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A study on Ricci soliton in S-manifolds.

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Abstract: *In this paper, we study semi symmetric and pseudo symmetric conditions in S-manifolds, those are $R\cdot R=0$, $R\cdot C=0$, $C\cdot R=0$, $C\cdot C=0$, $R\cdot R=L_1Q(g,R)$, $R\cdot C=L_2Q(g,C)$, $C\cdot R=L_3Q(g,R)$, and $C\cdot C=L_4Q(g,C)$, where C is the Concircular curvature tensor and L_1,L_2,L_3,L_4 are the smooth functions on M, further we discuss about Ricci soliton.*

Keywords: S*-manifold,* η*-Einstein manifold, Einstein manifold, Ricci soliton.*

I. Introduction

The notion of f-structure on a $(2n+s)$-dimensional manifold M, i.e., a tensor field of type $(1,1)$ on M of rank $2n$ satisfying $f^3+f=0$, was firstly introduced in 1963 by K. Yano [28] as a generalization of both (almost) contact (for s = 1) and (almost) complex structures (for s = 0). During the subsequent years, this notion has been furtherly developed by several authors [3], [4], [11], [12], [15], [16], [17]. Among them, H. Nakagawa in [16] and [17] introduced the notion of framed f-manifold, later developed and studied by S.I. Goldberg and K. Yano ([11], [12]) and others with the denomination of globally framed f-manifolds.

A Riemannian manifold M is called locally symmetric if its curvature tensor R is parallel, i.e., $\nabla R=0$, where ∇ denotes the Levi-Civita connection. As a generalization of locally symmetric manifolds the notion of semisymmetric manifolds was defined by

$(R(X,Y)\cdot R)(U,V)W=0,\ X,Y,U,V,W\in TM$

and studied by many authors [18], [19], [26], [20]. Z.I. Szabo [25] gave a full intrinsic classification of these spaces. R. Deszcz [8, 9] weakened the notion of semisymmetry and introduced the notion of pseudosymmetric manifolds by

$$(R(X,Y)\cdot R)(U,V)W=L_R[((X\wedge Y)\cdot R)(U,V)W], \quad (1.1)$$

where L_R is smooth function on M and $X\wedge Y$ is an endomorphism defined by

$$(X\wedge Y)Z=g(Y,Z)X-g(X,Z)Y. \quad (1.2)$$

Definition 1 *A Ricci soliton is a natural generalization of an Einstein metric and is defined on a Riemannian manifold (M,g). A Ricci soliton is a triple (g,V,λ) with g is a Riemannian metric, V is a vector field and λ is a real scalar such that*

$$(L_V)g(X,Y)+2S(X,Y)+2\lambda g(X,Y)=0. \quad (1.3)$$

where S is a Ricci tensor of M and L_V denotes the Lie derivative operator along the vector field V. The Ricci soliton is said to be shrinking, steady and expanding according as λ is negative, zero and positive respectively. The authors R.Sharma [22, 23, 24] and M.M.Tripathi [27] initiated the study of Ricci solitons in contact manifold. But Călin and Crasmareanu [6], Bagewadi and Ingalahalli [14, 1], S.Debnath and A.Battacharya [7] have studied the existence and also obtained results on Ricci solitons in f-kenmotsu manifolds, α-Sasakian manifolds, Lorentzian α-Sasakian manifolds, Trans-Sasakian manifolds using L.P.Eisenhart problem [10]. But C.S.Bagewadi, Ingalahalli and Ashok, C.S.Bagewadi and K.R.Vidyavathi have studied Ricci solitons in Kenmotsu manifolds, almost $C(\alpha)$ manifolds using semi-symmetric and pseudosymmetric conditions [2]. In the present paper, we study Ricci soliton in S-manifolds satisfying semi symmetric and pseudo symmetric conditions those are $R\cdot R=0$, $R\cdot C=0$, $C\cdot R=0$, $C\cdot C=0$, $R\cdot R=L_1Q(g,R)$, $R\cdot C=L_2Q(g,C)$, $C\cdot R=L_3Q(g,R)$, and $C\cdot C=L_4Q(g,C)$, where C is the

Concircular curvature tensor and L_1, L_2, L_3, L_4 are the smooth functions on M .

II. Preliminaries

Let M be a $(2n+s)$-dimensional manifold with an f-structure of rank $2n$. If there exists global vector fields $\xi_\alpha, \alpha = (1,2,3,......,s)$ on M such that;

$$f^2 = -I + \sum \xi_\alpha \otimes \eta_\alpha, \ \ \eta_\alpha(\xi_\beta) = \delta_\beta^\alpha, \quad (2.1)$$

$$f\xi_\alpha = 0, \ \ \eta_\alpha \circ f = 0, \quad (2.2)$$

$$g(X,\xi_\alpha) = \eta_\alpha(X), \ \ g(X, fY) = -g(fX,Y), \quad (2.3)$$

where η_α are the dual 1-forms of ξ_α, we say that the f-structure has complemented frames. For such a manifold there exists a Riemannian metric g such that

