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Amusing Properties of Odd Numbers Derived From Valuated Binary Tree

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Abstract: *The article reveals some new properties of odd numbers by means of valuated binary tree. It first systematically investigates properties of valuated binary subtrees, including calculation law of nodes, symmetric law of nodes and their common divisors as well as duplication and transition law of a subtree, then it digs out several new properties of the odd numbers that are bigger than 1. The new properties claim that, in an odd valuated binary subtree the sum of nodes on every level and the sum of nodes in the first finite levels are multiples of the root node, and division of the sum of the nodes in the first finite levels by the root node remains the same value for all odd valuated binary subtrees provided that the same number of levels are taken into the sum. The new properties can be valuable in studying and analyzing the security key in information system.*
Keywords: *Odd numbers, Integer division, Multiple*
MSC 2000*: 11A51,11A05*

I. Introduction

Article [1] put forward the concept of valuated binary tree and demonstrated that the valuated binary tree could certainly dig out new properties of integers. As follow-up work, this article presents some other new and amusing properties of odd numbers in term of valuated binary tree.

II. Preliminaries

2.1 Definitions and Notations

A valuated binary tree ***T***, which is defined in [1], is such a binary tree that each of its node is assign a value. An odd-number *N* rooted tree, which is also defined in [1], is a recursively–constructed valuated binary tree whose root is the odd number *N* with $2N-1$ and $2N+1$ being the root's left and right sons, respectively. For convenience, let symbol $N_{(k,j)}$ be the node on the j^{th} position of the k^{th} level in ***T***, where $k \ge 0$ and $0 \le j \le 2^k - 1$; let symbol $T_{N_{(k,j)}}$ denote a subtree whose root is $N_{(k,j)}$ and symbol $N_{(i,\omega)}^{N_{(k,j)}}$ denote the node on the ω^{th} position of the i^{th} level in $T_{N_{(k,j)}}$.

2.2 Lemmas

Let $N_{(0,0)}$ be an odd number and $T_{N_{(0,0)}}$ be the $N_{(0,0)}$-rooted binary tree, as shown in figure 1. Then the following lemma holds

Lemma 1. Let T be a recursively–constructed $N_{(0,0)}$ -rooted binary tree; then the following statements hold

(1) There are 2^k nodes on the k^{th} level, $k = 0,1,...$;

(2) Node $N_{(k,j)}$ is computed by

$$N_{(k,j)} = 2^k N_{(0,0)} - 2^k + 2j + 1; k = 0,1,2,...; j = 0,1,...,2^k - 1 \tag{1}$$

(3) The two nodes, $N_{(i,\omega)}$ and $N_{(i,2^i-1-\omega)}$, are at the symmetric position of the i^{th} level and it holds

$$N_{(i,\omega)} + N_{(i,2^i-1-\omega)} = 2^{i+1} N_{(0,0)} \tag{2}$$

Proof. See in [1].

□

Lemma 2. For arbitrary *m*>0, $2^{2m} \equiv 1 (\bmod 3)$.

Proof. $2^{2m} - 1 = (2^m + 1)(2^m - 1)$. Since the reminder of 2^m divided by 3 is either 1 or 2, it knows that one of $(2^m + 1)$ and $(2^m - 1)$ must be divided by 3. Hence the lemma holds.

□

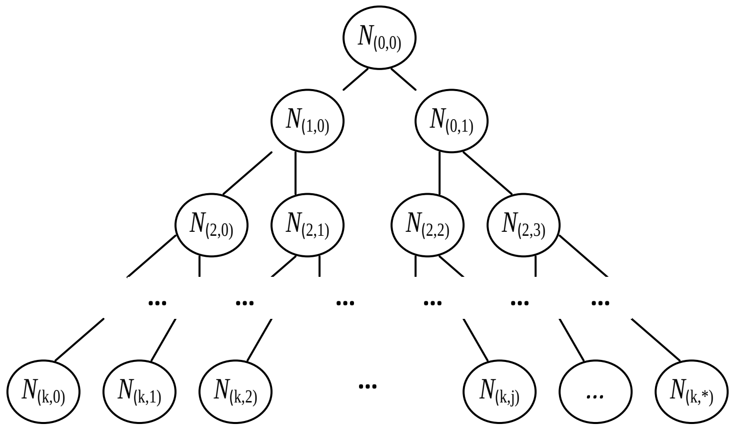

Fig.1. An $N_{(0,0)}$-rooted binary tree

III. Main Results and Proofs

New results include properties of subtrees and new traits of odd numbers. They are introduced separately in the following two sections.

