IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 12, Issue 6 Ver. III (Nov. - Dec.2016), PP 19-31
www.iosrjournals.org

Continued fraction expansion of the relative operator entropy and the Ts all is relative entropy

Ali Kacha*, Brahim Ounir & Salah Salhi

Abstract: *The aim of this paper is to provide some results and applications of continued fractions with matrix arguments. First, we recall some properties of matrix functions with real coefficients. Afterwards, we give a continued fraction expansion of the relative operator entropy and for the Ts all is relative operator entropy. At the end, we study some metrical equations.*

Keywords: *Continued fraction expansion, positive definite matrix, relative operator entropy, Ts all is relative operator entropy.*
AMS Classification Number: 40A15; 15A60; 47A63.

I. Introduction and Motivation

Over the last two hundred years, the theory of continued fractions has been a topic of extensive study. The basic idea of this theory over real numbers is to give an approximation of various real numbers by the rational ones. One of the main reasons why continued fractions are so useful in computation is that they often provide representation for transcendental functions that are much more generally valid than the classical representation by, say, the power series. Further; in the convergent case, the continued fractions expansions have the advantage that they converge more rapidly than other numerical algorithms. Recently, the extension of continued fractions theory from real numbers to the matrix case has seen several developments and interesting applications (see [6],[8], [13]). The real case is relatively well studied in the literature. However, in contrast to the theoretical importance, one can nd in mathe- matical literature only a few results on the continued fractions with matrix arguments. There have been some reasons why all this attention has been devoted to what is, in essence, a very humble idea. Since calculations involving matrix valued functions with matrix arguments are feasible with large computers, it will be an interesting attempt to develop such matrix theory.

The main difficulty arises from the fact that the algebra of square matrices is not commutative.

In 1850, Clausius, introduced the notion of entropy in thermodynamics. Since then several extensions and reformulations have been developed in various disciplines [11,12,14,15]. There have been investigated the so-called entropy inequalities by some mathematicans, see [2,3,10] and references therein. A relative operator entropy of strictly positive operators A, B was introduced in non commutative information theory by Fujii and Kamei [9] by

$$S(A|B) = A^{1/2}\ln(A^{-1/2}BA^{-1/2})A^{1/2},$$

as a generalization of the operator entropy

$$H(A) = S(A|I) = -A\ln A.$$

In the present paper, we also study a parametric extension of the relative operator entropy which is called Tsallis relative operator entropy. It is firstly introduced in [18] in the following manner.

Definition 1.1 For two invertible positive operators A and B on Hilbert space, and any real number $\lambda \in]0,1[$, the Tsallis relative operator entropy is defined by

$$T_\lambda(A|B) \equiv \frac{A^{1/2}(A^{-1/2}BA^{-1/2})^\lambda A^{1/2} - A}{\lambda}.$$

For simplicity and clearness, we restrict ourselves to positive definite matrices, but our results can be, without special difficulties, projected to the case of positive definite operators from an infinite dimensional Hilbert space into itself.

This article is organized as follows: The section 2 contains some basic notions and results about matrix continued fractions that are needed later. In Section 3, we give a continued fractions expansion of the relative operator entropy and the Tsallis relative operator entropy. However, the last result of this paper is devoted to provide the solution of a matrix algebraic equation.

II. Preleminary an notations

The functions of matrix arguments play a widespreased role in science and engineering, with applications areas ranging from nuclear magnetic resonance [1]. So for any scalar polynomial $p(z) = \sum_{i=0}^{k} \alpha^i z^i$ gives rise to a matrix polynomial with scalar coefficients by simply substituting A^i for z^i :

$$P(A) = \sum_{i=0}^{k} \alpha^i A^i$$

More generally, for a function f with a series representation on an open disk containing the eigenvalues of A, we are able to define the matrix function $f(A)$ via the Taylor series for f [7].

Alternatively, given a function f that is analytic inside a closed contour Γ which encloses the eigenvalues of A, $f(A)$ can be defined, by analogy with Cauchy's integral theorem by

$$f(A) = \frac{1}{2\pi i} \int_{\Gamma} f(z)(zI - A)^{-1} dz.$$

The definition is known as the matrix version of Cauchy's integral theorem. Let $\mathcal{M}_m$ be the algebra of real square matrices, we now mention an important result of matrix functions.

Lemma 2.1 *Let f be an analytic function in a domain D.*
(i) If two matrices $A \in \mathcal{M}_m$ and $B \in \mathcal{M}_m$ are similar, with $A = ZBZ^{-1}$, and $sp(A) \subset D$, then the matrices $f(A)$ and $f(B)$ are also similar, with $f(A) = Zf(B)Z^{-1}$.
(ii) If $A \in \mathcal{M}_m$ is a block diagonal matrix $A = diag(A_1, A_2, ..., A_r)$ then $f(A) = diag(f(A_1), f(A_2), ..., f(A_r))$.

Proof. Its proof is obvious.

