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Mathematical Model of HIV/Aids on Varying Population

*Achugamonu, Pius C[1], Nwobi, Felix Nonyanim[2], Inyama, Simeon Chioma[3]
[1]*Department of Mathematics, Alvan Ikoku Federal College of Education, Owerri, Imo State, Nigeria*
[2]*Department of Statistics, Imo State University, Owerri, Imo State, Nigeria*
[3]*Department of Mathematics, Federal University of Technology, Owerri, Imo State, Nigeria*

Abstract: *A mathematical model of HIV/AIDS was developed for a human population of five compartments. The dynamical systems theory was used to analyze the model. The steady states (equilibrium points) were determined. The stability of the steady states (equilibrium points) was established using Routh Hurtwitz criterion. It was shown that the model has two non-negative equilibra which are disease free equilibrium* $E_0(0,0,0,0)$ *and endemic equilibrium* $E_E(S^*, I^*, I_c^*, I_T^*, a^*)$. *It was found that the disease free equilibrium* $E_0(0,0,0,0)$ *is both locally and globally asymptotically stable if is* $R_0 < 1$ *which means the disease will die out from the system after a long time and it is unstable and the infection is maintained in the population if* $R_0 > 1$. *The endemic equilibrium,* $E_E(S^*, I^*, I_c^*, I_T^*, a^*)$, *is locally asymptotically stable the disease remains an endemic disease. A sensitivity analysis and Simulation of the model were done which enabled us to draw conclusion and three parameters proved to be very sensitive in the model which are Programmes run by the Government to stop the spread,* a_1, *rate of movement from infective group,* γ, *and rate of recruitment into susceptible group,* λ.

Keywords: *HIV/AIDS, Mathematical Model, Varying Population, Stability, Dynamical Systems Theory*

I. Introduction

It is a simplified representative of certain aspects of a real system, created using mathematical concept such as functions, graphs, diagrams and equations to solve problems in the real world. Other definitions of mathematical modeling have also been given by various authors; ECA (2013). Defines it as art of translating physical problems into tractable mathematical formulations whose theoretical and numerical analysis provides understanding of the real life phenomenon and solution to the problem. Modeling involves identifying and selecting relevant features symbolically, analyzing and reasoning about the model and characteristics of the situation and considering the accuracy and limitations of the model: ECA (2013). A mathematical model can be formulated either through intuitive reasoning about the phenomenon or from physical law based on evidence from the experiment. It is usually constructed in the language of mathematics, logic and computer following the algebraic rules of syntax. A mathematical model often takes the form of differential equation or system of differential equation. Since our goal is to use the equation to solve specific problems, we are interested in specific rather than general solutions. We obtain these specific solutions by boundary conditions; Aris(1994). Generally the success of a model depends on how easily it can be used and how accurate are its predictions.

The use of computers has extended modeling by allowing combination of data, interaction, repetition sound graphics and other displays for various types of mathematical modeling. However, model should have a limited range of validity and should not be applied outside this range; Guideto (2010). Modeling has been applied in virtually every sphere of man's existence and it is as wide as nature itself, Hartman (1980) the list is not exhaustible. Modeling has been used to solve problems of robotics in the area of artificial- intelligence; detection of planetary system in Astronomy; population dynamics and spread of infectious disease in biology; planning of production units in chemical engineering; stability of electric circuits, micro analysis and power supply network optimization in electric engineering; prediction of oil or ore deposits and earthquake in geosciences; stability of structures and structural optimization in civil engineering; and so on. There is hardly any problem that cannot be modeled mathematically if one is vast in modeling. But our goal/objective here is to use mathematical modeling to know the economic impact of HIV/AIDS on transmission from mother to child at birth and during the processes involved and to reduce the effect of this transmission. Other useful works here are the works of (Cooper, (2004); Chukwu and John, (2014) and UNAIDS, 2015)

II. The Model

2.1 Symbols and parameters

N = Total population
S = Susceptible group
I = Infective group
I_C = Incubation group
I_T = Treatment group
A = fully developed AIDS group
λ = Rate of recruitment into susceptible group.
h_i = Average number of sexual partners per unit time, $(i = 1,2,3,4)$
β = Sexual contact rate
γ = Rate of movement from infective group
k_1 = Fraction of babies infected with HIV that die after birth
k_2 = Rate of new babies infected with HIV
μ = Natural mortality rate
μ_A = Death rate due to AIDS
f_1 = fraction of γ moving into the incubation period group
ψ = Rate of movement of incubation period group individuals
Into AIDS group
f_2 = fraction of γ moving to treatment group
ω = Rate at which AIDS group get treatment.
g = Fraction of ψ who get treatment
α = Rate at which treatment group becomes full blow AIDS.
ξ = Rate of movement from the susceptible class to the incubation class.
θ = Rate of movement from the susceptible class to the Treatment class.
a_1 = Programs run by government to stop the spread
a_2 = Government total budget to stop the spread
a_3 = Rate of movement of treatment group with AIDS Sponsored by government from infected group to full blown AIDS

