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On Face Bimagic Labeling of Graphs

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Abstract: *Let $G=(V,E,F)$ be a simple, finite, plane graph with |V (G)| number of vertices, |E (G)| number of edges and |F (G)| number of faces. A labeling of type (1,1,0) is a bijective mapping from the vertex set and edge set of G to the set {1, 2,…, |V (G)| + |E (G)|}. Moreover, the labeling is called super if the vertices are labeled with the smallest numbers. The weight of a face under the labeling is the sum of labels of edges and vertices surrounding that face. In this paper we study face bimagic labeling of type (1,1,0) for wheels, cylinders and disjoint union of m copies of prism graphs.*

Keywords: *Wheel, cylinders, prism, face bimagic labeling.*

I. Introduction

In this paper we consider finite, simple, undirected and plane graphs. If *G* is a plane graph, with the vertex set *V* (*G*), edge set *E* (*G*) and face set *F* (*G*). A labeling of type (1,1,0) assigns labels from the set {1, 2,…, |*V* (*G*)| + |*E* (*G*)|} to the vertices and edges of plane graph *G* in such a way that each vertex and edge receive exactly one label and each number is used exactly once as a label. This labeling is called magic if for every positive integer *s* the set of *s*-sided faces have the same weights. The notion of magic labeling of plane graphs was defined by Ko-Wei Lih in [20], and some of the platonic family are given. The magic labeling of type (1,1,1) for fans, planar bipyramids and ladders, is given in [6] by Bača, who also proves that grids, hexagonal lattices, Mobius ladders and $P_2 \times P_3$ have magic labeling of type (1,1,1) in [7,8,9,10] respectively. Bača proves the cylinders $C_n \times P_m$ have magic labeling of type (1,1,0) when $m \geq 2$, $n \geq 3, n \neq 4$ in [11]. Kathiresan and Gokulakrishnan [19] provided magic labeling of type (1,1,1) for the families of planar graphs with 3-sided faces, 5-sided faces,6-sided faces, and one external infinite face. Ali, Hussain, Ahmed, and Miller [3] study magic labeling of type (1,1,1) for wheels and subdivided wheels. Bača [12, 13, 14, 15, 9, 16] and Bača and Holländer [17] gave magic labelling of type (1,1,1) and type (1,1,0) for certain classes of convex polytopes.

In 2004 J.B. Babujee [4] introduced the concept of edge bimagic total labeling and studied for some families of graphs in [5]. Abijection *g* from $V(G)\cup E(G)$ to the set $\{1,2,...,|V(G)|+|E(G)|\}$ is called edge bimagic total labeling if the edge weights are either equal to a constant k_1 or to a constant k_2, where the edge weight of an edge is the sum of the edge labels and the labels of its end vertices. More normal literature about edge bimagic is available in [18].

II. Main Results

In [2] authors introduced the concept of face bimagic labeling.

Definition 2.1: Let *G* = (*V* (*G*), *E* (*G*), *F* (*G*)) be a simple, finite, connected plane graph with the vertex set *V* (*G*), the edge set *E* (*G*) and the face set *F* (*G*). A bijection *g* from $V(G)\cup E(G)$ to the set $\{1,2,...,|V(G)|+|E(G)|\}$ is called *face bimagic* if for every positive integer *s* the weight of every *s*-sided face is equal either to k_s or to t_s. We allow different numbers k_s, t_s for different *s*. Moreover, if for every positive integer *s*, the numbers of *s*-sided faces with weights k_s and t_s differ by at most one, then this labeling is called equitable *face bimagic*.

This paper deals first, with the open problem that every even wheel W_n is (1,1,0) face magic labeling [3], the problem is still open, however we will prove that every even wheel W_n is (1,1,0) super equitable face bimagic labeling, and then we will study super face bimagic labeling for cylinders and disjoint union of *m* copies of prism graphs.

The next theorem deals with super equitable face bimagic labeling of type (1,1,0) for wheel graphs.