Let $A \in \mathcal{M}_m$, A is said to be positive semidefinite (resp. positive definite) if A is symmetric and

$$\forall x \in \mathbb{R}^m, \ < Ax, x > \geq 0 \quad (\text{ resp. } \forall x \in \mathbb{R}^m, \ x \neq 0 \ \ < Ax, x >> 0)$$

where $< .,. >$ denotes the standard scalar product of $\mathbb{R}^m$.
We observe that positive semidefiniteness induces a partial ordering on the space of symmetric matrices: if A and B are two symmetric matrices, we write $A \leq B$ if $B - A$ is positive semidefinite.
Henceforth, whenever we say that $A \in \mathcal{M}_m$ is positive semidefinite (or positive definite), it will be assumed that A is symmetric.

For any matrices A,$B \in \mathcal{M}_m$ with B invertible, we write $A/B = B^{-1}A$, in particular, if A=I, the matrix identity, then $I/B = B^{-1}$. It is easy to verify that for any invertible matrix X we have

$$\frac{A}{B} = \frac{XA}{XB} \neq \frac{AX}{BX}.$$

Definition 2.2 Let $\{A_n\}_{n\geq 0}$ and $\{B_n\}_{n\geq 1}$ be two sequences of matrices in $\mathcal{M}_m$. We denote the continued fraction expansion by

$$A_0 + \cfrac{B_1}{A_1 + \cfrac{B_2}{A_2 + \ldots}} := \left[A_0; \frac{B_1}{A_1}, \ldots, \frac{B_n}{A_n}\right].$$

Sometimes, we denote this continued fraction by $\left[A_0; \frac{B_n}{A_n}\right]_{n=1}^{+\infty}$ or $A_0 + K\left(B_n/A_n\right)$.
The fractions $\frac{B_n}{A_n}$ and $\frac{P_n}{Q_n} := \left[A_0; \frac{B_i}{A_i}\right]_{i=1}^{n}$ are called, respectively, the n^{th} partial quotient and the n^{th} convergent of the continued fraction $A_0 + K\left(B_n/A_n\right)$.

We note that the evaluation of n^{th} convergent according to the definition 2.1 is not practical because we have to repeatedly inverse matrices. The following proposition gives an adequate method to calculate $A_0 + K\left(B_n/A_n\right)$.

Proposition 2.3 [16]. *For the continued fraction $A_0 + K\left(B_n/A_n\right)$, define*

$$\begin{cases} P_{-1} = I, \ P_0 = A_0 \\ Q_{-1} = 0, \ Q_0 = I \end{cases} \quad \textit{and} \quad \begin{cases} P_n = A_n \ P_{n-1} + B_n P_{n-2} \\ Q_n = A_n \ Q_{n-1} + B_n Q_{n-2} \end{cases} \quad n \geq 1. \qquad (2.1)$$

Then $Q_n^{-1}P_n$ is the n^{th} convergent of the continued fraction $A_0 + K\left(B_n/A_n\right)$

The proof of the next proposition is elementary and we left it to the reader.

Proposition 2.4 *For any two matrices C and D with C invertible, we have*

$$C\left[A_0; \frac{B_k}{A_k}\right]_{k=1}^{n} D = \left[CA_0D; \frac{B_1 D}{A_1 C^{-1}}, \frac{B_2 C^{-1}}{A_2}, \frac{B_k}{A_k}\right]_{k=3}^{n}.$$

Definition 2.5 Let $\{A_n\}, \{B_n\}$ $\{C_n\}$ and $\{D_n\}$ be four sequences of matrices. We say that the continued fractions $A_0+K(B_n/A_n)$ and $C_0+K(D_n/C_n)$ are equivalent if we have $F_n = G_n$ for all $n \geq 1$, where F_n and G_n are the n^{th} convergents of $A_0 + K(B_n/A_n)$ and $C_0 + K(D_n/C_n)$ respectively.

In order to simplify the statements on some partial quotients of continued fractions with matrices arguments, we need the following proposition which is an example of equivalent continued fractions.

Proposition 2.6 *Let* $\left[A_0; \dfrac{B_k}{A_k}\right]_{k=1}^{+\infty}$ *be a given continued fraction. Then*

$$\frac{P_n}{Q_n} = \left[A_0; \frac{B_k}{A_k}\right]_{k=1}^{n} = \left[A_0; \frac{X_k B_k X_{k-2}^{-1}}{X_k A_k X_{k-1}^{-1}}\right]_{k=1}^{n},$$

where $X_{-1} = X_0 = I$ *and* $X_1, X_2, ..., X_n$ *are arbitrary invertible matrices.*