3.1 Properties of Subtrees

Theorem 1 (Law of Nodes' Calculation) The i^{th} ($i \geq 0$) level of subtree $T_{N_{(k,j)}}$ ($k \geq 0$) is the ($k+i$)$^{\text{th}}$ level of $T_{N_{(0,0)}}$, containing 2^i nodes, the smallest and biggest of which are in term of $T_{N_{(0,0)}}$ by

$N_{(k+i,2^i j)} = 2^i N_{(k,j)} - 2^i + 1$, $N_{(k+i,2^i j+2^i-1)} = 2^i N_{(k,j)} + 2^i - 1$

Node $N_{(i,\omega)}^{N_{(k,j)}}$ of $T_{N_{(k,j)}}$ $(0 \leq \omega \leq 2^i - 1)$ is corresponding to node $N_{(k+i,2^i j+\omega)}$ of $T_{N_{(0,0)}}$ by the following formula (3)

$$N_{(i,\omega)}^{N_{(k,j)}} = N_{(k+i,2^i j+\omega)} = 2^i N_{(k,j)} - 2^i + 2\omega + 1; j = 0,1,...,2^k - 1; i = 0,1,...; \omega = 0,1,...,2^i - 1 \quad (3)$$

Proof. By Theorem 3 in article [2], the node at the ω^{th} position of the i^{th} level in $T_{N_{(k,j)}}$ is just the node at the $(2^i j+\omega)^{\text{th}}$ position of the ($k+i$)$^{\text{th}}$ level in $T_{N_{(0,0)}}$, namely, $N_{(i,\omega)}^{N_{(k,j)}} = N_{(k+i,2^i j+\omega)}$. By (1) it yields

$$\begin{aligned} N_{(k+i,2^i j+\omega)} &= 2^{k+i} N_{(0,0)} - 2^{k+i} + 2(2^i j + \omega) + 1 \\ &= 2^{k+i} N_{(0,0)} - 2^{k+i} + 2^{i+1} j + 2\omega + 1 \\ &= 2^i (2^k N_{(0,0)} - 2^k + 2j + 1) - 2^i + 2\omega + 1 \\ &= 2^i N_{(k,j)} - 2^i + 2\omega + 1 \end{aligned}$$

Obviously, $N_{(k+i,2^i j+0)} = 2^i N_{(0,0)} - 2^i + 1$ is the smallest node and $N_{(k+i,2^i j+2^i-1)} = 2^i N_{(k,j)} + 2^i - 1$ is the biggest one.

□

Theorem 2 (Law of Symmetric Nodes) Nodes on each level of $T_{N_{(k,j)}}$ (k>1) are symmetric by the following laws

$$N_{(i,\omega)}^{N_{(k,j)}} + N_{(i,2^i-1-\omega)}^{N_{(k,j)}} = 2^{i+1} N_{(0,0)}^{N_{(k,j)}} \quad (4)$$

or

$$N_{(i,\omega)}^{N_{(k,j)}} + N_{(i,2^i-1-\omega)}^{N_{(k,j)}} = 2^{i+1} N_{(k,j)} \quad (5)$$

or

$$N_{(k+i,2^i j+\omega)} + N_{(k+i,2^i j+2^i-1-\omega)} = 2^{i+1} N_{(k,j)} \quad (6)$$

where $0 \leq \omega \leq 2^i - 1$.

Proof. By Theorem 1, $N_{(i,\omega)}^{N_{(k,j)}} = 2^i N_{(k,j)} - 2^i + 2\omega + 1$ and $N_{(i,2^i-1-\omega)}^{N_{(k,j)}} = 2^i N_{(k,j)} - 2^i + 2(2^i - 1 - \omega) + 1$. This immediately results in (5). Since $N_{(k,j)} = N_{(0,0)}^{N_{(k,j)}}$, (4) holds and (6) can be validated by (3).

□

Theorem 3 (Law of Symmetric Common Factor) Suppose node $N_{(k,j)}$ has a common factor d with $N_{(i,\omega)}^{N_{(k,j)}}$, then d is also a common factor of $N_{(k,j)}$ and $N_{(i,2^i-1-\omega)}^{N_{(k,j)}}$.