Theorem 2.2: *The wheel graph W_n admits super equitable face bimagic labeling of type* $(1,1,0)$ *for every even* $n, n \geq 4$.

Proof. Let the vertex set and the edge set of the wheel graph W_n be $V(W_n)=\{v_i : i=1,2,\ldots,n\}\cup\{u\}, E(W_n)=$

$\{v_iv_{i+1}, v_nv_1 : i = 1,2,\ldots,n-1\} \cup \{uv_i : i = 1,2,\ldots,n\}$. The set of faces is $F(W_n) = \{f_i : i = 1,2,\ldots,n\}$, where the boundaries of the faces f_i for $1 \le i \le n$ are defined as follows: for $1 \le i \le n-1$, the boundaries of the faces are uv_iv_{i+1}, and for $i = n$ the boundary of the face f_n is uu_nv_1.

For n is even, $n \ge 4$, we define a bijective mapping $g : V(W_n) \cup E(W_n) \to \{1,2,\ldots,3n+1\}$ such that

$$g(u) = 1,$$

$$g(v_i) = i+1 \text{ for } i = 1,2,\ldots,n,$$

$$g(uv_i) = \begin{cases} 2n+2-\dfrac{i+1}{2} & \text{for } i = 1,3,\ldots,n-1, \\ 2n+2-\dfrac{i+n}{2} & \text{for } i = 2,4,\ldots,n, \end{cases}$$

$$g(v_iv_{i+1}) = \begin{cases} 3n+2-i & \text{for } i = 1,2,\ldots,\dfrac{n}{2}, \\ 3n+1-i & \text{for } i = \dfrac{n}{2}+1, \dfrac{n}{2}+2, \ldots, n-1, \end{cases}$$

$$g(v_nv_1) = 2n+1+\frac{n}{2}.$$

For the face weights of faces f_i for $i = 1,2,\ldots,n$, we get

$$w_g(f_i) = g(u) + g(v_i) + g(v_{i+1}) + g(uv_i) + g(v_iv_{i+1}) + g(v_{i+1}u)$$

$$= \begin{cases} \dfrac{13n+18}{2} & \text{for } i = 1,2,\ldots,\dfrac{n}{2}, \\ \dfrac{13n+16}{2} & \text{for } i = \dfrac{n}{2}+1, \dfrac{n}{2}+2, \ldots, n-1, \end{cases}$$

$$w_g(f_n) = g(u) + g(v_n) + g(v_1) + g(uv_n) + g(v_nv_1) + g(v_1u) = \frac{13n+16}{2}.$$

Hence the wheel graph W_n, admits a super equitable face bimagic labeling of type (1,1,0) with two magic constants $\dfrac{13n+16}{2}, \dfrac{13n+18}{2}$ respectively .

Theorem 2.3: *For* $m \ge 2$, *the cylinders* $C_n \times P_m$ *have super face bimagic labeling of type* $(1,1,0)$ *for every even* $n, n \ge 6$.

Proof. Let the vertex set and the edge set of the cylinders $C_n \times P_m$ be $V(C_n \times P_m) = \{v_{ij} : i = 1,2,\ldots,n, j = 1,2,\ldots,m\}$, $E(C_n \times P_m) = \{v_{ij}v_{i+1\,j}, v_{nj}v_{1j} : i = 1,2,\ldots,n-1, j = 1,2,\ldots,m\} \cup \{v_{ij}v_{i\,j+1} : i = 1,2,\ldots,n, j = 1,2,\ldots,m-1\}$. The set of faces is $F(C_n \times P_m) = \{f_{ij} : i = 1,2,\ldots,n, j = 1,2,\ldots,m-1\} \cup \{f_{\text{inner,}} f_{\text{external}}\}$, where the boundaries of the faces f_{ij} for $i = 1,2,\ldots,n-1, j = 1,2,\ldots,m-1$ are defined as $v_{ij}v_{i\,j+1}v_{i+1\,j+1}v_{i+1\,j}$, and the boundary of the face f_{nj}, is $v_{nj}v_{n\,j+1}v_{1\,j+1}v_{1\,j}$.