Proof. Let $\frac{P_n}{Q_n}$ and $\frac{\widetilde{P_n}}{\widetilde{Q_n}}$, be the nth convergents of the continued fractions $\left[A_0; \frac{B_k}{A_k}\right]_{k=1}^{+\infty}$ and $\left[A_0; \frac{X_k B_k X_{k-2}^{-1}}{X_k A_k X_{k-1}^{-1}}\right]_{k=1}^{+\infty}$ respectively. By proposition 2, for all $n \geq 1$, we can write

$$\widetilde{P_n} = X_n A_n X_{n-1}^{-1} \widetilde{P_{n-1}} + X_n B_n X_{n-2}^{-1} \widetilde{P_{n-2}},$$

which is equivalent to

$$X_n^{-1} \widetilde{P_n} = A_n (X_{n-1}^{-1} \widetilde{P_{n-1}}) + B_n (X_{n-2}^{-1} \widetilde{P_{n-2}}).$$

This last result joined to the initial conditions prove that for all $n \geq 1$, $X_n^{-1} \widetilde{P_n} = P_n$.
A similar result can be obtained for Q_n. Consequently, both continued fractions have the same convergents and the proof of proposition 2.6 follows. We also recall the following proposition in real case.

Proposition 2.7 *Let* (r_n) *be a non-zero sequence of real numbers. We prove easily that the following continued fractions*

$$\left[a_0; \frac{b_1}{a_1}, \frac{b_2}{a_2}, ..., \frac{b_n}{a_n}, ...\right] \quad \text{and} \quad \left[a_0; \frac{r_1 b_1}{r_1 a_1}, \frac{r_2 r_1 b_2}{r_2 a_2}, .., \frac{r_{n-1} r_n b_n}{r_n a_n}, ...\right]$$

are equivalent.

Definition 2.8 (Contraction of a continued fraction)
Let B_n, A_n and f_n denote the n^{th} numerator, denominator and approximant, respectively of a continued fraction $a_0 + K(b_n/a_n)$ and we let D_n, C_n and g_n denote the n^{th} numerator, denominator and approximant, respectively of a continued fraction $c_0 + K(d_n/c_n)$. Then $c_0 + K(d_n/c_n)$ is called an even contraction or even part of $a_0 + K(b_n/a_n)$ if and only if

$$g_n = f_{2n} \quad \text{for all} \quad n \geq 1.$$

Proposition 2.9 [13] **i)** *The even canonical contraction of* $a_0 + K(b_n/a_n)$ *is given by*

$$\left[c_0; \frac{d_1}{c_1}, \frac{d_2}{c_2}, \frac{d_n}{c_n}\right]_{n=3}^{+\infty} =$$

$$\left[a_0; \frac{b_1 a_2}{a_1 a_2 + b_2}, \frac{-b_2 b_3 a_4}{(a_2 a_3 + b_3) a_4 + a_2 b_4}, \frac{-b_{2n-2} b_{2n-1} a_{2n-4} a_{2n}}{(a_{2n-2} a_{2n-1} + b_{2n-1}) a_{2n} + a_{2n-2} b_{2n}}\right]_{n=3}^{+\infty}.$$

ii) *The odd canonical contraction of* $a_0 + K(b_n/a_n)$ *is given by*

$$\left[c_0; \frac{d_1}{c_1}, \frac{d_2}{c_2}, \frac{d_n}{c_n}\right]_{n=3}^{+\infty} =$$

$$\left[\frac{a_0 a_1 + b_1}{a_1}; \frac{-b_1 b_2 a_3 / a_1}{b_2 a_3 + a_1 (b_3 + a_2 a_3)}, \frac{-b_{2n-1} b_{2n} a_{2n+1} a_{2n-3}}{b_{2n} a_{2n+1} + a_{2n-1} (b_{2n+1} + a_{2n} a_{2n+1})}\right]_{n=2}^{+\infty}.$$

We end this section by introducing some topological notions of continued fractions with matrix arguments. We provide $\mathcal{M}_m$ with the topology induced by the following classical norm:

$$\forall A \in \mathcal{M}_m, \ ||A|| = Sup_{x \neq 0} \frac{|Ax|}{|x|} = Sup_{|x|=1} |Ax|.$$

The continued fraction $\left[A_0, \frac{B_k}{A_k}\right]_{k=1}^{+\infty}$ is said to be convergent in $\mathcal{M}_m$ if the sequence $(F_n)_n = (P_n/Q_n)_n = (Q_n^{-1} P_n)_n$ converges in $\mathcal{M}_m$ in the sense that there exists a matrix $F \in \mathcal{M}_m$ such that
$\lim_{n \to +\infty} ||F_n - F|| = 0$.

III. Main Result

Our aim in this section is to give the continued fraction expansions of the relative operator entropy and of the Tsallis relative operator entropy for two invertible and positive definite matrices A and B.

3.1 Continued fractions expansion of a relative operator entropy.

For simplicity, we start with the real case and we begin by recalling La- guerre's continued fraction of ln x; where x is a strictly positive real number in the following lemma.

