function candidate fails to indentify an asymptotically stable equilibrium point by having $\dot{V}(x)$ *negative semi definite*.

In order to prove an asymptotical stability we need to define LaSalle's Invariance Principle.

LaSalle's Invariance Principle , developed in 1960 by J.P. LaSalle, the principle Basically verify that if there is a Lyapunov function within the neighborhood of the origin , has a negative semi-definite time derivative along the trajectories of the system which established that no trajectory can stay identically at point where $\dot{V}(x)=0$ except at the origin , then the origin is asymptotically stable .In order to understand that we present a definition and theorems related to LaSalle's Invariance Principle[7] .

Definition 3 [5]: A set M is said to be an invariant set with respect to the system (1) if $x(0)\in M \Rightarrow x(t)\in M \ \ \forall t \in R^+$.

Theorem 3 [10]:(**LaSalle's theorem**): Let $V: D \to R$ be a continuously differentiable function and assume that

i) $M \subset D$ is a compact set, invariant with respect to the solutions of (1).

ii) $\dot{V} \le 0$ in M

iii) $E: \{x : x \in M, and \dot{V} = 0\}$; that is, E is the set of all points of M such that $\dot{V} = 0$

iv) $N:$ is the largest invariant set in E .

Then every solution starting in M approaches N as $t \to \infty$.

Proof [10]:Consider a solution $x(t)$ in (1) starting in M. Since $\dot{V}(x) \le 0 \in M$, $V(x)$ is a decreasing function of t. Also, since $V(.)$ is a continuous function, it is bounded from below in the compact set M. It follows that $V(x(t))$ has a limit as $t \to \infty$. Let ω be the limit set of this trajectory. It follows that $\omega \subset M$ since M is (an invariant) closed set. For any $p \in \omega \ \exists$ a sequence t_n with $t_n \to \infty$ and $x(t_n) \to p$. By continuity of $V(x)$, we have that

$$V(p) = \lim_{n\to\infty} V(x(t_n)) = a \quad (a \text{ constant})$$

Hence, $V(x) = a$ on ω . Also, consider ω is an invariant set, and moreover $\dot{V}(x) = 0$ on ω (since $V(x)$ is constant on ω). It follows that

$$\omega \subset N \subset E \subset M$$

Since $x(t)$ is bounded this implies that $x(t)$ approaches ω (its positive limit set) as $t \to \infty$ .Hence $x(t)$ approaches N as $t \to \infty$.

Corollary [9]: Let $x = 0 \in D$ be an equilibrium point of the system (1). Let $V: D \to R$ be a continuously differentiable positive definite function on the domain D, such that $\dot{V} \le 0, \forall x \in D$. Let $S = \{x \in D \backslash \dot{V}(x) = 0\}$ and suppose that no solution can stay identically in S, other than the trivial solution $x(t) = 0$. Then ,the origin is asymptotically stable.

Theorem 4 [10]: The equilibrium point $x = 0$ of the autonomous system (1) is ***asymptotically stable*** if there exists a function $V(x)$ satisfying

i) $V(x)$ positive definite $\forall x \in D$, where we assume that $0 \in D$

ii) $\dot{V}(x)$ is negative semi definite in a bounded region $R \subset D$.

iii) $\dot{V}(x)$ does not vanish identically along any trajectory in R , other than the null solution $x = 0$.

Proof [10]:By (Theorem 1), we know that for each $\varepsilon \succ 0$ there exist $\delta \succ 0$

$$\|x_0\| \prec \delta \Rightarrow \|x(t)\| \prec \varepsilon$$

That is, any solution starting inside the closed ball B_δ will remain within the closed ball B_ε. Hence any solution $x(t, x_0, t_0)$ of (1) that starts in B_δ is bounded and tends to its limit set N that is contained in B_ε. Also $V(x)$ continuous on the compact set B_ε and thus is bounded from below in B_ε. It is also non increasing by assumption and thus tends to a non-negative limit L as $t \to \infty$. Notice also that $V(x)$ is continuous and thus, $V(x) = L \ \forall x$ in the limit set N . If N is an invariant set with respect to (1), which means that any solution that starts in N will remain there for all future time. But along that solution, $\dot{V}(x) = 0$ since $V(x)$ is constant $(= L)$ in N . Thus, by assumption, N is the origin of the state space and we conclude that any solution starting in $R \subset B_\delta$ converges to $x = 0$ as $t \to \infty$.

Example 4 [10]: In Example 2, when the origin of the nonlinear pendulum was stable by using Lyapunov direct method however , asymptotic stability could not be obtained .

Return to the system

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2 \qquad (10)$$

And the candidate Lyapunov function is:

$$V(x) = a(1 - \cos x_1) + \frac{1}{2}x_2^2 \ , \qquad (11)$$

$$\dot{V}(x) = -kl^2 x_2^2 \ . \qquad (12)$$

which is negative semi definite since $\dot{V}(x) = 0$ for $x = (x_1, 0)$, if we apply (theorem 4, conditions (i),(ii)) we satisfy in the origin

$R = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

With $-\pi \prec x_1 \prec \pi$, and $-a \prec x_2 \prec a$, for any $a \in R^+$. If we check condition (iii) which $\dot{V}$ can vanish identically along the trajectories trapped in R , other than the null solution. Using (12) we get

$\dot{V} = 0 \Rightarrow 0 = -kl^2 x_2^2 \; x_2 = 0 \;\; \forall t \Rightarrow \dot{x}_2 = 0$.

And using (11) we obtain:

$0 = \frac{g}{l}\sin x_1 - \frac{k}{m}x_2$ since $x_2 = 0 \Rightarrow \sin x_1 = 0$.

Restricting x_1 to $x_1 \in (-\pi, \pi)$ condition (iii) is satisfied if and only if $x_1 = 0$. So $\dot{V}(x) = 0$ does not vanish identically along any trajectory other than $x = 0$, therefore $x = 0$ is ***asymptotically stable*** by (Theorem 4).

V. Conclusion

When we use Lyapunov direct method, we can get from the system if it is stable or asymptotic or unstable, but in some cases, like our example, this method fails to achieve the stability and this does not mean that the system is not stable, just only means that such stability property cannot be established by using this method. In this case we applied Lasalle's invariance principle to obtain the asymptotic stability . when we saw the origin is stable, not asymptotic stable.

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