For n even, $m \ge 2$, we define a bijective mapping $g : V(C_n \times P_m) \cup E(C_n \times P_m) \to \{1,2,\ldots,3nm-n\}$, such that for $j = 1,2,\ldots,m$,

$$g(v_{ij}) = \begin{cases} nj-n+i & \text{for } i = 1,2,\ldots,n, \text{ if } j \text{ is odd} \\ nj+1-i & \text{for } i = 1,2,\ldots,n, \text{ if } j \text{ is even} \end{cases}$$

$$g(v_{ij}v_{i+1\,j}) = \begin{cases} 3nm-n-nj+i & \text{for } i = 1,2,\ldots,\dfrac{n}{2}, \\ 3nm-n-nj+1+i & \text{for } i = \dfrac{n}{2}+1, \dfrac{n}{2}+2, \ldots, n-1, \end{cases}$$

$$g(v_{n\,j}v_{1j}) = 3nm+1-jn-\frac{n}{2},$$

$$g(v_{ij}v_{i\,j+1}) = 2nm-nj+1-i \text{ for } i = 1,2,\ldots,n.$$

For the weights of faces f_{ij} for $i = 1,2,\ldots,n-1, j = 1,2,\ldots,m-1$, we get

$$w_g(f_{ij}) = g(v_{ij}) + g(v_{i\,j+1}) + g(v_{i+1\,j+1}) + g(v_{i+1\,j}) + g(v_{ij}v_{i\,j+1}) + g(v_{i\,j+1}v_{i+1\,j+1}) + g(v_{i+1\,j+1}v_{i+1\,j}) + g(v_{i+1}v_{ij})$$

$$= \begin{cases} (nj-n+i)+(n(j+1)+1-i)+(n(j+1)+1-(i+1))+(nj-n+i+1)+(2nm-nj+1-i)+ \\ (3nm-n-n(j+1)+i)+(2nm-nj+1-(i+1))+(3nm-n-nj+i) \quad \text{for } i=1,2,\dots,\frac{n}{2}, \text{if } j \text{ is odd} \\ (nj+1-i)+(n(j+1)-n+i)+(n(j+1)-n+i+1)+(nj+1-(i+1))+(2nm-nj+1-i)+ \\ (3nm-n-n(j+1)+i)+(2nm-nj+1-(i+1))+(3nm-n-nj+i) \quad \text{for } i=1,2,\dots,\frac{n}{2}, \text{if } j \text{ is even} \\ (nj-n+i)+(n(j+1)+1-i)+(n(j+1)+1-(i+1))+(nj-n+i+1)+(2nm-nj+1-i)+(3nm-n- \\ n(j+1)+i+1)+(2nm-nj+1-(i+1))+(3nm-n-nj+i+1) \quad \text{for } i=\frac{n}{2}+1,\frac{n}{2}+2,\dots,n-1, \text{if } j \text{ is odd} \\ (nj+1-i)+(n(j+1)-n+i)+(n(j+1)-n+i+1)+(nj+1-(i+1))+(2nm-nj+1-i)+(3nm-n- \\ n(j+1)+i+1)+(2nm-nj+1-(i+1))+(3nm-n-nj+i+1) \quad \text{for } i=\frac{n}{2}+1,\frac{n}{2}+2,\dots,n-1, \text{if } j \text{ is even} \end{cases}$$

$$= \begin{cases} 10nm-3n+3 \quad \text{for } i=1,2,\dots,\frac{n}{2}, \\ 10nm-3n+5 \quad \text{for } i=\frac{n}{2}+1,\frac{n}{2}+2,\dots,n-1, \end{cases}$$

$$w_g(f_{nj}) = g(v_{nj}) + g(v_{n\,j+1}) + g(v_{1\,j+1}) + g(v_{1j}) + g(v_{nj}v_{n\,j+1}) + g(v_{n\,j+1}v_{1\,j+1}) + g(v_{1\,j+1}v_{1j}) + g(v_{1j}v_{nj})$$