Lemma 3.1.1 *Let x be a real number such that $x > 0$. A continued fraction expansion of $\ln x$ is given by :*

$$\ln x = \left[0; \frac{2\left(\frac{x-1}{x+1}\right)}{1}, \frac{-\left(\frac{x-1}{x+1}\right)^2}{3}, \frac{-2^2\left(\frac{x-1}{x+1}\right)^2}{5}, \frac{-n^2\left(\frac{x-1}{x+1}\right)^2}{2n+1}\right]_{n=3}^{+\infty}. \tag{3.1}$$

Proof of lemma 3.1.1 Let z be a real number such $|z| < 1$, We know that (see [13]) a continued fraction expansion of $\ln(1+z)$ is

$$\ln(1+z) = \left[0; \frac{z}{1}, \frac{a_n z}{1}\right]_{n=2}^{+\infty},$$

where for $k \geq 1$

$$\begin{cases} a_{2k} = \dfrac{k}{2(2k-1)}, \\ a_{2k+1} = \dfrac{k}{2(2k+1)}. \end{cases}$$

The even canonical contraction of a previous continued fraction expansion of $\ln(1+z)$ bellow is given by

$$\ln(1+z) = \left[0; \frac{2z}{2+z}, \frac{-1^2 z^2}{3(2+z)}, \frac{-n^2 z^2}{(2n+1)(2+z)}\right]_{n=2}^{+\infty}. \tag{3.2}$$

Let z be a real number such that $|z| < 1$, according to the relationship (3.2), we have

$$\ln\left(\frac{1+z}{1-z}\right) = \left[0; \frac{2z}{1}, \frac{-1^2 z^2}{3}, \frac{-n^2 z^2}{2n+1}\right]_{n=2}^{+\infty}. \tag{3.3}$$

Let x be a real number such that $x > 0$, in order to conclude the proof it suffices to put $z = \frac{x-1}{x+1}$ which is equivalent to $x = \frac{1+z}{1-z}$.

The next lemma is a matrix version of the previous lemma 3.1.1.

Lemma 3.1.2 *Let $A \in \mathcal{M}_m$ be a positive definite matrix. Then a continued fraction expansion of $\ln(A)$ is*

$$\ln(A) = \left[0; \frac{2\left(\frac{A-I}{A+I}\right)}{I}, \frac{-\left(\frac{A-I}{A+I}\right)^2}{3I}, \frac{-2^2\left(\frac{A-I}{A+I}\right)^2}{5I}, \frac{-n^2\left(\frac{A-I}{A+I}\right)^2}{(2n+1)I}\right]_{n=3}^{+\infty}.$$

Now we establish a main theorem which gives a continued fraction expansions of the relative operator entropy $S(A|B)$.

Theorem 3.1.3 *Let A and B be two invertible and positive definite matrices in $\mathcal{M}_m$. A continued fraction expansion of the relative operator entropy $S(A|B)$ is given by*

$$S(A|B) = \left[0; \frac{2A\left(\frac{B-A}{B+A}\right)}{I}, \frac{-A\left(\frac{B-A}{B+A}\right)^2 A^{-1}}{3I}, \frac{-2^2 A\left(\frac{B-A}{B+A}\right)^2 A^{-1}}{5I}, \frac{-n^2 A\left(\frac{B-A}{B+A}\right)^2 A^{-1}}{(2n+1)I}\right]_{n=3}^{+\infty}. \tag{3.4}$$

Proof of lemma 3.1.2 Let $A \in \mathcal{M}_m$ be a positive definite matrix. Then there exists an invertible matrix X such that $A = XDX^{-1}$, where $D = diag(\lambda_1, \lambda_2, ..., \lambda_m)$ and $\lambda_i > 0$.
As the function $z \to \ln(z)$ is analytic in the open halfplane $\{z \in \mathbb{C},\ Re(z) > 0\}$, then

$$\ln(A) = X\ (\ln D)\ X^{-1} = X\ diag(\ln(\lambda_1), \ln(\lambda_2), ..., \ln(\lambda_m))\ X^{-1}.$$

Let us define the sequences $\{P_n\}$ and $\{Q_n\}$ by :

$$\begin{cases} P_{-1} = I, P_0 = 0, P_1 = 2\phi(D) \\ Q_{-1} = O, Q_0 = I, Q_1 = I \end{cases}$$

and for $n \geq 2$,

$$\begin{cases} P_n = (2n+1)P_{n-1} - n^2(\phi(D))^2 P_{n-2}, \\ \\ Q_n = (2n+1)Q_{n-1} - n^2(\phi(D))^2 Q_{n-2}, \end{cases}$$

where $\phi(D) = \frac{D-I}{D+I}$.

