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Farey Grids

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Abstract: *We define the Farey grid matrix and also, we introduce the concept of Farey grid graph and Farey grid from Farey sequence and present some observations.*
Keywords: *Farey Sequence, Farey matrix, Farey fractions, Farey grid, Farey grid graph ,Farey grid matrix.*

Notations:
F_n - Farey sequence of an order $'n'$
FM - Farey matrix
$|F_N|$ - Cardinality of Farey sequence of order $'N'$
$\varphi(N)$ - Euler's function
S_N - Sum of Farey fraction
FG - Farey grid
FGM - Farey Grid Matrix
D_N - $'N'$ th order farey grid
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I. Introduction

The Farey sequence ,sometimes called Farey series ,is a series of sequences in which each sequence consists of rational numbers ranging in value from 0 to 1.A natural quadratic generalization of the Farey sequence studied numerically by Brown and Mahler. Afterwards , trying to make some justice to Haros, Delmer and Deshouillers have called this new sequences the Haros sequence .Taking all these into account after two hundred years one may agree that the appropriate name for what is largely known as " the Farey sequence" should be " the Haros-Farossequence". Any reduced fraction with positive denominator $\leq n$ is a member of the Farey sequence of order n and can be called a Farey fraction of order n.

Farey sequences are very useful to find rational approximations of irrational numbers.The farey sequence of order n contains all of members of the Farey sequences of lower orders. In particular F_n contains all the members of F_{n-1} and also contains an additional fraction for each number that is less than n and co-prime to n. The middle term of a Farey sequence F_n is always is $\frac{1}{2}$, for $n > 1$. From this , we can relate the lengths of F_n and F_{n-1} ,using Euler's totient function $\varphi(n)$ is $|F_n| = |F_{n-1}| + \varphi(n)$, $|F_1| = 2$ using, we can derive an expression for length of F_n is $|F_n| = 1 + \sum_{m=1}^{n} \varphi(m)$. The Farey grid graph is formed from Farey fraction and the Farey grid from Farey Matrix.

Farey sequence:

The Farey sequence $F_n$for any positive integer$n$is the set of irreducible rational numbers a/b with $0 \leq a \leq b \leq n$ and $(a, b) = 1$ arranged in increasing order. The first five Farey sequence are,

$F_1 = \left\{\frac{0}{1}, \frac{1}{1}\right\}$

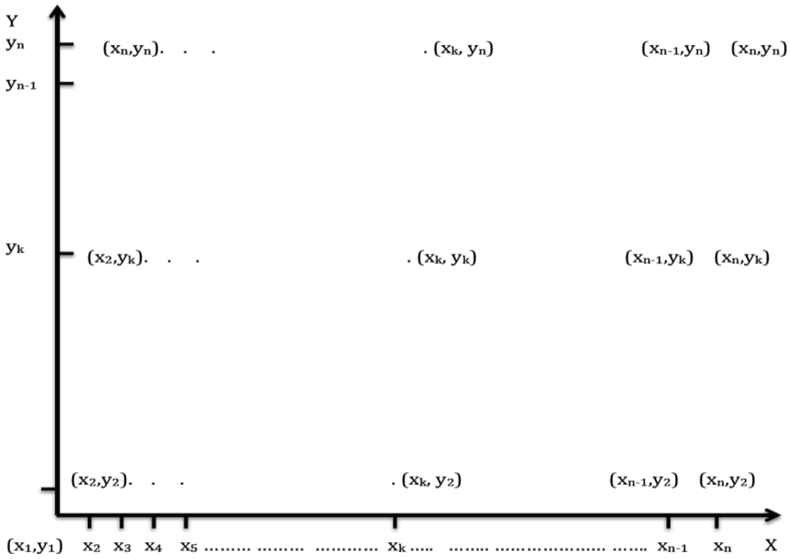
$F_2 = \left\{\frac{0}{1}, \frac{1}{2}, \frac{1}{1}\right\}$

$F_3 = \left\{\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\right\}$

$F_4 = \left\{\frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}\right\}$

$F_5 = \left\{\frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}\right\}$

Farey Graph:
Consider the rectangular XOY axis $X=(x_1,x_2,\ldots.,x_n), Y=(y_1,y_2,\ldots.,y_n)$ where $x_i,y_j \in F_N[0,1]$. The Farey graph is a graph of vertices (x_i,y_j) and it forms a grid whose graphical representation is given below:



Farey matrix:
A Farey matrix denoted by FM is defined as the square matrix of order n, whose elements are the sum of Farey fractions in the Farey graph.

$$FM=\begin{bmatrix} x_1+y_1 & x_1+y_2 & \ldots & x_1+y_n \\ x_2+y_1 & x_2+y_2 & \cdots & x_2+y_n \\ \vdots & \vdots & \ldots & \vdots \\ x_n+y_1 & x_n+y_2 & \ldots & x_n+y_n \end{bmatrix} = \left[x_i+y_j\right]_{n\times n}$$