2.2 Flow Diagram of the Model

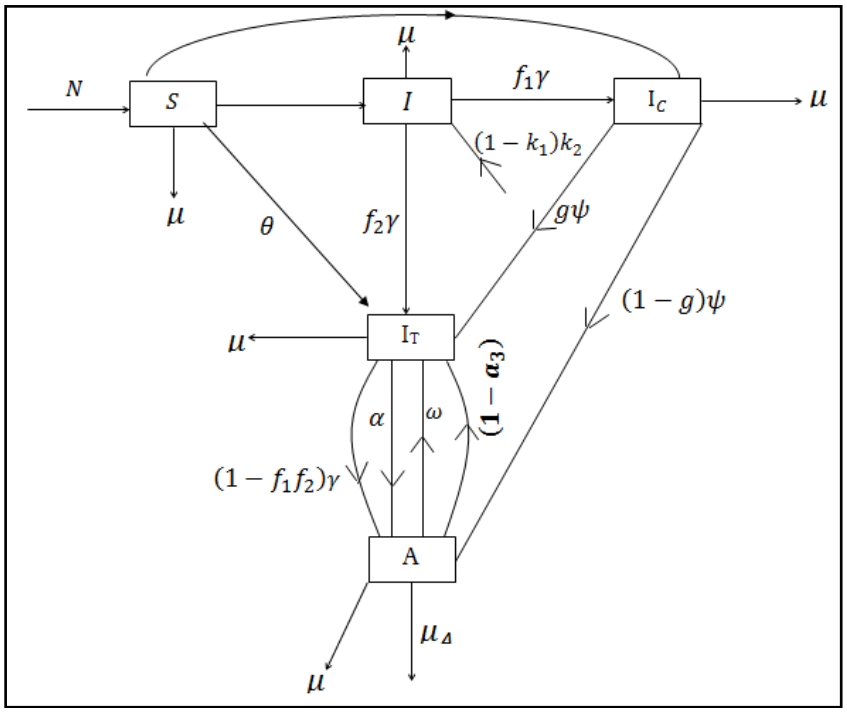

Fig 2.1 Flow Diagram of the Model

From the flowchart, the arrows points to the direction of contact with each compartment as shown in the diagram. Population moving in and going out of the compartments after contact were also shown. Arrows moving out from the compartments completely are death rate due to mortality.

2.3 Assumptions

1. There is no recovered group
2. AIDS individuals compartment were not sexually active and leave the population by AIDS related mortality.
3. The proportion of new recruits allocated to each compartment is held constant over the time period. (To maintain the system)
4. h_2 and h_4 are negligible, hence we ignore them
5. $(1 - k_1)k_2 (I_C + A) = 0$ [Basavarajaiah, et al 2012]
6. It is assumed that people in the incubation period group and those in the full developed AIDS group are exposed and capable of bearing children.
7. The populations (compartments) were closed to migration

2.4 The Model Formulation

Applying the flow diagram, symbols and parameters and the assumptions, we develop the model as follows:

$$\left.\begin{aligned}
\frac{dS}{dt} &= \lambda N - \frac{h_1(\beta_1 - a_1)IS}{N} - \frac{h_3(\beta_3 - a_2)I_C S}{N} - (\mu + \xi + \theta)S \\
\frac{dI}{dt} &= \frac{h_1(\beta_1 - a_1)IS}{N} + \frac{h_3(\beta_3 - a_2)I_C S}{N} - (\gamma + \mu)I + (1 - k_1)k_2 I \\
\frac{dI_C}{dt} &= f_1\gamma + \xi S I_C - (\Psi + \mu)I_C \\
\frac{dI_T}{dt} &= f_2\gamma + \theta S I_T + g\Psi I_C + \omega A - (\alpha + \mu)I_T + (1 - a_3)A \\
\frac{dA}{dt} &= (1 - f_1 - f_2)\gamma + (1 - g)\Psi I_C + \alpha I_T - (\omega - \mu_A + \mu)A - a_3 A
\end{aligned}\right\} (2.1)$$

We normalize the model by substituting (2.2) and (2.3) into (2.1)

$$\left.\begin{aligned}
& s = \frac{S}{N}, i = \frac{I}{N}, i_C = \frac{I_C}{N}, i_T = \frac{I_T}{N}, a = \frac{A}{N} \\
& \Rightarrow sN = S, iN = I, i_C N = I_C, i_T N = I_T, aN = A
\end{aligned}\right\} (2.2)$$

$$\left.\begin{aligned}
N\frac{ds}{dt} + s\frac{dN}{dt} &= \frac{dS}{dt} \Rightarrow \frac{ds}{dt} = \frac{1}{N}\left(\frac{dS}{dt} - s\frac{dN}{dt}\right) \\
N\frac{di}{dt} + i\frac{dN}{dt} &= \frac{dI}{dt} \Rightarrow \frac{di}{dt} = \frac{1}{N}\left(\frac{dI}{dt} - i\frac{dN}{dt}\right) \\
N\frac{di_C}{dt} + i_C\frac{dN}{dt} &= \frac{dI_C}{dt} \Rightarrow \frac{di_C}{dt} = \frac{1}{N}\left(\frac{dI_C}{dt} - i_C\frac{dN}{dt}\right) \\
N\frac{di_T}{dt} + i_T\frac{dN}{dt} &= \frac{dI_T}{dt} \Rightarrow \frac{di_T}{dt} = \frac{1}{N}\left(\frac{dI_T}{dt} - i_T\frac{dN}{dt}\right) \\
N\frac{da}{dt} + a\frac{dN}{dt} &= \frac{dA}{dt} \Rightarrow \frac{da}{dt} = \frac{1}{N}\left(\frac{dA}{dt} - a\frac{dN}{dt}\right)
\end{aligned}\right\} (2.3)$$