$$g(X,Y) = g(fX, fY) + \sum \eta_\alpha(X)\eta_\alpha(Y) \qquad (2.4)$$

for any vector fields X and Y on M .
An f-structure f is normal, if it has complemented frames and

$$[f, f] + 2\sum \xi_\alpha \otimes d\eta_\alpha = 0,$$

where $[f, f]$ is Nijenhuis torsion of f .
Let F be the fundamental 2-form defined by $F(X,Y) = g(X, fY), X,Y \in T(M)$. A normal f-structure for which the fundamental form F is closed, $\eta_1 \wedge,......,\eta_s \wedge (d\eta_\alpha)^n \neq 0$ for any α , and $d\eta_1 = ... = d\eta_s = F$ is called to be an S-structure. A smooth manifold endowed with an S-stucture will be called an S-manifold. These manifolds introduced by Blair [3].
We have to remark that if we take $s = 1$, S-manifolds are natural generalizations of Sasakian manifolds. In the case $s \geq 2$ some interesting examples are given [3], [13].
If M is an S-manifold, then the following relations holds true [3];

$$\nabla_X \xi_\alpha = -fX, \ \ X \in T(M), \alpha = 1,2,....,s \qquad (2.5)$$

$$(\nabla_X f)Y = \sum \{g(fX, fY)\xi_\alpha + \eta_\alpha(Y) f^2 X\}, \ \ X,Y \in T(M), \qquad (2.6)$$

where ∇ is the Riemannian connection of g. Let Ω be the distribution determined by the projection tensor-f^2 and let N be the complementry distribution which is determined by $f^2 + I$ and spanned by $\xi_1,....\xi_s$. It is clear that if $X \in \Omega$ then $\eta_\alpha(X) = 0$ for any α, and if $X \in N$, then fX =0. A plane section π on M is called an invariant f-section if it is determined by a vector $X \in \Omega(x), x \in M$, such that $\{X, fX\}$ is an orthonormal pair spanning the section. The sectional curvature of π is called the f-sectional curvature. If M is an S-manifold of constant f-sectional curvature k, then its curvature tensor has the form

$$R(X,Y,Z,W) = \sum_{\alpha,\beta} \{g(fX, fW)\eta_\alpha(Y)\eta_\beta(Z) - g(fX, fZ)\eta_\alpha(Y)\eta_\beta(W) + g(fY, fZ)\eta_\alpha(X)\eta_\beta(W)$$
$$- g(fY, fW)\eta_\alpha(X)\eta_\beta(Z)\} + \frac{1}{4}(k+3s)\{g(fX, fW)g(fY, fZ) - g(fX, fZ)g(fY, fW)\}$$
$$+ \frac{1}{4}(k-s)\{F(X,W)F(Y,Z) - F(X,Z)F(Y,W) - 2F(X,Y)F(Z,W)\}, \qquad (2.7)$$

where $X,Y,Z,W \in T(M)$. Such a manifold $N(K)$ will be called an S-space form. The Euclidean space E^{2n+s} and the hiperbolic space H^{2n+s} are examples of S-space forms.

Definition 2 *S -manifold $(M, f, \eta_\alpha, g, \xi_\alpha)$ is said to be η -Einstein if the Ricci tensor S of M is of the form*

$$S = ag + b\sum_{\alpha=1}^{s}\eta_\alpha \otimes \eta_\alpha,$$

where a, b are constants on M.
Now contracting equation (2.7) we get

$$S(Y,Z) = \left[\frac{4s+(k+3s)(2n-1)+3(k-s)}{4}\right]g(Y,Z) + \left[\frac{(2n+s-2)(4-k-3s)-3(k-s)}{4}\right]\sum_\alpha \eta_\alpha(Y)\eta_\alpha(Z), \quad (2.8)$$

$$S(Y,\xi_\alpha) = \frac{1}{4}[s^2(13-6n-k-3s)+2s(7n-5)+k(2-s)+2nk(1-s)]. \quad (2.9)$$

From (2.7) we have

$$R(X,Y)\xi_\alpha = s\sum_\alpha\{\eta_\alpha(Y)X - \eta_\alpha(X)Y\}, \quad (2.10)$$

$$R(\xi_\alpha,Y)Z = s\sum_\alpha\{g(Y,Z)\xi_\alpha - \eta_\alpha(Z)Y\}, \quad (2.11)$$

$$\eta_\alpha(R(X,Y)Z) = s\sum_\alpha\{g(Y,Z)\eta_\alpha(X) - g(X,Z)\eta_\alpha(Y)\}. \quad (2.12)$$

III. Ricci Soliton In Semi-Symmetric S -Manifolds

An S -manifold is said to be semi-symmetric if $R \cdot R = 0$.

$$(R(\xi_\alpha,Y)\cdot R)(U,V)W = 0, \quad (3.1)$$

$$R(\xi_\alpha,Y)R(U,V)W - R(R(\xi_\alpha,Y)U,V)W - R(U,R(\xi_\alpha,Y)V)W - R(U,V)R(\xi_\alpha,Y)W = 0. \quad (3.2)$$