Proof. (Omit)

□

Theorem 4 (Law of Subtrees' Duplication) Subtree $T_{N_{(k,j)}}$ can be duplicated from $T_{N_{(0,0)}}$. Let $N_{(i,\omega)}^{N_{(k,j)}}$ be node of $T_{N_{(k,j)}}$ and $N_{(i,\omega)}$ be node of $T_{N_{(0,0)}}$; then

$$N_{(i,\omega)}^{N_{(k,j)}} = 2^i(N_{(k,j)} - N_{(0,0)}) + N_{(i,\omega)}, i = 0,2,...; \omega = 0,1,...,2^i - 1 \quad (7)$$

Proof. By (1) it yields

$$N_{(i,\omega)} = 2^i N_{(0,0)} - 2^i + 2\omega + 1$$

And by Theorem 1, $N_{(i,\omega)}^{N_{(k,j)}} = N_{(k+i,2^i j+\omega)}$, and hence it yields

$$N_{(i,\omega)}^{N_{(k,j)}} = N_{(k+i,2^i j+\omega)} = 2^i N_{(k,j)} - 2^i + 2\omega + 1$$

$$= 2^i N_{(k,j)} - 2^i N_{(0,0)} + 2^i N_{(0,0)} - 2^i + 2\omega + 1$$

$$= 2^i(N_{(k,j)} - N_{(0,0)}) + 2^i N_{(0,0)} - 2^i + 2\omega + 1$$

$$= 2^i(N_{(k,j)} - N_{(0,0)}) + N_{(i,\omega)}$$

□

Take $\boldsymbol{T}_9$ and $\boldsymbol{T}_3$ as an example, the following figure 2 illustrates the duplication law of subtrees.

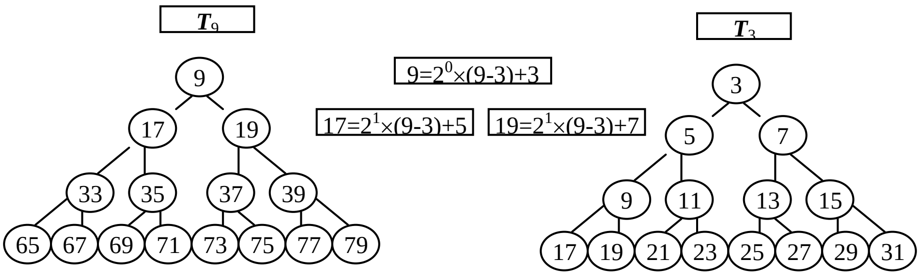

Fig.2. $\boldsymbol{T}_9$ duplicated from $\boldsymbol{T}_3$

Theorem 5 (Law of Subtrees' Transition) Let $T_{N_{(m,s)}}$ and $T_{N_{(k,j)}}$ ($m \ge k$) be two subtrees of $T_{N_{(0,0)}}$; then it holds

$$N_{(i,\omega)}^{N_{(m,s)}} = 2^i(N_{(m,s)} - N_{(k,j)}) + N_{(i,\omega)}^{N_{(k,j)}}, i = 1,2,...; \omega = 0,1,...,2^i - 1 \quad (8)$$

Proof. By Theorem 4, it holds

$$N_{(i,\omega)}^{N_{(m,s)}} = 2^i(N_{(m,s)} - N_{(0,0)}) + N_{(i,\omega)}, i = 1,2,...; \omega = 0,1,...,2^i - 1$$

$$N_{(i,\omega)}^{N_{(m,s)}} = 2^i(N_{(m,s)} - N_{(0,0)}) + N_{(i,\omega)}, i = 1,2,...; \omega = 0,1,...,2^i - 1$$

Consequently it results in

$$N_{(i,\omega)}^{N_{(m,s)}} = 2^i(N_{(m,s)} - N_{(k,j)}) + N_{(i,\omega)}^{N_{(k,j)}}, i = 1,2,...; \omega = 0,1,...,2^i - 1$$

□

3.2 New Traits of Odd Numbers

Theorem 6 (Law of Sum by Level) Sum of the i^{th}($i \ge 0$) level in Subtree $T_{N_{(k,j)}}$ fits the following identity

$$\sum_{\omega=0}^{2^i-1} N_{(i,\omega)}^{N_{(k,j)}} = 2^{2i} N_{(0,0)}^{N_{(k,j)}} = 2^{2i} N_{(k,j)} \quad (9)$$

Proof. By (7) and symmetry of nodes it yields

$$\sum_{\omega=0}^{2^i-1} N_{(i,\omega)}^{N_{(k,j)}} = \frac{1}{2}\sum_{i=0}^{2^i-1}(N_{(i,\omega)}^{N_{(k,j)}} + N_{(i,2^i-1-\omega)}^{N_{(k,j)}}) = \frac{1}{2}2^i \times 2^{i+1} N_{(k,j)} = 2^{2i} N_{(k,j)}$$

□

Theorem 7 (Law of Root Division) The root of subtree $T_{N_{(k,j)}}$ can divide the sum of the first m ($m \ge 0$) levels of $T_{N_{(k,j)}}$, namely, $N_{(k,j)}$ can divide $\sum_{i,\omega} N_{(i,\omega)}^{N_{(k,j)}} \Rightarrow \sum_{i=0}^{m-1}\sum_{\omega=0}^{2^i-1} N_{(i,\omega)}^{N_{(k,j)}}$. Or $\dfrac{\sum_{i=0}^{m-1}\sum_{\omega=0}^{2^i-1} N_{(i,\omega)}^{N_{(k,j)}}}{N_{(k,j)}}$ is an integer.