The normalization results in the following

$$\left.\begin{aligned}
&\frac{ds}{dt} = -(\mu+\xi+\theta+\lambda-\mu-\theta)s + (\lambda - h_1(\beta_1 - a_1))is - h_3(\beta_3 - a_2)i_c s - (\lambda-\gamma-\mu+k_1-k_1k_2)i - (\lambda-\mu)i_c \\
&-(\lambda+\theta S-\mu)i_T - (\lambda+1-2a_3+\mu_A-\mu)a \\
&\frac{di}{dt} = h_1(\beta_1-a_1)is + h_3(\beta_3-a_2)i_c s - (\gamma+\mu)i + (1-K_1)K_2 i - (\lambda-\mu-\theta)si \\
&-(\lambda-\gamma-\mu+k_1-k_1k_2)i^2 - (\lambda-\mu)i_c i - (\lambda+\theta S-\mu)i_T i - (\lambda+1-2a_3+\mu_A-\mu) \\
&\frac{di_c}{dt} = \begin{bmatrix} (f_1\gamma+\xi s i_c - (\Psi+\mu)i_c) - (\lambda-\mu-\theta)si_c - (\lambda-\gamma-\mu+k_1-k_1k_2)ii_c - (\lambda-\mu){i_c}^2 \\ -(\lambda+\theta S-\mu)i_T i_c - (\lambda+1-2a_3+\mu_A-\mu)ai_c \end{bmatrix} \\
&\frac{di_T}{dt} = \begin{pmatrix} f_2\gamma+\theta S-(\lambda-\gamma-\mu+k_1-k_1k_2)i + (g\Psi-\lambda+\mu)i_c - (\alpha+\mu+\lambda+\theta S-\mu)i_T - \\ (\lambda-\mu-\theta)si_T + (\omega+\lambda+1-2a_3+\mu_A-\mu)a \end{pmatrix} \\
&\frac{da}{dt} = \begin{pmatrix} (1-f_1-f_2)\gamma + (1-g)\Psi i_C + \alpha i_T - (\omega-\mu_A+\mu-a_3)a - (\lambda-\mu-\theta)sa - (\lambda-\gamma-\mu+k_1-k_1k_2)ia \\ -(\lambda-\mu)i_c a - (\lambda+\theta S-\mu)i_T a - (\lambda+1-2a_3+\mu_A-\mu)a^2 \end{pmatrix}
\end{aligned}\right\} \quad (2.4)$$

$Y = (\lambda+1-2a_3+\mu_A-\mu)$ and Z = $(\lambda-\gamma-\mu+k_1-k_1k_2)$

We state two theorems that will aid us in the analysis as follows:

Theorem 2.1(Beltrami, 1983)

The steady state (equilibrium) of $\frac{d\underline{U}}{dt} = A\underline{U}$ is globally asymptotically stable if and only if the real parts of the eigen-values of A are negative and unstable otherwise**.**

Theorem 2.2 (Routh Hurwitz Criteria)

Given the characteristic equation $P(k) = \lambda^k + a_1\lambda^{k-1} + a_2\lambda^{k-2} + \ldots + a_{k-1}\lambda + a_k$

Form the following k Hurwitz matrices

$$H_1 = (a_1),\ H_2 = \begin{pmatrix} a_1 & 1 \\ a_3 & a_2 \end{pmatrix},\ H_3 = \begin{pmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{pmatrix}, \ldots,\ H_k = \begin{pmatrix} a_1 & 1 & 0 & 0 & 0 & 0 & 0 \\ a_3 & a_2 & a_1 & 1 & 0 & 0 & 0 \\ a_5 & a_4 & a_3 & a_2 & 0 & 0 & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & a_k \end{pmatrix}$$

if $a_i = 0\ if\ \ j > n$. The steady state is stable:

$n = 2 : a_1 > 0$ and $a_2 > 0$

$n = 3 : a_1 > 0, a_3 > 0$ and $a_1 a_2 > a_3$

$n = 4 : a_1 > 0, a_3 > 0, a_4 > 0 and\ \ a_1 a_2 a_3 > a_3^2 + a_1^2 a_4$

$n = 5 : a_i > 0, i = 1,2,3.4,5; a_1 a_2 a_3 > a_3^2 + a_1^2 a_4 and$

$$(a_1 a_4 - a_5)(a_1 a_2 a_3 - a_3^2 - a_1^2 a_4) > a_5(a_1 a_2 - a_3)^2 + a_1 a_5^2$$

Proof: See Gantmacher, (1964)

Lemma 2.1 (Invariant Region)

The following biologically-feasible region of the HIV/AIDS model (2.4)

$\Omega = \left\{ (S, I, I_C, I_T, A) \in \mathrm{R}_+^5 : S + I + I_C + I_T + A \le \frac{\lambda}{\mu} \right\}$ is positively –invariant and attracting.

Proof: From model (2.3),

$$S + I + I_c + I_T + A = N \Rightarrow \frac{dN}{dt} = \frac{dS}{dt} + \frac{dI}{dt} + \frac{dI_C}{dt} + \frac{dI_T}{dt} + \frac{dA}{dt}$$

$$\therefore \frac{dN}{dt} = \lambda N - (\mu + \xi + \theta)S - (\gamma + \mu)I + (1 - k_1)k + f_1\gamma + \xi SI_C - (\Psi + \mu)I_C + f_2\gamma + \theta SI_T + g\Psi I_C + \omega A - (\alpha + \mu)I_T$$
$$+ (1 - a_3)A + (1 - f_1 - f_2)\gamma + (1 - g)\Psi I_C + \alpha I_T - (\omega - \mu_A + \mu)A - a_3 A$$

$$\frac{dN}{dt} = \lambda N - (S + I + I_C + I_T + A)\mu - \{(\xi + \theta)S - \gamma I + (1 - k_1)k + \xi SI_C + \theta SI_T + A - a_3 A + \gamma + \mu_A A - a_3 A\}$$

$$\frac{dN}{dt} = \lambda N - \mu N - \{-(\xi + \theta)S + \gamma I - \xi SI_C - \theta SI_T - (1 - 2a_3 + \mu_A)A - (1 - k_1)k - \gamma\}$$

$$\frac{dN}{dt} \leq \lambda N - \mu N$$

$$\Rightarrow \frac{dN}{dt} \leq (\lambda - \mu)N \Rightarrow \frac{dN}{N} \leq (\lambda - \mu)dt$$