Using (2.11) in (3.2), we get

$$s\sum_\alpha\{g(Y,R(U,V)W)\xi_\alpha - \eta_\alpha(R(U,V)W)Y - g(Y,U)R(\xi_\alpha,V)W + \eta_\alpha(U)R(Y,V)W$$
$$- g(Y,V)R(U,\xi_\alpha)W + \eta_\alpha(V)R(U,Y)W - g(Y,W)R(U,V)\xi_\alpha + \eta_\alpha(W)R(U,V)Y\} = 0 \quad (3.3)$$

By taking an inner product with ξ_α then we get

$$\sum_\alpha\{sR(U,V,W,Y) - \eta_\alpha(R(U,V)W)\eta_\alpha(Y) - g(Y,U)\eta_\alpha(R(\xi_\alpha,V)W) + \eta_\alpha(U)\eta_\alpha(R(Y,V)W)$$
$$- g(Y,V)\eta_\alpha(R(U,\xi_\alpha)W) + \eta_\alpha(V)\eta_\alpha(R(U,Y)W) - g(Y,W)\eta_\alpha(R(U,V)\xi_\alpha) + \eta_\alpha(W)\eta_\alpha(R(U,V)Y)\} = 0. \quad (3.4)$$

By using (2.10), (2.12) in (3.4) we have

$$sR(U,V,W,Y) + s^2 g(Y,V)g(U,W) - s^2 g(Y,U)g(V,W) = 0. \quad (3.5)$$

Taking $U = Y = e_i$ in (3.5) and summing over $i = 1,2,.......2n+s$ we get

$$S(V,W) = s(2n+s-1)g(V,W) \quad (3.6)$$

Thus we state the following;

Theorem 1 *Semi symmetric S -manifold is an Einstein manifold.*

If V is co-linear with ξ, then Ricci soliton along ξ is given by

$$(L_\xi g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y)$$

Definition 3 *Let $(f, \xi_1, \xi_2, ..., \xi_s, \eta_1, \eta_2, ..., \eta_s, g)$ is the contact S -frame manifold, if V is in the linear span (combination) of $\xi_1, \xi_2, ...\xi_s$ then $V = c_1\xi_1 + c_2\xi_2 + ... + c_s\xi_s$ and the Ricci soliton is a* *triple* (g, ξ_α, λ) *with* g *is a Riemannian metric,* $\xi_\alpha, (\alpha = 1,2,....s)$ *is a vector field and* λ *is a real scalar such that*

$$(\sum_{i=1}^{s} c_i L_{\xi_i} g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) = 0 \qquad (3.7)$$

From (3.7) we have

$$c_i g(\nabla_X \xi_\alpha, Y) + c_i g(\nabla_Y \xi_\alpha, X) + 2S(X,Y) + 2\lambda g(X,Y) = 0. \qquad (3.8)$$

Using (2.5) in (3.8) we get

$$c_i g(-fX,Y) + c_i g(-fY,X) + 2S(X,Y) + 2\lambda g(X,Y) = 0 \qquad (3.9)$$

From (3.6) and (3.9) we have

$$(s(2n+s-1)+\lambda) g(X,Y) = 0 \; (3.10)$$

Taking $X = Y = e_i$ in (3.10) and summing over $i = 1,2,.......2n+s$, we get the value of λ

$$\lambda = -s(2n+s-1)(<0)$$

Thus we state the following;

Theorem 2 *Ricci soliton in semi-symmetric* S *-manifold is shrinking.*

Corollary 1 *Ricci soliton in semi symmetric* S *-manifold is steady if* $s = 0$ *(Kaehler manifold) and is shrinking if* $s = 1$ *(Sasakian manifold).*

IV. Ricci soliton in S -manifolds satisfying $R \cdot C = 0$.

The Concircular curvature tensor C is given by

$$C(X,Y)Z = R(X,Y)Z - \frac{r}{2n(2n+1)}\{g(Y,Z)X - g(X,Z)Y\} \qquad (4.1)$$

Using (2.10), (2.11) and (2.12) in (4.1) we get

$$C(X,Y)\xi_\alpha = \left[s - \frac{r}{2n(2n+1)} \right] \sum_\alpha \{X\eta_\alpha(Y) - \eta_\alpha(X)Y\}, \qquad (4.2)$$

$$C(\xi_\alpha, Y)Z = \left[s - \frac{r}{2n(2n+1)} \right] \sum_\alpha \{g(Y,Z)\xi_\alpha - Y\eta_\alpha(Z)\}, \qquad (4.3)$$

$$\eta_\alpha(C(X,Y)Z) = \left[s - \frac{r}{2n(2n+1)} \right] \sum_\alpha \{g(Y,Z)\eta_\alpha(X) - g(X,Z)\eta_\alpha(Y)\}. \qquad (4.4)$$