Theorem:
If $[A_{ij}]$ denote 2×2 submatrix of the Farey matrix F_n which are the vertices of the Farey grid then

$$\sum_{j=1}^{n}\sum_{i=1}^{n}[A_{ij}] = \begin{bmatrix} n[|F_n|-1] & n|F_n| \\ n[|F_n|-2] & n[|F_n|-1] \end{bmatrix}$$

Proof:

$$\sum_{j=1}^{n}\sum_{i=1}^{n}[A_{ij}] = \sum_{j=1}^{n}\left[A_{1j}+A_{2j}{}_{+\cdots+}A_{nj}\right]$$

$$\sum_{j=1}^{n}\sum_{i=1}^{n}[A_{ij}] = \sum_{j=1}^{n}[A_{1j}]+\sum_{j=1}^{n}[A_{2j}]+\cdots+\sum_{j=1}^{n}[A_{nj}]$$

$$=\begin{bmatrix} n\{x_1\}+S_N & n\{x_2\}+S_N \\ n\{x_1\}+S_{N-1} & n\{x_2\}+S_N-1 \end{bmatrix}+\begin{bmatrix} n\{x_2\}+S_N & n\{x_3\}+S_N \\ n\{x_2\}+S_N-1 & n\{x_3\}+S_{N-1} \end{bmatrix}+\cdots$$

$$+\begin{bmatrix} n\{x_{n-1}\}+S_N & n\{x_n\}+S_N \\ n\{x_{n-1}\}+S_N-1 & n\{x_n\}+S_N-1 \end{bmatrix}$$

$$=\begin{bmatrix} n\{x_1+x_2+\cdots+x_{n-1}\}+nS_N & n\{x_2+x_3+\cdots+x_{n-1}\}+nS_N \\ n\{x_1+x_2+\cdots+x_{n-1}\}+nS_N-n & n\{x_2+x_3+\cdots+x_{n-1}\}+nS_N-n \end{bmatrix}$$

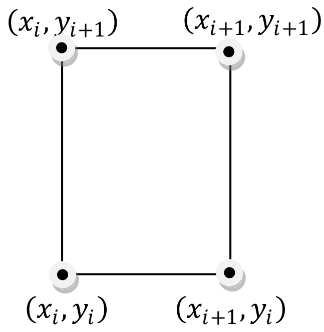
$$=\begin{bmatrix} n\left(x_1+x_{2}{}_{+\cdots+}x_n\right)-n\{x_n\}+nS_N & 2nS_N \\ n\left(x_1+x_{2}{}_{+\cdots+}x_n\right)-n\{x_n\}+nS_N-n & 2nS_N-n \end{bmatrix}$$

$$=\begin{bmatrix} 2nS_N-n & 2nS_N \\ 2nS_N-n-n & 2nS_N-n \end{bmatrix}$$

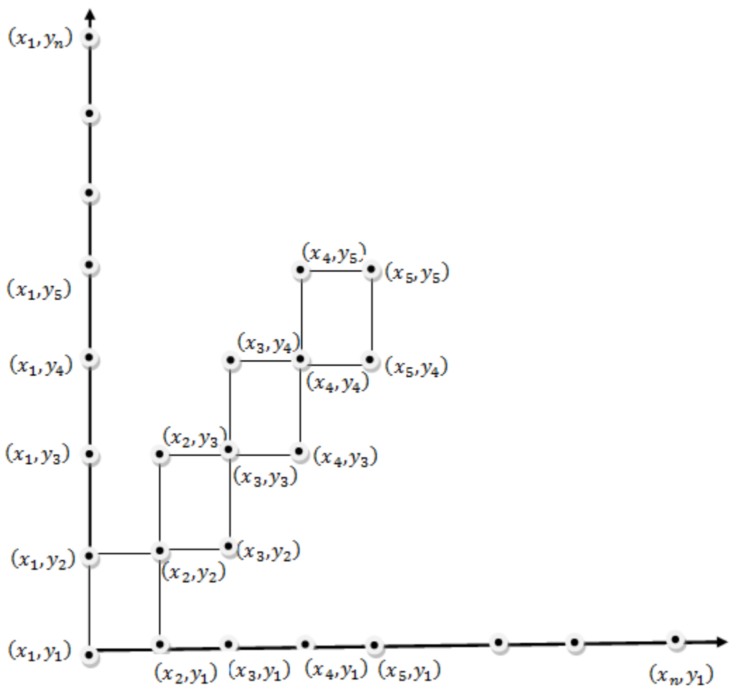
$$=\begin{bmatrix} n(2S_N-1) & 2nS_N \\ 2n(S_N-1) & (2S_N-1)n \end{bmatrix}$$

$$\therefore \sum_{j=1}^{n}\sum_{i=1}^{n}[A_{ij}] = \begin{bmatrix} n[|F_n|-1] & n|F_n| \\ n[|F_n|-2] & n[|F_n|-1] \end{bmatrix}$$

Farey Grid:
A Farey grid is a sequence of rectangles formed from Farey graph along the main diagonal in a Farey graph.