Integrating both sides we get,

$$\ln N \leq (\lambda - \mu)t + c \Rightarrow N(t) \leq Ae^{(\lambda - \mu)t}$$

Applying the initial boundary condition

$$N(t) \leq N_0 e^{(\lambda - \mu)t}$$

$$N(t) \geq 0 \ \ whenever \ \ \lambda \geq \mu$$

Thus Ω is positively invariant. Hence, it is sufficient to consider the dynamics of the model (2.4) in Ω. In this region, the model can be considered as been epidemiologically and mathematically well-posed, [8]

III. Positivity of Solution

Theorem 3.1

Let the initial data be $\{s(o) > 0, i(o) > 0, i_c(o) > 0, i_T(o) > 0, a(o) \geq o\} \epsilon \phi$, then the solution set $\{s(t), i(t), i_c(t), i_t(t), a(t)\}$ of the system (2.4) is positive for all $t > 0$.

Proof

From the first equation of the model system (1.3) we have

$$\frac{ds}{dt} = \lambda N - \frac{h_1(\beta_1 - a_1)IS}{N} - \frac{h_3(\beta_3 - a_2)I_c S}{N} - (\mu + \xi + \theta)S \geq -(\mu + \xi + \theta)S.$$

$$\frac{ds}{dt} \geq -(\mu + \xi + \theta)S.$$

Integrating by separation of variables give

$$\int \frac{ds}{s} \geq \int -(\mu + \xi + \theta)dt$$

$$\therefore \ s(\mathrm{t}) \geq s(o)e^{-(\mu + \xi + \theta)t} \geq 0.$$

From the second equation of (2.4) we have

$$\frac{dI}{dt} = \frac{h_1(\beta_1 - a_1)IS}{N} - \frac{h_3(\beta_3 - a_2)I_c S}{N} - (\gamma + \mu)I + (1 - k_1)k_2 I \geq -(\gamma + \mu)I.$$

$$\therefore \ \frac{dI}{dt} \geq -(\gamma + \mu)I.$$

Integrating by separation of variables give

$$\int \frac{dI}{I} \geq -\int (\gamma + \mu)dt$$

$$\therefore I(t) \geq \ I(o)e^{-(\gamma + \mu)t} \geq 0.$$

From the third equation of (2.4) we have

$\frac{dI_c}{dt} = f_1\gamma + \xi SI_c \geq -(\Psi + \mu)I_c \geq -(\Psi + \mu)I_c.$

$\therefore \frac{dI_c}{dt} \geq -(\Psi + \mu)I_c.$

Integrating by separation of variables give

$\int \frac{dI_c}{I_c} \geq -\int (\Psi + \mu)dt$

$\therefore\ I_c(t) \geq I_c(o)\ e^{-(\Psi+\mu)t} \geq 0.$

From the fourth equation of (2.4) we have

$\frac{dI_t}{dt} = f_2\gamma + \theta SI_t + g\Psi I_c + \omega A - (\alpha + \mu)I_t + (1 - a_3)A \geq -(\alpha + \mu)I_t.$

$\frac{dI_t}{dt} \geq -(\alpha + \mu)I_t.$

Integrating by separation of variables gives

$\int \frac{dI_t}{I_t} \geq -\int (\alpha + \mu)dt$

$\therefore\ I_t(t) \geq I_t(o)\ \ e^{-(\alpha+\mu)t} \geq 0.$

And lastly the fifth equation of (2.4) gives

$\frac{dA}{dt} = (1 - f_1 - f_2)\gamma + (1 - g)\Psi I_c + \alpha I_t - (\omega - \mu_A + \mu)A - a_3A \geq -(\omega - \mu_A + \mu + a_3)A.$

$\frac{dA}{dt} \geq -(\omega - \mu_A + \mu + a_3)A.$

Integrating both by separation of variables give

$\int \frac{dA}{A} \geq -\int (\omega - \mu_A + \mu + a_3)dt$

$A(t) \geq A(o)e^{-(\omega-\mu_A+\mu+a_3)t} \geq 0.$

Hence the solution set

$\{s(t), i(t), i_c(t), i_T, a(t)\}$ of the system (2.4) is positive for all $t \geq 0$. ■

Remark 1: $e^k > 0$ for all real values of k.

IV. Stability Analysis

4.1 Disease-free Steady State

The disease-free steady state occurs at the point where $i = i_c = i_T = a = 0$ which is at

$E_0(s^*,0,0,0,0)$

$$\text{where } s^* = \frac{(\lambda - \gamma - \mu + k_1 - k_1k_2)i^* + (\lambda - \mu)i_c^* + (\lambda + \theta S - \mu)i_T^* + (\lambda + 1 - 2a_3 + \mu_A - \mu)a^*}{-(\mu + \xi + \theta + \lambda - \mu - \theta) + (\lambda - h_1(\beta_1 - a_1))i^* - h_3(\beta_3 - a_2)i_c^*}$$

From system (2.4) we define F_i and V_i as

$$F_i = \begin{bmatrix} h_1(\beta_1 - a_1)is & h_3(\beta_3 - a_2)i_cs \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and

$$V_{i=} \begin{bmatrix} -(\gamma + \mu)i + (1 - K_1)K_2i - (\lambda - \mu - \theta)si - (Z)i^2 - (\lambda - \mu)i_ci - (\lambda + \theta S - \mu)i_Ti - Y \\ (f_1\gamma + \xi si_c - (\Psi + \mu)i_c) - (\lambda - \mu - \theta)si_c - (Z)ii_c - (\lambda - \mu)i_c^2 - (\lambda + \theta S - \mu)i_Ti_c - (Y)ai_c \\ f_2\gamma + \theta S - (Z)i + (g\Psi - \lambda + \mu)i_c - (\alpha + \mu + \lambda + \theta S - \mu)i_T - (\lambda - \mu - \theta)si_T + \omega + (Y)a \\ (1 - f_1 - f_2)\gamma + (1 - g)\Psi i_C + \alpha i_T - (\omega - \mu_A + \mu - a_3)a - (\lambda - \mu - \theta)sa - (Z)ia - (\lambda - \mu)i_ca - (\lambda + \theta S - \mu)i_Ta - (Y)a^2 \end{bmatrix}$$