Let us assume that the condition $R((\xi_\alpha, Y) \cdot C)(U,V)W = 0$ hold on M, then

$$R(\xi_\alpha, Y)C(U,V)W - C(R(\xi_\alpha, Y)U,V)W - C(U, R(\xi_\alpha, Y)V)W - C(U,V)R(\xi_\alpha, Y)W = 0. \qquad (4.5)$$

Using (2.11) in (4.5), we get

$$s\sum_\alpha \{g(Y, C(U,V)W)\xi_\alpha - \eta_\alpha(C(U,V)W)Y - g(Y,U)C(\xi_\alpha, V)W + \eta_\alpha(U)C(Y,V)W$$
$$-g(Y,V)C(U,\xi_\alpha)W + \eta_\alpha(V)C(U,Y)W - g(Y,W)C(U,V)\xi_\alpha + \eta_\alpha(W)C(U,V)Y\} = 0 \qquad (4.6)$$

By taking an inner product with ξ_α then we get

$$\sum_\alpha \{sC(U,V,W,Y) - \eta_\alpha(C(U,V)W)\eta_\alpha(Y) - g(Y,U)\eta_\alpha(C(\xi_\alpha, V)W) + \eta_\alpha(U)\eta_\alpha(C(Y,V)W)$$
$$-g(Y,V)\eta_\alpha(C(U,\xi_\alpha)W) + \eta_\alpha(V)\eta_\alpha(C(U,Y)W) - g(Y,W)\eta_\alpha(C(U,V)\xi_\alpha) + \eta_\alpha(W)\eta_\alpha(C(U,V)Y)\} = 0. \qquad (4.7)$$

By using (4.2), (4.4) in (4.7) we have

$$C(U,V,W,Y)=\left[s-\frac{r}{2n(2n+1)}\right]\{g(Y,U)g(V,W)-g(Y,V)g(U,W)\}. \qquad (4.8)$$

Taking $U=Y=e_i$ in (4.8) and summing over $i=1,2,.......2n+s$ and using (4.1) we get

$$S(V,W)=s(2n+s-1)g(V,W) \qquad (4.9)$$

Thus we state the following;

Theorem 3 S *-manifold satisfying the condition* $R\cdot C=0$ *is an Einstein manifold.*

From (4.9) and (3.9) we have

$$(s(2n+s-1)+\lambda)g(X,Y)=0 \quad (4.10)$$

Taking $X=Y=e_i$ in (4.10) and summing over $i=1,2,.......2n+s$, we get the value of λ

$$\lambda=-s(2n+s-1)(<0)$$

Thus we state the following;

Theorem 4 *Ricci soliton in* S *-manifold satisfying the condition* $R\cdot C=0$ *is shrinking.*

Corollary 2 *Ricci soliton in* S *-manifold satisfying* $R\cdot C=0$ *is steady if* $s=0$ *(Kaehler manifold) and is shrinking if* $s=1$ *(Sasakian manifold).*

V. Ricci soliton in S -manifolds satisfying $C\cdot R=0$.

Let us assume that the condition $C((\xi_\alpha,Y)\cdot R)(U,V)W=0$ hold on M, then

$$C(\xi_\alpha,Y)R(U,V)W-R(C(\xi_\alpha,Y)U,V)W-R(U,C(\xi_\alpha,Y)V)W-R(U,V)C(\xi_\alpha,Y)W=0. \qquad (5.1)$$

Using (4.3) in (5.1), we get

$$\left[s-\frac{r}{2n(2n+1)}\right]\sum_\alpha\{g(Y,R(U,V)W)\xi_\alpha-\eta_\alpha(R(U,V)W)Y-g(Y,U)R(\xi_\alpha,V)W+\eta_\alpha(U)R(Y,V)W$$
$$-g(Y,V)R(U,\xi_\alpha)W+\eta_\alpha(V)R(U,Y)W-g(Y,W)R(U,V)\xi_\alpha+\eta_\alpha(W)R(U,V)Y\}=0 \qquad (5.2)$$

By taking an inner product with ξ_α then we get

$$\sum_\alpha\{sR(U,V,W,Y)-\eta_\alpha(R(U,V)W)\eta_\alpha(Y)-g(Y,U)\eta_\alpha(R(\xi_\alpha,V)W)+\eta_\alpha(U)\eta_\alpha(R(Y,V)W)$$

$$-g(Y,V)\eta_\alpha(R(U,\xi_\alpha)W)+\eta_\alpha(V)\eta_\alpha(R(U,Y)W)-g(Y,W)\eta_\alpha(R(U,V)\xi_\alpha)+\eta_\alpha(W)\eta_\alpha(R(U,V)Y)\}=0. \qquad (5.3)$$

By using (4.2), (4.4) in (5.3) we have

$$R(U,V,W,Y)=s\{g(Y,U)g(V,W)-g(Y,V)g(U,W)\}. \qquad (5.4)$$

Taking $U=Y=e_i$ in (5.4) and summing over $i=1,2,.......2n+s$ we get

$$S(V,W)=s(2n+s-1)g(V,W) \qquad (5.5)$$

Thus we state the following;