$$= \begin{cases} (nj)+(n(j+1)+1-n)+(n(j+1))+(nj-n+1)+(2nm-nj+1-n)+ \\ (3nm+1-(j+1)n-\frac{n}{2})+(2nm-nj)+(3nm+1-jn-\frac{n}{2}) \quad \text{if } j \text{ is odd} \\ (nj+1-n)+(n(j+1))+(n(j+1)-n+1)+(nj)+(2nm-nj+1-n)+ \\ (3nm+1-(j+1)n-\frac{n}{2})+(2nm-nj)+(3nm+1-jn-\frac{n}{2}) \quad \text{if } j \text{ is even} \end{cases}$$

$$= 10nm-3n+5 \quad \text{for } j=1,2,\dots,m-1,$$

$$w_g(f_{\text{inner}}) = \sum_{i=1}^{n} g(v_{i1}) + \sum_{i=1}^{n-1} g(v_{i1}v_{i+1\,1}) + g(v_{n1}v_{11})$$

$$= \sum_{i=1}^{n} i + \sum_{i=1}^{n/2}(3nm-2n+i) + (\sum_{i=(n/2)+1}^{n-1}(3nm-2n+1+i)) + (3nm+1-n-\frac{n}{2}) = 3n^2m+n-n^2$$

$$w_g(f_{\text{external}}) = \sum_{i=1}^{n} g(v_{im}) + \sum_{i=1}^{n-1} g(v_{im}v_{i+1m}) + g(v_{nm}v_{1m})$$

$$= \begin{cases} \sum_{i=1}^{n}(nm-n+i) + \sum_{i=1}^{n/2}(2nm-n+i) + \sum_{i=(n/2)+1}^{n-1}(2nm-n+1+i)+ \\ (2nm+1-\frac{n}{2}) \quad \text{if } m \text{ odd} \\ \sum_{i=1}^{n}(nm+1-i) + \sum_{i=1}^{n/2}(2nm-n+i) + \sum_{i=(n/2)+1}^{n-1}(2nm-n+1+i)+ \\ (2nm+1-\frac{n}{2}) \quad \text{if } m \text{ even} \end{cases}$$

$$= 3n^2m+n-n^2 \quad \text{for both } m \text{ odd and even}$$

Now, by swapping the labeling of the edge $v_{n\,m-1}v_{nm}$ with the labeling of the vertex v_{nm}, we can easily change the weight of external face from $3n^2m+n-n^2$ to $3n^2m+n-n^2+1$, if *m* is odd, and to $3n^2m+2n-n^2$, if *m* is even, This swapping does not have any effect on the weights of 4-sided faces, hence the cylinders $C_n \times P_m$ admits a super face bimagic labeling of type (1,1,0) with two magic constants $10nm-3n+3$, $10nm-3n+5$, for 4-sided faces, and $3n^2m+n-n^2$, $3n^2m+n-n^2+1$, if *m* is odd, $3n^2m+n-n^2$, $3n^2m+2n-n^2$, if *m* is even for *n*-sided faces respectively.

Corollary 2.4: *For* $m \geq 2$, *the cylinders* $C_4 \times P_m$ *have super face bimagic labeling of type* $(1,1,0)$.

Proof. Let the vertex set, edge set and face set of cylinders $C_4 \times P_m$ as were defined in the previous theorem.