Evaluating the partial derivative of F_i with respect to $(i,\ i_c, i_t, a)$ at the disease free point

$E_0(s^*,0,0,0,0)$

$$F = \begin{bmatrix} h_1(\beta_1 - a_1) & h_3(\beta_3 - a_2) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Similarly, partial differentiation of V_i with respect to (i, i_c, i_t, a) at the disease free equilibrium gives

$$V = \begin{pmatrix} -(\gamma+\mu)+(1-K_1)K_2-(\lambda-\mu-\theta) & 0 & 0 & 0 \\ 0 & \xi-(\Psi+\mu)-(\lambda-\mu-\theta)-Ya & 0 & 0 \\ Z & (g\Psi-\lambda+\mu) & (\alpha+\mu+\lambda+\theta-\mu)-w & Y \\ 0 & -(1-g)\gamma & \alpha i_t & \theta_1-w \end{pmatrix}$$

where $w = (\lambda - \mu - \theta)$

The eigen values of FV^{-1} for the equation $|FV^{-1} - Ie| = 0$ are given by $e^3 = 0$ and $e = \frac{p}{a}$ where $p = h_1(\beta_1 - a_1)$ and $a = \theta - \gamma - \lambda + k_2 - k_1 k_2$.The value of e is equal to R_0. This is gotten from the inverse of V when pre-multiplied by F and Ie subtracted from the result. But,

$$FV^{-1} = \begin{bmatrix} \frac{P}{a^{**}} & \frac{S}{b} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where S = $h_3(\beta_3 - a_2)$ and b = $\xi - (\Psi + \mu) - (\lambda - \mu - \theta) - (Y)a$, P = $h_1(\beta_1 - a_1)$ and $a{**} = \theta - \gamma - \lambda + k_2 - k_1 k_2$.

$$R_0 = \left|FV^{-1} - Ie\right| = \begin{vmatrix} \frac{P}{a^{**}} - e & \frac{S}{b} & 0 & 0 \\ 0 & -e & 0 & 0 \\ 0 & 0 & -e & 0 \\ 0 & 0 & 0 & -e \end{vmatrix} = 0$$

$$R_0 = \frac{h_1(\beta_1 - a_1)}{\theta - \gamma - \lambda + k_2 - k_1 k_2}$$

Hence, the Lemma below:

Lemma 4.1

The disease free steady state $E_0(s^*,0,0,0,0)$ is locally asymptotically stable if $R_0 < 0$ and unstable if $R_0 > 0$

4.2 Endemic steady state

Lemma 4.2: The modified model (2.4) has a unique positive endemic steady state of the form $E_E(s^*, i^*, i_C^*, i_T^*, a^*)$ whenever $R_E > 1$

The endemic steady states are $E_E(s^*, i^*, i_C^*, i_T^*, a^*)$ and we establish its stability. At the steady states, the Jacobian of the system with respect to s, i, i_c, i_t, a is given by

$$J_E = \begin{bmatrix} (\xi+\lambda) & -h_1(\beta_1-a_1)-z & -h_3(\beta_3-a_2)-\lambda+\mu & -(\lambda+\theta S-\mu) & -y \\ 0 & h_1(\beta_1-a_1)+\theta_2 & h_3(\beta_3-a_2) & 0 & 0 \\ 0 & 0 & \xi-(\Psi+\mu)-(\lambda-\mu-\theta) & 0 & 0 \\ \theta & Z & (g\Psi-\lambda+\mu) & -\alpha-\theta-\theta S+\mu & \omega+Y \\ 0 & 0 & (1-g)\Psi & \alpha & -(\omega+\mu_A+\mu-a_3) \end{bmatrix}$$

where $\theta_2 = -(\gamma+\mu)+(1-k)k_2-(\lambda-u)$

Applying Theorem 2.1 on J_E, we have

$$\left|J_E - Ie\right| =$$

$$\begin{vmatrix} (\xi+\lambda)-e & -h_1(\beta_1-a_1)-z & -h_3(\beta_3-a_2)-\lambda+\mu & -(\lambda+\theta S-\mu) & -Y \\ 0 & h_1(\beta_1-a_1)+\theta_2-e & h_3(\beta_3-a_2) & 0 & 0 \\ 0 & 0 & \xi-(\Psi+\mu)-(\lambda-\mu-\theta)-e & 0 & 0 \\ 0 & Z & (g\Psi-\lambda+\mu) & -\alpha-\theta-\theta S+\mu-e & \omega+Y \\ 0 & 0 & (1-g)\Psi & \alpha & -(\omega+\mu_A+\mu-a_3-e \end{vmatrix} = 0$$

where $\theta_2 = -(\gamma+\mu)+(1-k)k_2-(\lambda-u)$

The eigen-values of $\left|J_{E-Ie}\right| = 0$ are

$$e^5 + b_1e^4 + b_2e^3 + b_3e^2 + b_4e + b_5$$

where

$$a_1 = (\xi-(\Psi+\mu)-(\lambda-\mu-\theta)-e+(\xi+\lambda)-e + h_1(\beta_1-a_1)+\theta_2-e-\alpha-\theta-\theta s+\mu-e-(\omega+\mu_A+\mu-a_3)-e)),$$