Theorem 5 S *-manifold satisfying the condition* $C\cdot R=0$ *is an Einstein manifold.*

From (5.5) and (3.9) we have

$$(s(2n+s-1)+\lambda)g(X,Y)=0 \quad (5.6)$$

Taking $X=Y=e_i$ in (5.6) and summing over $i=1,2,.......2n+s$, we get the value of λ

$$\lambda=-s(2n+s-1)(<0)$$

Thus we state the following;

Theorem 6 *Ricci soliton in* S *-manifold satisfying the condition* $C\cdot R=0$ *is shrinking.*

Corollary 3 *Ricci soliton in S -manifold satisfying $C \cdot R = 0$ is steady if $s = 0$ (Kaehler manifold) and is shrinking if $s = 1$ (Sasakian manifold).*

Ricci soliton in S -manifolds satisfying $C \cdot C = 0$.

Let us assume that the condition $C((\xi_\alpha, Y) \cdot C)(U,V)W = 0$ hold on M, then

$$C(\xi_\alpha, Y)C(U,V)W - C(C(\xi_\alpha, Y)U,V)W - C(U, C(\xi_\alpha, Y)V)W - C(U,V)C(\xi_\alpha, Y)W = 0. \quad (6.1)$$

Using (4.3) in (6.1), we get

$$\left[s - \frac{r}{2n(2n+1)} \right] \sum_\alpha \{ g(Y, C(U,V)W)\xi_\alpha - \eta_\alpha(C(U,V)W)Y - g(Y,U)C(\xi_\alpha, V)W + \eta_\alpha(U)C(Y,V)W$$
$$- g(Y,V)C(U,\xi_\alpha)W + \eta_\alpha(V)C(U,Y)W - g(Y,W)C(U,V)\xi_\alpha + \eta_\alpha(W)C(U,V)Y \} = 0 \quad (6.2)$$

By taking an inner product with ξ_α then we get

$$\sum_\alpha \{ sC(U,V,W,Y) - \eta_\alpha(C(U,V)W)\eta_\alpha(Y) - g(Y,U)\eta_\alpha(C(\xi_\alpha, V)W) + \eta_\alpha(U)\eta_\alpha(C(Y,V)W)$$

$$- g(Y,V)\eta_\alpha(C(U,\xi_\alpha)W) + \eta_\alpha(V)\eta_\alpha(C(U,Y)W) - g(Y,W)\eta_\alpha(C(U,V)\xi_\alpha) + \eta_\alpha(W)\eta_\alpha(C(U,V)Y) \} = 0. \quad (6.3)$$

By using (4.2), (4.4) in (6.3) we have

$$C(U,V,W,Y) = \left[s - \frac{r}{2n(2n+1)} \right] \{ g(Y,U)g(V,W) - g(Y,V)g(U,W) \}. \quad (6.4)$$

Taking $U = Y = e_i$ in (4.8) and summing over $i = 1,2,.......2n+s$ and using (4.1) we get

$$S(V,W) = s(2n+s-1)g(V,W) \quad (6.5)$$

Thus we state the following;

Theorem 7 *S -manifold satisfying the condition $C \cdot C = 0$ is an Einstein manifold.*

From (6.5) and (3.9) we have

$$(s(2n+s-1) + \lambda)g(X,Y) = 0 \quad (6.6)$$

Taking $X = Y = e_i$ in (6.6) and summing over $i = 1,2,.......2n+s$, we get the value of λ

$$\lambda = -s(2n+s-1)(< 0)$$

Thus we state the following;

Theorem 8 *Ricci soliton in S -manifold satisfying the condition $C \cdot C = 0$ is shrinking.*

Corollary 4 *Ricci soliton in S -manifold satisfying $C \cdot C = 0$ is steady if $s = 0$ (Kaehler manifold) and is shrinking if $s = 1$ (Sasakian manifold).*

VI. Ricci soliton in Pseudo-symmetric S -manifolds

An S -manifold is said to be Pseudo-symmetric if $R \cdot R = L_1 Q(g,R)$.

$$(R(\xi_\alpha, Y) \cdot R)(U,V)W = L_1[((\xi_\alpha \wedge Y) \cdot R)(U,V)W], \quad (7.1)$$

$$R(\xi_\alpha, Y)R(U,V)W - R(R(\xi_\alpha, Y)U,V)W - R(U, R(\xi_\alpha, Y)V)W - R(U,V)R(\xi_\alpha, Y)W$$
$$= L_1[(\xi_\alpha \wedge Y)R(U,V)W - R((\xi_\alpha \wedge Y)U,V)W - R(U,(\xi_\alpha \wedge Y)V)W - R(U,V)(\xi_\alpha \wedge Y)W] \quad (7.2)$$