Farey Grid graph:



Farey Grid 2×2 Matrix:
A farey Grid $\mathbf{2 \times 2}$ matrix denoted by FGM is defined as of an order two whose elements are the sum of Farey fractions. The Vertices of the Farey Grid are clearly Farey Fractions.

$$FGM = \begin{bmatrix} x_i + y_{i+1} & x_{i+1} + y_{i+1} \\ x_i + y_i & x_{i+1} + y_i \end{bmatrix}, where\ \ i = 1,2,\dots,n\,.$$

Theorem:
The sum of the determinants of the Farey grid matrix of order N is unity.
Proof:
In the Farey grid, the vertices constituting the grids are $\{(x_i, y_i), (x_i, y_{i+1})(x_{i+1}, y_{i+1}), (x_{i+1}, y_i)\}$ where (x_i, y_i) denote the Farey fractions. Denoting the matrix of each grid by D_i , the number of grids for the sequence F_N is $|F_{N-1}| + \varphi(N)$.
By observation, the matrix $D_1 + D_2 + \cdots + D_N$ for the farey sequence F_N is

$$\begin{bmatrix} |F_{N-1}| + \varphi(N) & |F_{N-1}| + \varphi(N) + 1 \\ |F_{N-1}| + \varphi(N) - 1 & |F_{N-1}| + \varphi(N) \end{bmatrix}$$

The determinant of this matrix is

$$|D_1 + D_2 + \cdots + D_N| = \left(\left(|F_{N-1}| + \varphi(N)\right)^2\right) - \left((|F_{N-1}| + \varphi(N) - 1 \times |F_{N-1}| + \varphi(N) + 1)\right) \; = 1.$$

Numerical Illustration:

Consider the Farey sequence (F_2)

$$D_1 + D_2 = \begin{bmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{2} \end{bmatrix} + \begin{bmatrix} \frac{3}{2} & 2 \\ 2 & \frac{3}{2} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

$$|D_1 + D_2| = 1$$

Consider the Farey sequence (F_3)

$$D_1 + D_2 + D_3 + D_4 = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ 0 & \frac{1}{3} \end{bmatrix} + \begin{bmatrix} \frac{5}{6} & 1 \\ \frac{2}{3} & \frac{5}{6} \end{bmatrix} + \begin{bmatrix} \frac{7}{6} & \frac{4}{3} \\ 1 & \frac{7}{6} \end{bmatrix} + \begin{bmatrix} \frac{5}{3} & 2 \\ \frac{4}{3} & \frac{5}{3} \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix}$$

$$|D_1 + D_2 + D_3 + D_4| = 1$$

Consider the Farey sequence (F_4)

$$D_1 + D_2 + D_3 + D_4 + D_5 + D_6 = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{4} \end{bmatrix} + \begin{bmatrix} \frac{3}{2} & 2 \\ 2 & \frac{3}{2} \end{bmatrix} + \begin{bmatrix} \frac{5}{6} & 1 \\ \frac{2}{3} & \frac{5}{6} \end{bmatrix} + \begin{bmatrix} \frac{7}{6} & \frac{4}{3} \\ 1 & \frac{7}{6} \end{bmatrix} + \begin{bmatrix} \frac{17}{12} & \frac{3}{2} \\ \frac{4}{3} & \frac{17}{12} \end{bmatrix} + \begin{bmatrix} \frac{7}{4} & 2 \\ \frac{3}{2} & \frac{7}{4} \end{bmatrix} = \begin{bmatrix} 6 & 7 \\ 5 & 6 \end{bmatrix}$$

$$|D_1 + D_2 + D_3 + D_4 + D_5 + D_6| = 1.$$

References

[1]. A.Gnanam, M.A.Gopalan, C.Dinesh, Farey Matrix, International Journal of Mathematics Trends and Technology-Volume 19 Number 1 Mar 2015.

[2]. Ivan Niven, Herbert S.Zuckerman, Hugh L.Montgomery,An Introduction to The Theory of Numbers (Fifth edition),Wiley-India Edition 2011.

[3]. Don Redmond, Number Theory An Introduction, Marcel Dekker,Inc,1996.

[4]. JozseSandor,Dragoslav S.Mitrinovic and BorislaCrstici, Handbook of Number Theory I,Springer 2006.

[5]. G.H. Hardy and E.M. Wright,An Introduction to The Theory of Numbers(Fourth edition),Oxford at The Clarendon Press 1975.

[6]. Jin HoKwak,Sungpyo Hong, Linear Algebra, Springer International Edition 2010.

[7]. Crigtian cobeli and Alexandru Zaharescu , The Haros Farey sequence at two Hundred years a Survey, Acta universitatis Apulensis.