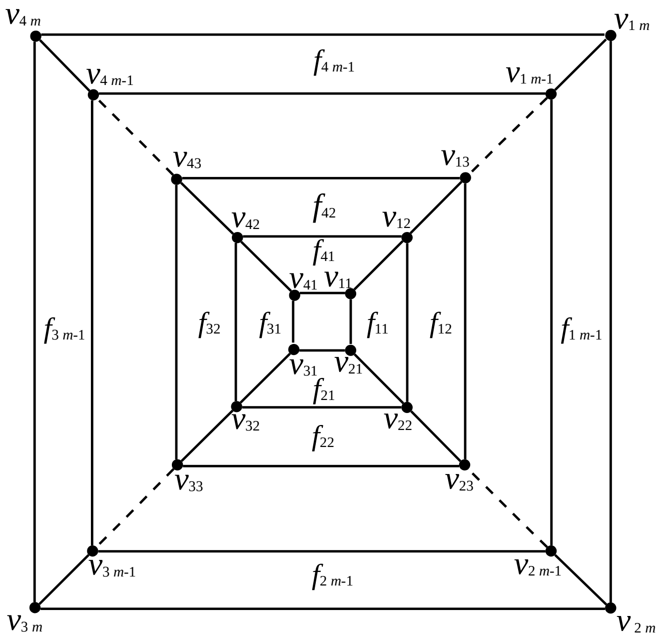

Fig. 1 (Cylinders $C_4 \times P_m$)

For $m \geq 2$, we define a bijective mapping $g: V(C_4 \times P_m) \cup E(C_4 \times P_m) \rightarrow \{1, 2, \ldots, 12m-4\}$ such that for $j = 1, 2, \ldots, m$,

$$g(v_{ij}) = \begin{cases} 4j - 4 + i & \text{for } i = 1,2,3,4, \text{ if } j \text{ is odd} \\ 4j + 1 - i & \text{for } i = 1,2,3,4, \text{ if } j \text{ is even} \end{cases}$$

$$g(v_{ij} v_{i+1\,j}) = \begin{cases} 8m - 4j + i & \text{for } i = 1,2,3, \text{ if } j \text{ is odd} \\ 8m - 4j + i + 1 & \text{for } i = 1,2,3, \text{ if } j \text{ is even} \end{cases}$$

$$g(v_{4j} v_{1j}) = \begin{cases} 8m - 4j + 4 & \text{if } j \text{ is odd} \\ 8m - 4j + 1 & \text{if } j \text{ is even} \end{cases}$$

And for $j = 1, 2, \ldots, m-1$,

$$g(v_{ij} v_{i\,j+1}) = 12m + 1 - i - 4j \quad \text{for } i = 1,2,3,4.$$

For the weights of faces f_{ij} for $i = 1,2,3,4,\ j = 1,2,\ldots,m-1$, we get

$$w_g(f_{ij}) = g(v_{ij}) + g(v_{i\,j+1}) + g(v_{i+1\,j+1}) + g(v_{i+1\,j}) + g(v_{ij} v_{i\,j+1}) + g(v_{i\,j+1} v_{i+1\,j+1}) + g(v_{i+1\,j+1} v_{i+1\,j}) + g(v_{i+1} v_{ij})$$

$$= \begin{cases} (4j - 4 + i) + (4(j+1) + 1 - i) + (4(j+1) + 1 - (i+1)) + (4j - 4 + i + 1) + \\ (12m + 1 - i - 4j) + (8m - 4(j+1) + 1 + i) + (12m + 1 - (i+1) - 4j) + (8m - 4j + i) \ \text{ for } i = 1,2,3, \text{ if } j \text{ is odd} \\ (4j + 1 - i) + (4(j+1) - 4 + i) + (4(j+1) - 4 + (i+1)) + (4j + 1 - (i+1)) + \\ (12m + 1 - i - 4j) + (8m - 4(j+1) + i) + (12m + 1 - (i+1) - 4j) + (8m - 4j + i + 1) \ \text{ for } i = 1,2,3, \text{ if } j \text{ is even} \end{cases}$$

$$= \left. \begin{cases} 40m \text{ for } i = 1,2,3, & \text{if } j \text{ is odd} \\ 40m \text{ for } i = 1,2,3, & \text{if } j \text{ is even} \end{cases} \right\} = 40m \quad \text{for } i = 1,2,3,\ j = 1,2,\ldots,m$$