We see that P_n and Q_n are diagonal matrices. By setting $p_n = diag(p_n^1, p_n^2, ..., p_n^m)$ and $q_n = diag(q_n^1, q_n^2, ..., q_n^m)$, we obtain for each i where $1 \leq i \leq m$,

$$\begin{cases} p_{-1}^i = 1,\ p_0^i = 0,\ p_1^i = 2\phi(\lambda_i) \\ q_{-1}^i = 0,\ q_0^i = 1,\ q_1^i = 1 \end{cases}$$

and for $n \geq 2$,

$$\begin{cases} p_n^i = (2n+1)p_{n-1}^i - n^2(\phi(\lambda_i))^2 p_{n-2}^i, \\ q_n^i = (2n+1)q_{n-1}^i - n^2(\phi(\lambda_i))^2 q_{n-2}^i. \end{cases}$$

By lemma 3.1.1, the convergent (p_n^i/q_n^i) converges to $\ln \lambda_i$. It follows that P_n/Q_n converges to the matrix $\ln(D)$, so that

$$\ln D = \left[0; \frac{2\phi(D)}{I}, \frac{-1^2(\phi(D))^2}{3I}, \frac{-n^2(\phi(D))^2}{(2n+1)I}\right]_{n=2}^{+\infty}.$$

By proposition 2.4, we get

$$\begin{aligned} \ln A = X(\ln D)X^{-1} &= X\left[0; \frac{2\phi(D)}{I}, \frac{-1^2(\phi(D))^2}{3I}, \frac{-n^2(\phi(D))^2}{(2n+1)I}\right]_{n=2}^{+\infty} X^{-1} \\ &= \left[0; \frac{2\phi(D)X^{-1}}{X^{-1}}, \frac{-1^2(\phi(D))^2X^{-1}}{3I}, \frac{-n^2(\phi(D))^2}{(2n+1)I}\right]_{n=2}^{+\infty}. \end{aligned}$$

Let us define the sequence $(X_n)_{n\geq -1}$ by

$$\begin{cases} X_{-1} = X_0 = I, \\ X_n = X, \text{ for } n \geq 1. \end{cases}$$

Then

$$\begin{cases} \dfrac{X_1B_1X_{-1}^{-1}}{X_1A_1X_0^{-1}} = \dfrac{2X_1\phi(D)X^{-1}X_{-1}^{-1}}{X_0X^{-1}X_0^{-1}} = \dfrac{2\phi(A)}{I}, \\ \dfrac{X_2B_2X_0^{-1}}{X_2A_2X_1^{-1}} = \dfrac{X_2(-1^2(\phi(D)^2))X^{-1})X_0^{-1}}{X_2(3I)X_1^{-1}} = \dfrac{-1^2(\phi(A))^2}{3I}. \end{cases}$$

For $n \geq 3$, we have

$$\frac{X_nB_nX_{n-2}^{-1}}{X_nA_nX_{n-1}^{-1}} = \frac{-(n-1)^2(\phi(A))^2}{(2n-1)I}.$$

By applying the result of proposition 2.6 to the sequence $(X_n)_{n\geq -1}$, we finish the proof of lemma 3.1.2

Proof of theorem 3.1.3 Let A and B be two invertible and positive definite matrices in $\mathcal{M}_m$. In order to apply lemma 3.1.2, we have

$$\begin{aligned} \phi(A^{-1/2}BA^{-1/2}) = \frac{A^{-1/2}BA^{-1/2} - I}{A^{-1/2}BA^{-1/2} + I} &= \frac{A^{-1/2}(B-A)A^{-1/2}}{A^{-1/2}(B-A)A^{-1/2}}. \\ &= A^{1/2}\frac{B-A}{B+A}A^{-1/2}. \end{aligned}$$

So,

$$(\phi(A^{-1/2}BA^{-1/2}))^2 = A^{1/2}\left(\frac{B-A}{B+A}\right)^2 A^{-1/2}.$$

Then, according to lemma 3.1.2, we obtain

$$\ln(A^{-1/2}BA^{-1/2}) = \left[0; \frac{2A^{1/2}\left(\frac{B-A}{B+A}\right)A^{-1/2}}{I}, \frac{-A^{1/2}\left(\frac{B-A}{B+A}\right)^2A^{-1/2}}{3I}, \frac{-n^2A^{1/2}\left(\frac{B-A}{B+A}\right)^2A^{-1/2}}{(2n+1)I}\right]_{n=2}^{+\infty}.$$

Due to proposition 2.4, we deduce that

$$S(A|B) = A^{1/2}\ln(A^{-1/2}BA^{-1/2})A^{1/2} =$$

$$\left[0; \frac{2A^{1/2}\left(\frac{B-A}{B+A}\right)}{A^{-1/2}}, \frac{-A^{1/2}\left(\frac{B-A}{B+A}\right)^2A^{-1}}{3I}, \frac{-n^2A^{1/2}\left(\frac{B-A}{B+A}\right)^2A^{-1/2}}{(2n+1)I}\right]_{n=2}^{+\infty}.$$

In order to achieve the proof of theorem 3.1.3, let us take

$$\begin{cases} X_{-1} = X_0 = I, \\ X_n = A^{1/2}, \ \forall n \geq 1. \end{cases}$$

Thanks to proposition 2.6, we see that

$$S(A|B) = \left[0; \frac{2A\left(\frac{B-A}{B+A}\right)}{I}, \frac{-A\left(\frac{B-A}{B+A}\right)^2 A^{-1}}{3I}, \frac{-n^2 A\left(\frac{B-A}{B+A}\right)^2 A^{-1}}{(2n+1)I}\right]_{n=2}^{+\infty}.$$

3.1.4 Examples of applications

This sections is devoted to illustrate our above theoretical result (3.4) with some examples.