Using (2.11) L.H.S of (??) is

$$s\sum_{\alpha}\{g(Y,R(U,V)W)\xi_\alpha-\eta_\alpha(R(U,V)W)Y-g(Y,U)R(\xi_\alpha,V)W+\eta_\alpha(U)R(Y,V)W$$
$$-g(Y,V)R(U,\xi_\alpha)W+\eta_\alpha(V)R(U,Y)W-g(Y,W)R(U,V)\xi_\alpha+\eta_\alpha(W)R(U,V)Y\}. \quad (7.3)$$

By taking an inner product with ξ_α then we get

$$s\sum_{\alpha}\{sR(U,V,W,Y)-\eta_\alpha(R(U,V)W)\eta_\alpha(Y)-g(Y,U)\eta_\alpha(R(\xi_\alpha,V)W)+\eta_\alpha(U)\eta_\alpha(R(Y,V)W)$$
$$-g(Y,V)\eta_\alpha(R(U,\xi_\alpha)W)+\eta_\alpha(V)\eta_\alpha(R(U,Y)W)-g(Y,W)\eta_\alpha(R(U,V)\xi_\alpha)+\eta_\alpha(W)\eta_\alpha(R(U,V)Y)\}.$$
(7.4)

By using (2.10), (2.12) in (??) we have

$$s\{sR(U,V,W,Y)+s^2g(Y,V)g(U,W)-s^2g(Y,U)g(V,W)\}. \quad (7.5)$$

Again using (2.11) R.H.S of (??), we get

$$L_1[\sum_{\alpha}\{g(Y,R(U,V)W)\xi_\alpha-\eta_\alpha(R(U,V)W)Y-g(Y,U)R(\xi_\alpha,V)W+\eta_\alpha(U)R(Y,V)W$$
$$-g(Y,V)R(U,\xi_\alpha)W+\eta_\alpha(V)R(U,Y)W-g(Y,W)R(U,V)\xi_\alpha+\eta_\alpha(W)R(U,V)Y\}]. \quad (7.6)$$

By taking an inner product with ξ_α then we get

$$L_1[\sum_{\alpha}\{sR(U,V,W,Y)-\eta_\alpha(R(U,V)W)\eta_\alpha(Y)-g(Y,U)\eta_\alpha(R(\xi_\alpha,V)W)+\eta_\alpha(U)\eta_\alpha(R(Y,V)W)$$
$$-g(Y,V)\eta_\alpha(R(U,\xi_\alpha)W)+\eta_\alpha(V)\eta_\alpha(R(U,Y)W)-g(Y,W)\eta_\alpha(R(U,V)\xi_\alpha)+\eta_\alpha(W)\eta_\alpha(R(U,V)Y)\}].$$
(7.7)

By using (2.10), (2.12) in (7.7) we have

$$L_1[sR(U,V,W,Y)+s^2g(Y,V)g(U,W)-s^2g(Y,U)g(V,W)]. \quad (7.8)$$

From (7.5) and (7.8) we get

$$[L_1-s][sR(U,V,W,Y)+s^2g(Y,V)g(U,W)-s^2g(Y,U)g(V,W)]=0. \quad (7.9)$$

Therefore either $L_1=s$ or

$$R(U,V,W,Y)=s\{g(Y,U)g(V,W)-g(Y,V)g(U,W)\}. \quad (7.10)$$

Taking $U=Y=e_i$ in (7.10) and summing over $i=1,2,.......2n+s$ we get

$$S(V,W)=s(2n+s-1)g(V,W) \quad (7.11)$$

Thus we state the following;

Theorem 9 *Pseudo symmetric S -manifold is an Einstein manifold provided $L_1\neq s$*

From (7.11) and (3.9) we have

$$(s(2n+s-1)+\lambda)g(X,Y)=0 \quad (7.12)$$

Taking $X=Y=e_i$ in (7.12) and summing over $i=1,2,.......2n+s$, we get the value of λ

$$\lambda=-s(2n+s-1)(<0)$$

Thus we state the following;

Theorem 10 *Ricci soliton in pseudo symmetric S -manifold is shrinking.*

Corollary 5 *Ricci soliton in pseudo symmetric S -manifold is steady if $s=0$ (Kaehler manifold) and is shrinking if $s=1$ (Sasakian manifold).*

VII. Ricci soliton in S -manifolds satisfying $R\cdot C=L_2Q(g,C)$.

Let us assume that the condition $R((\xi_\alpha,Y)\cdot C)(U,V)W=L_2[(\xi_\alpha\wedge Y)\cdot C](U,V)W$ hold on M, then

$$R(\xi_\alpha,Y)C(U,V)W-C(R(\xi_\alpha,Y)U,V)W-C(U,R(\xi_\alpha,Y)V)W-C(U,V)R(\xi_\alpha,Y)W$$
$$=L_2[(\xi_\alpha\wedge Y)C(U,V)W-C((\xi_\alpha\wedge Y)U,V)W-C(U,(\xi_\alpha\wedge Y)V)W-C(U,V)(\xi_\alpha\wedge Y)W]$$

(8.1)