$a_2 = ((\xi+\lambda)-e)(\xi-(\Psi+\mu)-(\lambda-\mu-\theta)-e)+h_1(\beta_1-a_1)+\theta_2-e)(\xi-(\Psi+\mu)-(\lambda-\mu-\theta)-e)+(\xi-(\Psi+\mu)-(\lambda-\mu-\theta)-e)(-\alpha-\theta-\theta s+\mu-e)+(\xi-(\Psi+\mu)-(\lambda-\mu-\theta)-e)(-(\omega+\mu A+\mu-a3)-e)+(\xi+\lambda-e)(h1(\beta1-a1)+\theta2-e))+(\xi+\lambda-e)(-\alpha-\theta-\theta s+\mu-e)+(\xi+\lambda-e)(-\omega+\mu A+\mu-a3)-e)+(h1(\beta1-a1)+\theta2-e)(-\alpha-\theta-\theta s+\mu-e)+(h1(\beta1-a1)+\theta2-e)(-\omega+\mu A+\mu-a3)-e)+(-\alpha-\theta-\theta s+\mu-e)(-(\omega+\mu_A+\mu-a_3)-e))+(\alpha)(\omega+Y)),$

$a a_3 = (((\xi+\lambda)-e)(h_1(\beta_1-a_1)+\theta_2-e)(\xi-(\Psi+\mu)-(\lambda-\mu-\theta)-e)+((\xi+\lambda)-e)(\xi-(\Psi+\mu)-(\lambda-\mu-\theta)-e)(-\alpha-\theta-\theta s+\mu-e)+((\xi+\lambda)-e)(\xi-(\Psi+\mu)-(\lambda-\mu-\theta)-e)(-(\omega+\mu_A+\mu-a_3)-e))+(h_1(\beta_1-a_1)+\theta_2-e)(\xi-(\Psi+\mu)-(\lambda-\mu-\theta)-e)(-\alpha-\theta-\theta s+\mu-e)+(h_1(\beta_1-a_1)+\theta_2-e)(\xi-(\Psi+\mu)-(\lambda-\mu-\theta)-e)(-(\omega+\mu_A+\mu-a_3)-e))+(\xi-(\Psi+\mu)-(\lambda-\mu-\theta)-e)(-\alpha-\theta-\theta s+\mu-e)(-(\omega+\mu_A+\mu-a_3)-e))+(\xi-(\Psi+\mu)-(\lambda-\mu-\theta)-e)(\alpha)(\omega+Y)+((\xi+\lambda)-e)(h_1(\beta_1-a_1)+\theta_2-e)(-\alpha-\theta-\theta s+\mu-e)+((\xi+\lambda)-e)(h_1(\beta_1-a_1)+\theta_2-e)(-(\omega+\mu_A+\mu-a_3)-e))+((\xi+\lambda)-e)(-\alpha-\theta-\theta s+\mu-e)(-(\omega+\mu_A+\mu-a3)-e)-(\xi+\lambda-e)(\alpha)(\omega+Y)-(h1(\beta1-a1)+\theta2-e)(-\alpha-\theta-\theta s+\mu-e)(-\omega+\mu A+\mu-a3)-e)-(h1(\beta1-a1)+\theta_2-e)(\alpha)(\omega+Y)),$

$$\begin{aligned} a_4 = \Big(&((\xi+\lambda)-e)(h_1(\beta_1-a_1)+\theta_2-e)(\xi-(\Psi+\mu)-(\lambda-\mu-\theta)-e)(-\alpha-\theta-\theta s+\mu-e) \\ &+((\xi+\lambda)-e)(h_1(\beta_1-a_1)+\theta_2-e)(\xi-(\Psi+\mu)-(\lambda-\mu-\theta) \\ &-e)(-(\omega+\mu_A+\mu-a_3)-e))+(\xi+\lambda)-e)(\xi-(\Psi+\mu)-(\lambda-\mu-\theta)-e)(-\alpha-\theta \\ &-\theta s+\mu-e)(-(\omega+\mu_A+\mu-a_3)-e)) \\ &-((\xi+\lambda)-e)(\xi-(\Psi+\mu)-(\lambda-\mu-\theta)-e)(\alpha)(\omega+Y) \\ &+(h_1(\beta_1-a_1)+\theta_2-e)(\xi-(\Psi+\mu)-(\lambda-\mu-\theta)-e)(-\alpha-\theta-\theta s+\mu \\ &-e)(-(\omega+\mu_A+\mu-a_3)-e)) \\ &-(h_1(\beta_1-a_1)+\theta_2-e)(\xi-(\Psi+\mu)-(\lambda-\mu-\theta)-e)(\alpha)(\omega+Y) \\ &+((\xi+\lambda)-e)(h_1(\beta_1-a_1)+\theta_2-e)(-\alpha-\theta-\theta s+\mu-e)-((\xi+\lambda)-e)(h_1(\beta_1 \\ &-a_1)+\theta_2-e)(\alpha)(\omega+Y)), and \end{aligned}$$

$a_5 = ((\xi+\lambda)-e)(h_1(\beta_1-a_1)+\theta_2-e)(\xi-(\Psi+\mu)-(\lambda-\mu-\theta)-e)(-\alpha-\theta-\theta s+\mu-e)(-(\omega+\mu_A+\mu-a_3)-e)).$

Applying Theorem 2.2 (Routh-Hurwitz criterion) for a 5x 5 system, the conditions for stability are: $(i) a_1, a_2, a_3, a_4, a_5 > 0$ and (ii) $a_1a_2a_3 > a_3^2 + a_1^2a_4$

$(iii) a_i > 0, i = 1,2,3.4,5; a_1a_2a_3 > a_3^2 + a_1^2a_4 and$

$(a_1a_4 - a_5)(a_1a_2a_3 - a_3^2 - a_1^2a_4) > a_5(a_1a_2 - a_3)^2 + a_1a_5^2$. From the analysis above (i), (ii) and (iii) are satisfied, hence the endemic steady state is globally asymptotic stable.