$$w_g(f_{4j}) = g(v_{4j}) + g(v_{4\,j+1}) + g(v_{1\,j+1}) + g(v_{1j}) + g(v_{4j} v_{4\,j+1}) + g(v_{4\,j+1} v_{1\,j+1}) + g(v_{1\,j+1} v_{1j}) + g(v_{1j} v_{4j})$$

$$= \begin{cases} (4j) + (4(j+1) - 3) + (4(j+1)) + (4j - 3) + (12m - 3 - 4j) + \\ (8m - 4(j+1) + 1) + (12m - 4j) + (8m - 4j + 4) & \text{if } j \text{ is odd} \\ (4j - 3) + (4(j+1)) + (4j + 1) + (4j) + (12m - 3 - 4j) + \\ (8m - 4(j+1) + 4) + (12m - 4j) + (8m - 4j + 1) & \text{if } j \text{ is even} \end{cases}$$

$$= 40m \quad \text{for } j = 1, 2, \ldots, m-1,$$

$$w_g(f_{\text{inner}}) = \sum_{i=1}^{4} g(v_{i1}) + \sum_{i=1}^{3} g(v_{i1}v_{i+1\,1}) + g(v_{41}v_{11})$$
$$= \sum_{i=1}^{4} i + \sum_{i=1}^{3}(8m-4+i) + 8m = 32m+4$$
$$w_g(f_{\text{external}}) = \sum_{i=1}^{4} g(v_{im}) + \sum_{i=1}^{3} g(v_{im}v_{i+1\,m}) + g(v_{4m}v_{1m})$$
$$= \begin{cases} \sum_{i=1}^{4}(4m-4+i) + \sum_{i=1}^{3}(4m+i) + (4m+4) & \text{if } m \text{ odd} \\ \sum_{i=1}^{4}(4m+1-i) + \sum_{i=1}^{3}(4m+i+1) + (4m+1) & \text{if } m \text{ even} \end{cases}$$
$$= 32m+4 \quad \text{for } m \text{ odd and even}$$

Hence the cylinders graph $C_4 \times P_m$, admits a super face bimagic labeling of type (1,1,0) with two magic constants $40m, 32m+4$ respectively.

Theorem 2.5 *For every even $n, n \geq 6,\ m \geq 2,$ the disjoint union of m copies of a prism graph $(C_n \times P_2)$ admits super face bimagic labeling of type* $(1,1,0)$.

Proof. Let the vertex set and edge set of a graph $m(C_n \times P_2)$ be
$V(m(C_n \times P_2)) = \{v_i^j : i = 1,2,\ldots,2n,\ j = 12,\ldots,m\}$,
$E(m(C_n \times P_2)) = \{v_i^j v_{i+1}^j : i = 1,2,\ldots,n-1,n+1,n+2,\ldots,2n-1,\ j = 1,2,\ldots,m\} \cup \{v_i^j v_{n+i}^j : i = 1,2,\ldots,n,\ j = 1,2,\ldots,m\}$
$\cup \{v_n^j v_1^j, v_{2n}^j v_{n+1}^j\}$. The set of faces is $F(m(C_n \times P_2)) = \{f_i^j : i = 1,2,\ldots,n, j = 1,2,\ldots,m\} \cup \{f_{\text{inner}}^j, f_{\text{external}}^j : j = 1,2,\ldots,m\}$, where the boundaries of the faces f_i^j for $i = 1,2,\ldots n-1,\ j = 1,2,\ldots,m,$ are $v_i^j v_{n+i}^j v_{n+i+1}^j v_{i+1}^j$ and the boundary of the face f_n^j is $v_n^j v_{2n}^j v_{n+1}^j v_1^j$.