Example 1. Consider the matrix A such that

$$A = \begin{pmatrix} 80/99 & -8/99 \\ -8/99 & 80/99 \end{pmatrix}.$$

A is a diagonal matrices and we have $A = PDP^{-1}$ where

$$P = \begin{pmatrix} 80/99 & -8/99 \\ -8/99 & 80/99 \end{pmatrix}, D = \begin{pmatrix} 8/9 & 0 \\ 0 & 8/9 \end{pmatrix}.$$

The exact valeur of $S(A|I) = -A\ln(A)$ is

$$\begin{aligned} S(A|I) &= \begin{pmatrix} -(4/9)\ln(8/9) - (4/11)\ln(8/11) & (4/9)\ln(8/9) - (4/11)\ln(8/11) \\ (4/9)\ln(8/9) - (4/11)\ln(8/11) & -(4/9)\ln(8/9) - (4/11)\ln(8/11) \end{pmatrix} \qquad (3.5) \\ &= \begin{pmatrix} 0.168149372 & 0.06345334 \\ 0.06345334 & 0.168149372 \end{pmatrix}. \end{aligned}$$

Applying the theorem 3.1.3, the first convergents of $S(A|B)$ are given by:

$$F_1 = \begin{pmatrix} 19/99 & 8/99 \\ 8/99 & 19/99 \end{pmatrix} = \begin{pmatrix} 0.191919191 & 0.08080808 \\ 0.08080808 & 0.191919191 \end{pmatrix},$$

$$F_2 = \begin{pmatrix} 5344/31977 & 2000/31977 \\ 2000/31977 & 5344/31977 \end{pmatrix} = \begin{pmatrix} 0.167120117 & 0.062544954 \\ 0.062544954 & 0.167120117 \end{pmatrix},$$

$$F_3 = \begin{pmatrix} 4331/25740 & 409/6435 \\ 409/6435 & 4331/25740 \end{pmatrix} = \begin{pmatrix} 0.168259518 & 0.063558663 \\ 0.063558663 & 0.168259518 \end{pmatrix},$$

$$F_4 = \begin{pmatrix} 0.168142779 & 0.063446859 \\ 0.063446859 & 0.168142779 \end{pmatrix},$$

$$F_5 = \begin{pmatrix} 0.168150016 & 0.063453981 \\ 0.063453981 & 0.168150016 \end{pmatrix},$$

$$F_6 = \begin{pmatrix} 0.168149330 & 0.063453298 \\ 0.063453298 & 0.168149330 \end{pmatrix},$$

We observe that very good approximations of $S(A|B)$ are obtained from the first iterations. And this example explains the fast convergence of the continued fraction expansion of $S(A|I)$ given in (3.4).

Example 2. Now, let us Ccnsider the matrices A and B such that

$$A = \begin{pmatrix} 5 & -4 & 0 \\ -4 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 24 & -12 & -3 \\ -12 & 15 & 0 \\ -3 & 0 & 4 \end{pmatrix},$$

We will compute a relative operator entropy $S(A|B)$. It is not hard to see that

$$\begin{aligned} S(A|B) &= \begin{pmatrix} \frac{13}{2}\ln 3 & -4\ln 3 & \frac{-1}{2}\ln 3 \\ -4\ln(3) & 5\ln 3 & 0 \\ \frac{-1}{2}\ln 3 & 0 & \frac{7}{6}\ln 3 \end{pmatrix} \\ &= \begin{pmatrix} 7.14097978 & -4.394449156 & -0.5493061445 \\ -4.394449156 & 5.493061445 & 0 \\ -0.5493061445 & 0 & 1.281714338 \end{pmatrix}. \end{aligned} \tag{3.6}$$

Applying the theorem 3.1.3 the first convergents of $S(A|B)$ are given by:

$$F_2 = \begin{pmatrix} 6.869029276 & -4.3636364 & -0.47149461 \\ -4.3636364 & 5.490196078 & 0 \\ -0.47149461 & 0 & 1.248073960 \end{pmatrix},$$

$$F_3 = \begin{pmatrix} 7.074306086 & -4.392156863 & -0.5280366692 \\ -4.392156863 & 5.4930196078 & 0 \\ -0.5280366692 & 0 & 1.274051439 \end{pmatrix},$$

$$F_4 = \begin{pmatrix} 7.124146853 & -4.394281415 & -0.5437650283 \\ -4.394281415 & 5.492851768 & 0 \\ -0.5437650283 & 0 & 1.279825363 \end{pmatrix},$$

$$F_5 = \begin{pmatrix} 7.136715652 & -4.394436967 & -0.5478898144 \\ -4.394436967 & 5.493046209 & 0 \\ -0.5478898144 & 0 & 1.281239180 \end{pmatrix},$$