Theorem 4.2: The unique positive endemic steady state of the model (2.4), given by $E_E(s^*, i^*, i_C^*, i_T^*, a^*)$ is GAS in $\Omega_r \setminus \Omega_1$ if $R_E > 1$.

Proof:

V. Numeric Simulation Explanation

Numeric simulation is a very important aspect of dynamical system analysis in the study of non-linear differential equations whose exact solutions are very difficult to determine. As a result of this, we resort to numerical simulation in order to get an idea of the behavior of the model as time moves or progresses.
In this section we perform a numerical simulation of the model using the MATLAB, and the RUNGE-KUTTA method (algorithm) which aided in the running of the program.

The Table 1.1 contains the initial values set for the numerical simulations. These values were chosen to achieve numeric stability and the values of the b_1, b_2, b_3, b_4, b_5 calculated guided us in the selection.

Table 5.1 Parameter values for the simulation

Parameter	Value	References	Parameter	Value	References
X (1)	0.0	Lakshmikantham(1989	f1	0.2	Obasi (2013)
Y(1)	0.10	Basavarajaiah, et al 2012	f2	0.01	Assumed
Z(1)	0.05	Basavarajaiah, et al 2012	K	0.08	Obasi (2013)
P(1)	0.52	Assumed	Qq	0.9	Assumed
hh(1)	0.07	Assumed	C1	0.03	Basavarajaiah, et al 2012
a(1)	0.01	Assumed	C2	2	Lakshmikantham(1989
H	0.1	Assumed	M	0.4	Assumed
Q	0.3	Basavarajaiah, et al 2012	Pi	0.0	Lakshmikantham(1989
E	0.20	Assumed	Sm	0.6	Obasi (2013)
V	0.1	Assumed	Af	2	Lakshmikantham(1989
b1	0.4	Basavarajaiah, et al 2012	a1	0.3	Assumed
a2	2.0	Lakshmikantham(1989	Zy	V-C2+C1 * C2	Equation (4.2)
a3	0.5	Obasi (2013)	Yy	1-W-Va-u-2* a3	Equation (4.2)
d1	0.7	Obasi (2013)			
d3	1.0	Lakshmikantham(1989			
Va	0.005	Assumed			
W	0.7	Assumed			
U	0	Lakshmikantham(1989			
b3	0.005	Assumed			

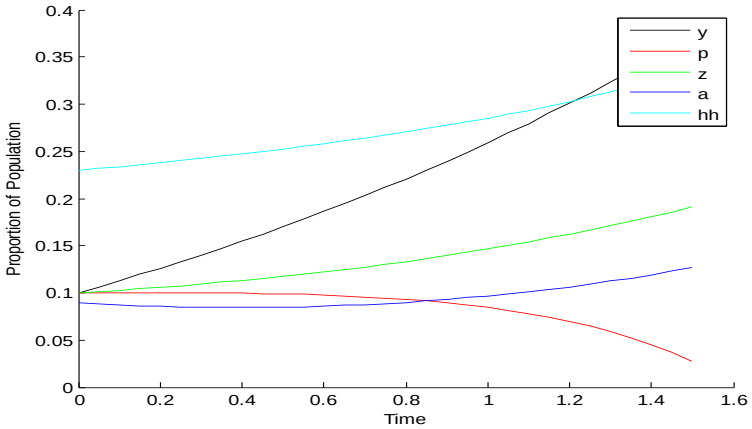

Fig 5.1: Behavior of the proportion of the populations over time.