For even $n,\ n \geq 6, m \geq 2$, we define a bijective mapping $g : V(m(C_n \times P_2)) \cup E(m(C_n \times P_2)) \to \{1,2,\ldots,5mn\}$ such that

$$g(v_i^j) = nj - n + i \qquad \text{for } i = 1,2,...,n,\ j = 1,2,...,m,$$
$$g(v_{n+i}^j) = mn + nj + 1 - i \quad \text{for } i = 1,2,...,n,\ j = 1,2,...,m,$$
$$g(v_i^j v_{n+i}^j) = 3mn + n - nj + 1 - i \quad \text{for } i = 1,2,...,n,\ j = 1,2,...,m,$$
$$g(v_{n+i}^j v_{n+i+1}^j) = \begin{cases} 4mn - nj + i & \text{for } i = 1,2,...,\frac{n}{2},\ j = 1,2,...,m, \\ 4mn - nj + 1 + i & \text{for } i = \frac{n}{2}+1, \frac{n}{2}+2,...,n-1,\ j = 1,2,...,m, \end{cases}$$
$$g(v_i^j v_{i+1}^j) = \begin{cases} 5mn - nj + i & \text{for } i = 1,2,...,\frac{n}{2},\ j = 1,2,...,m, \\ 5mn - nj + 1 + i & \text{for } i = \frac{n}{2}+1, \frac{n}{2}+2,...,n-1,\ j = 1,2,...,m, \end{cases}$$
$$g(v_n^j v_1^j) = 4mn - nj + 1 + \frac{n}{2},$$
$$g(v_{2n}^j v_{n+1}^j) = 5mn - nj + 1 + \frac{n}{2}.$$

For the weights of faces f_i^j for $i = 1,2,\ldots,n-1,\ j = 1,2,\ldots,m,$ we get

$$w_g(f_i^j) = g(v_i^j) + g(v_{n+i}^j) + g(v_{n+i+1}^j) + g(v_{i+1}^j) + g(v_i^j v_{n+i}^j) + g(v_{n+i}^j v_{n+i+1}^j) + g(v_{n+i+1}^j v_{i+1}^j) + g(v_{i+1}^j v_i^j)$$
$$= \begin{cases} (nj-n+i) + (mn+nj+1-i) + (mn+nj-i) + (nj-n+i+1) + (3mn+1-nj+n-i) + \\ (4mn-nj+i) + (3mn+n-nj-i) + (5mn-nj+i) \qquad \text{for } i = 1,2,...,\frac{n}{2},\ j = 1,2,...,m, \\ (nj-n+i) + (mn+nj+1-i) + (mn+nj-i) + (nj-n+i+1) + (3mn+1-nj+n-i) + \\ (4mn-nj+1+i) + (3mn+n-nj-i) + (5mn-nj+1+i) \quad \text{for } i = \frac{n}{2}+1, \frac{n}{2}+2,...,n-1,\ j = 1,2,...,m, \end{cases}$$

$$= \begin{cases} 17mn+3 & \text{for } i=1,2,\ldots,\frac{n}{2},\ j=1,2,\ldots,m, \\ 17mn+5 & \text{for } i=\frac{n}{2}+1,\frac{n}{2}+2,\ldots,n-1,\ j=1,2,\ldots,m, \end{cases}$$

$$w_g(f_n^j) = g(v_n^j)+g(v_{2n}^j)+g(v_{n+1}^j)+g(v_1^j)+g(v_n^j v_{2n}^j)+g(v_{2n}^j v_{n+1}^j)+g(v_{n+1}^j v_1^j)+g(v_1^j v_n^j)$$

$$= (nj-n+n)+(mn+nj+1-n)+(mn+nj)+(nj-n+1)+(3mn-nj+1)+(5mn-nj+1+\frac{n}{2})+$$

$$(3mn+n-nj)+(4mn-nj+1+\frac{n}{2})$$

$$= 17mn+5 \quad \text{for } i=n,\ j=1,2,\ldots,m,$$

For the weights of n-sided faces, we get

$$w_g(f_{\text{inner}}^j) = \sum_{i=1}^{n} g(v_i^j)+\sum_{i=1}^{n-1} g(v_i^j v_{i+1}^j)+g(v_n^j v_1^j)$$