$$F_6 = \begin{pmatrix} 7.139901896 & -4.394448272 & -0.5489471762 \\ -4.394448272 & 5.493060341 & 0 \\ -0.5489471762 & 0 & 1.281684124 \end{pmatrix},$$

$$F_7 = \begin{pmatrix} 7.140708024 & -4.394449091 & -0.5492155533 \\ -4.394449091 & 5.493061364 & 0 \\ -0.5492155533 & 0 & 1.281684124 \end{pmatrix},$$

$$F_8 = \begin{pmatrix} 7.140911453 & -4.394449150 & -0.5492833386 \\ -4.394449150 & 5.493061438 & 0 \\ -0.5492833386 & 0 & 1.281706734 \end{pmatrix},$$

$$F_9 = \begin{pmatrix} 7.140962679 & -4.394449154 & -0.5493004121 \\ -4.394449154 & 5.493061443 & 0 \\ -0.5493004121 & 0 & 1.281712426 \end{pmatrix},$$

$$F_{10} = \begin{pmatrix} 7.140975559 & -4.394449155 & -0.5493047051 \\ -4.394449155 & 5.493061443 & 0 \\ -0.5493047051 & 0 & 1.281713857 \end{pmatrix}.$$

We see that F_{10} is approximately the exact valeur of $S(A|B)$. This example justify the importance of our approach.

Theorem 3.2.2 *Let A and B be two invertible and positive definite matrices in $\mathcal{M}_m$. Then a continued fraction expansion of Tsallis relative operator entropy is defined by*

$$T_\lambda(A|B) = \left[0; \frac{B_n}{A_n}\right]_{n=1}^{+\infty}, \tag{3.7}$$

where

$$\begin{cases} B_1 = 2\lambda A^{1/2}\dfrac{A-B}{A+B}, B_2 = \lambda(\lambda^2-1)A^{1/2}\left(\dfrac{A-B}{A+B}\right)^2 A^{-1}, \\ A_1 = -\lambda A^{-1/2} - \lambda^2 A^{1/2}\dfrac{A-B}{A+B}A^{-1}, \\ B_n = \lambda(\lambda^2-(n-1)^2)A^{1/2}\left(\dfrac{A-B}{A+B}\right)^2 A^{-1/2} \ \ \textit{for all} \ \ n \geq 3, \\ A_n = -(2n-1)I \ \ \textit{for all} \ \ n \geq 2. \end{cases}$$

In order to prove theorem 3.2.2, we recall the following lemma.

Lemma 3.2.3 *[17]. Let A and B be two invertible and positive matrix in $\mathcal{M}_m$, λ a real number such that $0 < \lambda < 1$. The continued fraction expansion of $(A^{-1/2}BA^{-1/2})^\lambda$ is given by*

$$(A^{-1/2}BA^{-1/2})^\lambda = \left[I; \frac{B'_n}{A'_n}\right]_{n=1}^{+\infty},$$

where we set

$$\begin{cases} B'_1 = 2\lambda A^{1/2}\dfrac{A-B}{A+B}A^{-1/2}, \\ A'_1 = -I - \lambda A^{1/2}\dfrac{A-B}{A+B}A^{-1/2} \\ B'_n = (\lambda^2-(n-1)^2)A^{1/2}\left(\dfrac{A-B}{A+B}\right)^2 A^{-1/2}, \quad n \geq 2, \\ A'_n = -(2n-1)I, \quad n \geq 2. \end{cases} \tag{3.8}$$

Proof of theorem 3.2.2 With the same notations as in lemma 3.2.3, we have

$$(A^{-1/2}BA^{-1/2})^{\lambda} = \left[I; \frac{B'_1}{A'_1}, \frac{B'_2}{A'_2}, \frac{\lambda B'_k}{A'_k}\right]_{k=1}^{+\infty}.$$

By adding $(-I)$, dividing by lambda from the both sides and using the proposition 2.7, we get

$$\frac{(A^{-1/2}BA^{-1/2})^{\lambda} - I}{\lambda} = \left[0; \frac{B'_1}{\lambda A'_1}, \frac{\lambda B'_2}{A'_2}, \frac{B'_k}{A'_k}\right]_{k=3}^{+\infty} \tag{3.9}$$

Multiplying $A^{1/2}$ from both sides of (3.9), we have

$$A^{1/2}\frac{(A^{-1/2}BA^{-1/2})^{\lambda} - I}{\lambda}A^{1/2} = A^{1/2}\left[0; \frac{B'_1}{\lambda A'_1}, \frac{\lambda B'_2}{A'_2}, \frac{B'_k}{A'_k}\right]_{k=3}^{+\infty} A^{1/2}.$$

By propositions 2.6 and 2.7, we get the result of theorem 3.2.2.