Proof. By Theorem 6, sum of each level of $T_{N_{(k,j)}}$ is a multiple of $N_{(k,j)}$. Hence the theorem holds.

□

Theorem 8 (Law of Uniform Sum) Arbitrary two subtrees, $T_{N_{(k,j)}}$ and $T_{N_{(m,s)}}$, satisfy

$$\frac{\sum_{i,\omega} N_{(i,\omega)}^{N_{(m,s)}}}{N_{(m,s)}} = \frac{\sum_{i,\omega} N_{(i,\omega)}^{N_{(k,j)}}}{N_{(k,j)}}, i = 1,2,...; \omega = 0,1,...,2^i - 1 \qquad (10)$$

Proof. By Theorem 7, $\frac{\sum_{i,\omega} N_{(i,\omega)}^{N_{(m,s)}}}{N_{(m,s)}}$ and $\frac{\sum_{i,\omega} N_{(i,\omega)}^{N_{(k,j)}}}{N_{(k,j)}}$ are two integers. Note that

$$\sum_{i=0}^{m-1}\sum_{\omega=0}^{2^i-1} N_{(l,\omega)}^{N_{(k,j)}} = \sum_{i=0}^{m-1} 2^{2i} N_{(k,j)} = \frac{2^{2m}-1}{3} \times N_{(k,j)}$$

and

$$\sum_{i=1}^{m-1}\sum_{\omega=0}^{2^i-1} N_{(i,\omega)}^{N_{(k,j)}} = \frac{2^{2m}-1}{3} \times N_{(m,s)}$$

It yields，

$$\frac{\sum_{l,\omega} N_{(l,\omega)}^{N_{(m,s)}}}{N_{(m,s)}} = \frac{\sum_{l,\omega} N_{(l,\omega)}^{N_{(k,j)}}}{N_{(k,j)}} = \frac{2^{2m}-1}{3}$$

By Lemma 2, $\frac{2^{2m}-1}{3}$ is an integer, which validates the theorem.

□

Take T_5, T_7 and T_{35} as examples, Theorem 7 and 8 can be illustrated intuitionally by following figure 3.

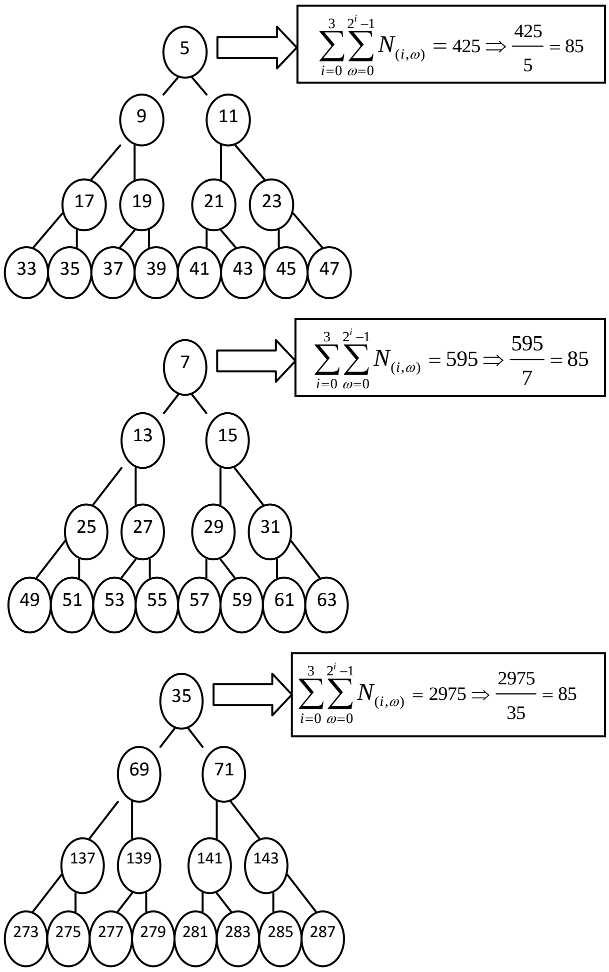

Fig.3. Root division and uniform sum in subtrees

IV. Conclusions

As stated in article [1], putting odd numbers on a binary tree is a new approach to study integers and it can derive many new properties of the odd numbers. Like the results derived in this article and in article [1], the new properties can be helpful to understand more traits of odd numbers and might be useful in studying and analyzing security keys. In addition, binary tree's excellent behavior in parallel computing is sure to make the new approach to obtain brilliant achievements in future work.

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