Using (2.11) L.H.S of (??) is

$$s\sum_{\alpha}\{g(Y,C(U,V)W)\xi_\alpha-\eta_\alpha(C(U,V)W)Y-g(Y,U)C(\xi_\alpha,V)W+\eta_\alpha(U)C(Y,V)W$$
$$-g(Y,V)C(U,\xi_\alpha)W+\eta_\alpha(V)C(U,Y)W-g(Y,W)C(U,V)\xi_\alpha+\eta_\alpha(W)C(U,V)Y\}. \quad (8.2)$$

By taking an inner product with ξ_α then we get

$$s\sum_{\alpha}\{sC(U,V,W,Y)-\eta_\alpha(C(U,V)W)\eta_\alpha(Y)-g(Y,U)\eta_\alpha(C(\xi_\alpha,V)W)+\eta_\alpha(U)\eta_\alpha(C(Y,V)W)$$

$$-g(Y,V)\eta_\alpha(C(U,\xi_\alpha)W)+\eta_\alpha(V)\eta_\alpha(C(U,Y)W)-g(Y,W)\eta_\alpha(C(U,V)\xi_\alpha)+\eta_\alpha(W)\eta_\alpha(C(U,V)Y)\}.$$

(8.3)

By using (4.2), (4.4) in (??) we have

$$s^2\left\{C(U,V,W,Y)-\left[s-\frac{r}{2n(2n+1)}\right][g(Y,U)g(V,W)-g(Y,V)g(U,W)]\right\}. \quad (8.4)$$

Again using (2.11) R.H.S of (8.1) is

$$L_2\sum_{\alpha}\{g(Y,C(U,V)W)\xi_\alpha-\eta_\alpha(C(U,V)W)Y-g(Y,U)C(\xi_\alpha,V)W+\eta_\alpha(U)C(Y,V)W$$
$$-g(Y,V)C(U,\xi_\alpha)W+\eta_\alpha(V)C(U,Y)W-g(Y,W)C(U,V)\xi_\alpha+\eta_\alpha(W)C(U,V)Y\}. (8.5)$$

By taking an inner product with ξ_α then we get

$$L_2\sum_{\alpha}\{sC(U,V,W,Y)-\eta_\alpha(C(U,V)W)\eta_\alpha(Y)-g(Y,U)\eta_\alpha(C(\xi_\alpha,V)W)+\eta_\alpha(U)\eta_\alpha(C(Y,V)W)$$

$$-g(Y,V)\eta_\alpha(C(U,\xi_\alpha)W)+\eta_\alpha(V)\eta_\alpha(C(U,Y)W)-g(Y,W)\eta_\alpha(C(U,V)\xi_\alpha)+\eta_\alpha(W)\eta_\alpha(C(U,V)Y)\}.$$

(8.6)

By using (4.2), (4.4) in (8.6) we have

$$sL_2\left\{C(U,V,W,Y)-\left[s-\frac{r}{2n(2n+1)}\right][g(Y,U)g(V,W)-g(Y,V)g(U,W)]\right\} \quad (8.7)$$

From (8.4) and (8.7) we get

$$[sL_2-s^2]\left\{C(U,V,W,Y)-\left[s-\frac{r}{2n(2n+1)}\right][g(Y,U)g(V,W)-g(Y,V)g(U,W)]\right\}=0$$

(8.8)

Therefore either $L_2=s$ or

$$C(U,V,W,Y)=\left[s-\frac{r}{2n(2n+1)}\right][g(Y,U)g(V,W)-g(Y,V)g(U,W)] \quad (8.9)$$

Taking $U=Y=e_i$ in (8.9) and summing over $i=1,2,.......2n+s$ we get

$$S(V,W)=s(2n+s-1)g(V,W) \quad (8.10)$$

Thus we state the following;

Theorem 11 S *-manifold satisfying the condition* $R\cdot C=L_2Q(g,C)$ *is an Einstein manifold provided* $L_2\neq s$.

From (8.10) and (3.9) we have

$$(s(2n+s-1)+\lambda)g(X,Y)=0 \quad (8.11)$$

Taking $X=Y=e_i$ in (8.11) and summing over $i=1,2,.......2n+s$, we get the value of λ

$$\lambda=-s(2n+s-1)(<0)$$

Thus we state the following;

Theorem 12 *Ricci soliton in S -manifold satisfying the condition $R \cdot C = L_2 Q(g,C)$ is shrinking.*

Corollary 6 *Ricci soliton in S -manifold satisfying $R \cdot C = L_2 Q(g,C)$ is steady if $s = 0$ (Kaehler manifold) and is shrinking if $s = 1$ (Sasakian manifold).*

VIII. Ricci soliton in S -manifolds satisfying $C \cdot R = L_3 Q(g,R)$.

Let us assume that the condition $C((\xi_\alpha, Y) \cdot R)(U,V)W = L_3[(\xi_\alpha \wedge Y) \cdot R](U,V)W$ hold on M, then

$$C(\xi_\alpha, Y)R(U,V)W - R(C(\xi_\alpha, Y)U,V)W - R(U,C(\xi_\alpha, Y)V)W - R(U,V)C(\xi_\alpha, Y)W$$
$$= L_3[(\xi_\alpha \wedge Y)R(U,V)W - R((\xi_\alpha \wedge Y)U,V)W - R(U,(\xi_\alpha \wedge Y)V)W - R(U,V)(\xi_\alpha \wedge Y)W]$$