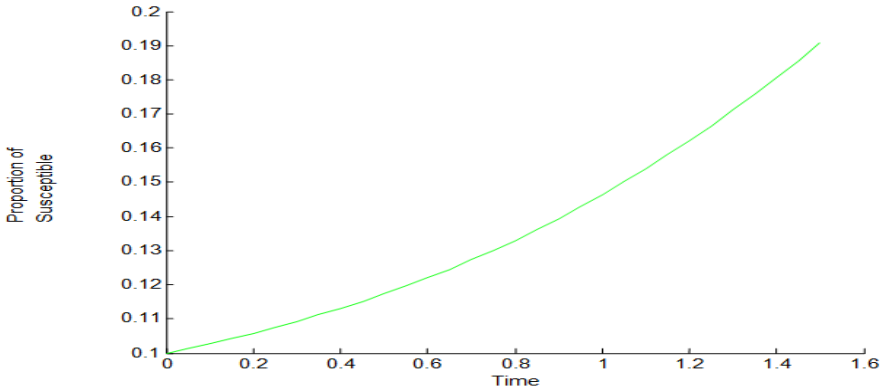

Fig 5.2: The plot of susceptible proportion against time

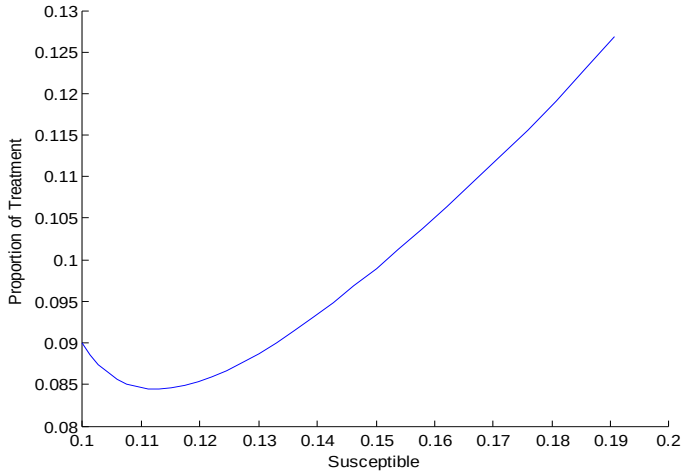

Fig 5.3: The plot of treated proportion against susceptible proportion

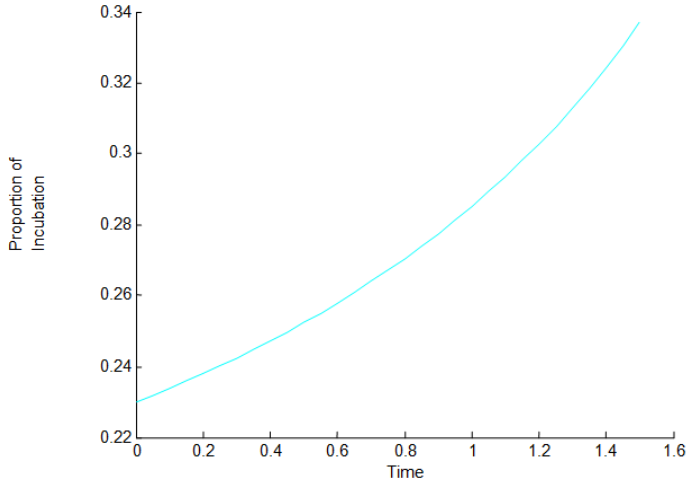

Fig. 5.4: The plot of incubation proportion against time

VI. Summary, Conclusion/Results and Recommendations

6.1 Summary

We were able to formulate the Mathematical Model of HIV/AIDS on a varying population and this resulted in five Compartments: Susceptible, Infective, Infected but in Incubation, Treated and Fully developed AIDS. The model was analyzed using dynamical systems theory. It was found that the model has two steady states- the disease- free and endemic steady states. We also determined their stability. The positivity of the solution of the model as investigated.

The solution set $\{s(t), i(t), i_c(t), i_T, a(t)\}$ of the system (2.4) is positive for all $t \geq 0$. We also proved that $\Omega = \left\{(S, I, I_C, I_T, A) \in \mathrm{R}_+^5 : S + I + I_C + I_T + A \leq \frac{\lambda}{\mu}\right\}$ is positively-invariant and attracting. A simulation (numerical solution) was carried out using MATLAB. This resulted in plotting four graphs. The graphs plotted were behavior of the proportion of the populations over time (Fig. 5.1), the plot of susceptible proportion against time (Fig. 5.2), the plot of treated proportion against susceptible proportion (Fig. 5.3) and the plot of incubation proportion against time (Fig. 5.4).

6.2 Conclusion/Results

From the dynamical systems theory analysis carried out, the model has two nonnegative equilibra which are disease-free equilibrium, $E_0(s^*, 0, 0, 0)$, and endemic equilibrium $E_E(s^*, i^*, i_c^*, i_T^*, a^*)$. It was found that the disease-free equilibrium, $E_0(s^*, 0, 0, 0)$, is locally asymptotically stable if $R_0 < 1$ but it is

unstable if $R_0 > 1$ and the infection is maintained in the population. The Endemic equilibrium, $E_E(s^*, i^*, i_c^*, i_T^*, a^*)$, is locally asymptotically stable.

The plot of proportion of the populations of $s^*, i^*, i_c^*, i_T^*, a^*$ over time (Fig. 5.1) shows the behaviour of the five compartments over time. The graphs reveal that while other populations are increasing at different rates, the infected group is decreasing. The plot of susceptible proportion against time (Fig. 5.2) shows that the susceptible population increases over time. The plot of treated group against the susceptible group (Fig. 5.3) shows a decrease for a little time, it got to its minimum and started increasing again. The plot of incubation group against time (Fig. 5.4) shows an increase over time, in-fact, it looks like a parabola in the positive first quadrant.

6.3 Recommendations

AIDS has the potential to cause severe deterioration in the economic conditions on many countries. However, this is not inevitable. There is much that can be done now to keep the epidemic from getting worse and to mitigate the negative effects. Among the responses that are necessary are:

1. Prevent new infections: The most effective response will be to support programs to reduce the number of new infections in the future. After more than decade of research and pilot programs, we now know how to prevent most new infections. An effective national response should include information, education and availability, expended and improved services to prevent and treat sexually transmitted diseases; and efforts to protect human rights and reduce stigma and discrimination. Governments, NGOs and the commercial sector, working together in a multi-sectoral effort can make a difference. Workplace-based programs can prevent new infections among experienced workers.
2. Design major development project appropriately: Some major development activities may inadvertently facilitate the spread of HIV. Major construction projects often require large numbers of male workers to live apart from their families for extended periods of time, leading to increased opportunities for commercial sex. A World Bank funded pipeline construction project in Cameroon was redesigned to avoid this problem by creating special villages where workers could live with their families. Special prevention programs can be put in place from the very beginning in projects such as mines or new ports where commercial sex might be expected to flourish.
3. Programs to address specific problems. (Centre for Development and Population Activities)(CEDPA) Special programs can mitigate the impact of AIDS by addressing some of the most severe problems. Reduced school fees or free education can help children from poor families and AIDS, orphans stay in school longer and avoid deterioration in the education level of the workforce. Tax benefits or other incentives for training can encourage firms to maintain worker productivity in spite of the loss of experienced workers.
4. Mitigate the effect of AIDS on poverty: The impacts of AIDS on households can be reduced to some extent by publicly funded programs to address the most severe problems. Such programs have included home care for people with HIV/AIDS support for the basic needs of the household coping with AIDS, foster care for AIDS orphans, food programs for children and support for educational expenses. Such programs can help families and particularly children survive some of the consequences of an adult AIDS death that occur when families are poor or become poor as a result of the cost of AIDS.

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