$$= \sum_{i=1}^{n}(nj-n+i)+\sum_{i=1}^{n/2}(5mn-nj+i)+\sum_{i=(n/2)+1}^{n-1}(5mn-nj+1+i)+(4mn-nj+1+\frac{n}{2})$$

$$= 5mn^2-mn+n \quad \text{for } j=1,2,\ldots,m,$$

$$w_g(f_{\text{external}}^j) = \sum_{i=1}^{n} g(v_{n+i}^j)+\sum_{i=1}^{n-1} g(v_{n+i}^j v_{n+i+1}^j)+g(v_{2n}^j v_{n+1}^j)$$

$$= \sum_{i=1}^{n}(mn+nj+1-i)+\sum_{i=1}^{n/2}(4mn-nj+i)+\sum_{i=(n/2)+1}^{n-1}(4mn-nj+1+i)+(5mn-nj+1+\frac{n}{2})$$

$$= 5mn^2+mn+n \quad \text{for } j=1,2,\ldots,m,$$

Observe that the difference between the vertex label $g(v_{n/2}^j)$ and the vertex label $g(v_{3n/2}^j)$ is $mn+1$, so that if we swap the vertex label $g(v_{n/2}^j)$ with the vertex label $g(v_{3n/2}^j)$, then the weights of n-sided faces will be

$w_g(f_{\text{inner}}^j) = 5mn^2+n+1 \quad \text{for } j=1,2,\ldots,m,$

$w_g(f_{\text{external}}^j) = 5mn^2+n-1 \quad \text{for } j=1,2,\ldots,m,$

This swapping does not have any effect on the weights of 4-sided faces, hence the graph $m(C_n \times P_2)$ admits super face bimagic labeling of type (1,1,0) with two magic constants $17mn+3,\ 17mn+5,$ for 4-sided faces and $5mn^2+n+1,\ 5mn^2+n-1,$ for n-sided faces respectively.

Corollary 2.6 *For $n=4, m\geq 2$ the disjoint union of m copies of a prism graph $(C_4 \times P_2)$ admits super face bimagic labeling of type* $(1,1,0)$.

Proof. The vertex set, edge set and face set as were defined in the previous theorem. For n = 4, $m \geq 2$, we define a bijective mapping $g: V(m(C_4 \times P_2)) \cup E(m(C_4 \times P_2)) \to \{1,2,\ldots,20m\}$, such that

$$g(v_i^j) = 4j-4+i \quad \text{for } i=1,2,3,4,\ j=1,2,\ldots,m,$$
$$g(v_{4+i}^j) = 4m+4j+1-i \quad \text{for } i=1,2,3,4,\ j=1,2,\ldots,m,$$
$$g(v_i^j v_{4+i}^j) = 12m+5-4j-i \quad \text{for } i=1,2,3,4,\ j=1,2,\ldots,m,$$
$$g(v_{4+i}^j v_{4+i+1}^j) = 16m-4j+i \quad \text{for } i=1,2,3,\ j=1,2,\ldots,m,$$
$$g(v_i^j v_{i+1}^j) = 20m-4j+i+1 \quad \text{for } i=1,2,3,\ j=1,2,\ldots,m,$$
$$g(v_4^j v_1^j) = 20m+1-4j \quad \text{for } j=1,2,\ldots,m,$$
$$g(v_8^j v_5^j) = 16m+4-4j \quad \text{for } j=1,2,\ldots,m.$$

It is easy to observe that, the weights of the faces f_i^j, for $i=1,2,3,4,\ j=1,2,\ldots,m,$ is $68m+4$ and the weights of the inner and external faces are $80m+4$, for $j=1,2,\ldots,m.$

Hence the graph $m(C_4 \times P_2)$ admits super face bimagic labeling of type (1,1,0) for every $m \geq 2$.

III. Conclusion

In the foregoing section we investigated the existence of face bimagic labeling of type $(1,1,0)$ for certain families of graphs, mainly for wheels, cylinders $C_n \times P_m$ and disjoint union of *m* copies of prism graph.

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