(9.1)

Using (5.2), (5.3), (7.6) and (7.7) in (9.1) we get

$$\left\{ sL_3 - \left[s - \frac{r}{2n(2n+1)} \right] \right\} \{ R(U,V,W,Y) - s[g(Y,U)g(V,W) - g(Y,V)g(U,W)] \} = 0$$

(9.2)

Therefore, either $L_3 = s - \frac{r}{2n(2n+1)}$ or

$$R(U,V,W,Y) = s\{g(Y,U)g(V,W) - g(Y,V)g(U,W)\}. \qquad (9.3)$$

Taking $U = Y = e_i$ in (9.3) and summing over $i = 1,2,.......2n+s$ we get

$$S(V,W) = s(2n+s-1)g(V,W) \qquad (9.4)$$

Thus we state the following;

Theorem 13 *S -manifold satisfying the condition $C \cdot R = L_3 Q(g,R)$ is an Einstein manifold provided $L_3 \neq s - \frac{r}{2n(2n+1)}$.*

From (9.4) and (3.9) we have

$$(s(2n+s-1) + \lambda)g(X,Y) = 0 \qquad (9.5)$$

Taking $X = Y = e_i$ in (9.5) and summing over $i = 1,2,.......2n+s$, we get the value of λ

$$\lambda = -s(2n+s-1)(<0)$$

Thus we state the following;

Theorem 14 *Ricci soliton in S -manifold satisfying the condition $C \cdot R = L_3 Q(g,R)$ is shrinking.*

Corollary 7 *Ricci soliton in S -manifold satisfying $C \cdot R = L_3 Q(g,R)$ is steady if $s = 0$ (Kaehler manifold) and is shrinking if $s = 1$ (Sasakian manifold).*

IX. Ricci soliton in S -manifolds satisfying $C \cdot C = L_4 Q(g,C)$.

Let us assume that the condition $C((\xi_\alpha, Y) \cdot C)(U,V)W = L_4[(\xi_\alpha \wedge Y) \cdot C](U,V)W$ hold on M, then

$$C(\xi_\alpha, Y)C(U,V)W - C(C(\xi_\alpha, Y)U,V)W - C(U,C(\xi_\alpha, Y)V)W - C(U,V)C(\xi_\alpha, Y)W$$
$$= L_4[(\xi_\alpha \wedge Y)C(U,V)W - C((\xi_\alpha \wedge Y)U,V)W - C(U,(\xi_\alpha \wedge Y)V)W - C(U,V)(\xi_\alpha \wedge Y)W]$$

(10.1)

Using (6.2), (6.3), (8.5) and (8.6) in (10.1) we get

$$\left\{ sL_4 - \left[s - \frac{r}{2n(2n+1)} \right] \right\} \left\{ C(U,V,W,Y) - \left[s - \frac{r}{2n(2n+1)} \right] [g(Y,U)g(V,W) - g(Y,V)g(U,W)] \right\} = 0$$

(10.2)

Therefore, either $L_4 = s - \frac{r}{2n(2n+1)}$ or

$$C(U,V,W,Y) = \left[s - \frac{r}{2n(2n+1)} \right] \{g(Y,U)g(V,W) - g(Y,V)g(U,W)\}. \quad (10.3)$$

Taking $U = Y = e_i$ in (10.3) and summing over $i = 1,2,.......2n+s$, using (4.1) we get

$$S(V,W) = s(2n+s-1)g(V,W) \quad (10.4)$$

Thus we state the following;

Theorem 15 *S -manifold satisfying the condition $C \cdot C = L_4 Q(g,C)$ is an Einstein manifold provided $L_4 \neq s - \frac{r}{2n(2n+1)}$.*

From (10.4) and (3.9) we have

$$(s(2n+s-1)+\lambda)g(X,Y) = 0 \quad (10.5)$$

Taking $X = Y = e_i$ in (10.5) and summing over $i = 1,2,.......2n+s$, we get the value of λ

$$\lambda = -s(2n+s-1)(<0)$$

Thus we state the following;

Theorem 16 *Ricci soliton in S -manifold satisfying the condition $C \cdot C = L_4 Q(g,C)$ is shrinking.*

Corollary 8 *Ricci soliton in S -manifold satisfying $C \cdot C = L_4 Q(g,C)$ is steady if $s = 0$ (Kaehler manifold) and is shrinking if $s = 1$ (Sasakian manifold).*

X. Conclusion

It is shown that Ricci soliton in S -manifold satisfying semi-symmetric and pseudo-symmetric conditions are shrinking. Hence if $S = 1$, then Sasakian manifolds are shrinking which in accordance with [1], [5], [14], and if $S = 0$, then Kaehler manifolds are steady [21]

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