3.3 The solution of a matrix algebraic equation and its continued fraction expansion.

Let A and B be two positive definite matrices in $\mathcal{M}_m$. As it is known, the explicit form of the geometric mean of A and B is given by

$$g_2(A,B) = f_{1/2}(A,B) = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}.$$

Definition 3.3.1 Let $m \geq 2$ be an integer, the geometric mean of m positive definte matrices $A_1,\ A_2, .., A_m$ is recursively defined by the relationship:

$$g_m(A_1, A_2, ..., A_m) = f_{1/m}(A_1, g_{m-1}(A_2, A_3, ..., A_m))$$

where

$$f_{1/m}(A,B) = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/m}A^{1/2}.$$

We consider the following matrix equation: Find a positive matrix X such that

$$X(AX)^3 = B. \tag{3.10}$$

It is well known that equation (3.10) has a unique solution given by

$$X = g_4(B, A^{-1}, A^{-1}, A^{-1}) = A^{-1/2}(A^{1/2}BA^{1/2})^{1/4}A^{-1/2}.$$

With the appearance of the term $A^{1/2}$ and $(A^{1/2}BA^{1/2})^{1/4}$, it is hard to calculate $g_4(B, A^{-1}, A^{-1}, A^{-1})$ directly.

The following theorem approximates the solution of (3.10) in term of continued fraction.

Theorem 3.3.3 *Let A and B be two positive definite matrices in $\mathcal{M}_m$, the solution of equation (8) has the following continued fraction expansion:*

$$X = \left[A^{-1}, \frac{B_n}{A_n}\right]_{n=1}^{+\infty},$$

where

$$\begin{cases} B_1 = \frac{1}{2}\frac{I-AB}{I+AB},\ B_2 = \frac{-15}{16}\left(\frac{I-AB}{I+AB}\right)^2 A \\ A_1 = -A - \frac{1}{4}\frac{I-AB}{I+AB}A, \end{cases}$$

$$\begin{cases} B_n = (\frac{1}{16} - (n-1)^2)\left(\frac{I-AB}{I+AB}\right)^2,\ \ for\ \ n \geq 3, \\ A_n = -(2n-1)I,\ \ for\ all\ \ n \geq 2. \end{cases}$$

Proof. From lemma 3.2.3 and classical transformation, we easily find this result.

Acknowledgement

The authors would like to express their sincere gratitude to the referee for a very careful reading of this paper and for all his/her insightful comments, which lead a number of improvements to this paper.

References

[1]. T.Ando, Topics on operators inequalities, Ryukyu Univ. Lecture Note Series, 1 (1978).
[2]. N.Bebiano, R.Lemos and J.da Providencia, Inequalities for quantum relative entropy, Linear Algebra Appl. 401 (2005), 159-172.
[3]. V. P.Belavkin and P.Staszewski, C————————————————-algebra generalisation of relative entropy and entropy. Ann, Inst. H. Poincare Sect. A 37 (1982), 51-58.
[4]. S.M.Cox and P. C.Matthews. Exponential time dierencing for sti systems. J. Comp. Phys., 176 (2); 430-455, 2002.
[5]. I.Csizar and J.Korner, Information Theory: Coding Theorems for Dis- crete Memoy-less Systems, Academic Press, New York, 1981.
[6]. A.Cuyt, V.Brevik Petersen, Handbook of continued fractions for special functions, Springer (2007).
[7]. F. R.Gantmacher. The Theory of Matrices, Vol. I. Chelsa, New York, Elsevier Science Publishers, (1992).
[8]. V. Gen H.Golub and Charles F.Van Loan, Matrix Computations, Johns Hopking University Press, Baltimore, MD, USA, third edition (1996).
[9]. T.Furuta, Reverse inequalities involving two relative operator entropies and two relative entropies, Linear Algebra Appl. 403 (2005), 24-30.
[10]. E.H.Lieb and M. B.Ruskai, Proof of the strong subadditivity of quantum- mechanical entropy, J. Math. Phys. 14 (1973) 1938-1941.
[11]. G.Lindbad. Entropy, information and quantum measurements, Comm. Math. Phys. 33 (1973), 305-322.
[12]. L.Lorentzen, H.Wadeland, Continued fractions with application, Elseiver Science Publishers, (1992).
[13]. Mathematical theory of entropy, Foreword by James K.Brooks. Reprint 19 of the 1981 hardbedition. Cambridge: Cambridge University Press.xli, 2011.
[14]. M.Nakamura and H.Umegaki, A note on the entropy for operator al- gebra, Proc. Jpn.Acad. 37 (1961), 149-154.
[15]. M.Raissouli, A.Kacha, Convergence for matrix continued fractions. Linear Algebra and its Applications, 320 (2000), pp. 115-129.
[16]. M.Raissouli, A.Kacha and S.Salhi, Continued fraction expansion of real power of positive de nite matrices with applications to matrix mean, The Arabian journal for sciences and engineering, Vol 31, number 1 (2006), pp. 1-15.
[17]. K.Yanagi, K.Kuriyama, Generalised Shanon inequalities based in Tsal- lis relative operator entropy, Linear Algebra Appl., vol.394 (2005) 109-118