IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 12, Issue 6 Ver. II (Nov. - Dec.2016), PP 47-54
www.iosrjournals.org

Fixed Point Result Satisfying *Φ* - Maps in G-Metric Spaces

Madhu Shrivastava[1], Dr.K.Qureshi[2], Dr.A.D.Singh[3]
[1]*TIT Group of Institution, Bhopal*
[2]*Ret.Additional Director, Bhopal*
[3]*Govt.M.V.M.College, Bhopal*

Abstract: *In this paper, we elaborate some existing result of fixed point theorems, that fulfill the nature of G-metric space and satisfy the ∅-maps. Previously Erdal Karapinar and Ravi Agrawal [1] have modified some existing result of fixed point theory of Samet et.al. Int. J. Anal (2013:917158, 2013) [2] and Jleli-Samet (Fixed point theory application. 2012: 210,2012) [3] in a different way.*

I. Introduction

The concept of G-metric spaces was introduced by Mustafa and Sims [4]. G-metric spaces is generalization of a metric spaces(X, d). Mustafa and Sims characterized the Banach contraction mapping principal [5] in the context of G-metric spaces .Subsequently many fixed point result on such spaces appeared. Since one is adapted from other. The G-metric spaces is to understand the geometry of three points instead of two, many result are obtained by contraction condition.

In 2013, Samet et al [2] and Jleli Samet [3] observed that some fixed point theorems in the context of a G-metric space in literature can be concluded by some existing results in the setting of (quashi-) metric spaces. Also the contraction condition of the fixed point theorem on a G-metric space can be reduced to two variables instead of three. In [2,3] the authors find $d(x,y) = G(x,y,y)$ form a quasi-metric .Erdal Karapinar and Ravi Agrawal modified some existing results of fixed point theorem .

II. Preliminaries

Definition 2.1 Let X be a non-empty set and let $G : X \times X \times X \to R^+$ be a function Satisfying the following properties:
(G1) $G(x,y,z) = 0 \; if \; x = y = z$,
(G2) $0 < G(x,x,y) \; for \; all \; x,y \in X \; with \; x \neq y$,
(G3) $G(x,x,y) \leq G(x,y,z) \; for \; all \; x,y,z \in X \; with \; y \neq z$,
(G4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \cdots$ (symmetry in all three variables),
(G5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ for all $x,y,z,a \in X$ (rectangle inequality).
Then the function G is called a generalized metric or, more specifically, a G-metric on X, and the pair (X,G) is called a G-metric space.
Every G-metric on X defines a metric d_G on X by

$$d_G(x,y) = G(x,y,y) + G(y,x,x) \text{, for all } x,y \in X.$$

Example 2.1 Let (X,d) be a metric space. The function $G: X \times X \times X \to [0,+\infty)$, defined as

$$G(x,y,z) = max\{d(x,y), d(y,z), d(z,x)\}$$

Or

$$G(x,y,z) = d(x,y) + d(y,z) + d(z,x) \quad ,$$

for all $x,y,z \in X$, is a G-metric on X.

Definition 2.2 Let (X,G) be a G-metric space, and let $\{x_n\}$ be a sequence of points of X. We say that $\{x_n\}$ is G-convergent to $x \in X$, if

$$\lim_{n,m\to+\infty} G(x,x_n,x_m) = 0$$

That is, for any $\varepsilon > 0$, there exists N ∈ ℕ such that $G(x,x_n,x_m) < \varepsilon$ for all $n,m \geq N$. We call x the limit of the sequence and write $x_n \to x$ or $\lim_{n\to\infty} x_n = x$.

Proposition 2.1 Let (X,G) be a G-metric space. The following are equivalent:
(1) $\{x_n\}$ is G-convergent to x,
(2) $G(x_n,x_n,x) \to 0$ as $n \to +\infty$,
(3) $G(x_n,x,x) \to 0 \; as \; n \to +\infty$,
(4) $G(x_n,x_m,x) \to 0 \; as \; n,m \to +\infty$.

Definition 2.3 Let (X, G) be a G-metric space. A sequence $\{x_n\}$ is called a G-Cauchy sequence if, for any $\varepsilon > 0$, there is N ∈ ℕ such that $G(x_n, x_m, x_l) < \varepsilon$ for all $m, n, l \geq N$, that is, $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to +\infty$.

Proposition 2.2 Let (X, G) be a G-metric space. Then the following are equivalent:
(1) the sequence $\{x_n\}$ is G-Cauchy,
(2) for any $\varepsilon > 0$, there exists N ∈ N such that $G(x_n, x_m, x_m) < \varepsilon$ for all $m, n \geq N$.

Definition 2.4 A G-metric space (X, G) is called G-complete if every G-Cauchy sequence is G-convergent in (X, G).

Lemma 2.1 Let (X, G) be a G-metric space. Then $G(x, x, y) \leq 2\, G(x, y, y)$ for all $x, y \in X$.

Definition2.5 Let (X, G) be a G-metric space. A mapping $T: X \to X$ is said to be G-continuous if $\{T(x_n)\}$ is G-convergent to $T(x)$ where $\{x_n\}$ is any G-convergent sequence Converging to x.

In [26] , Mustafa characterized the well-known Banach contraction mapping principle in the context of G-metric spaces in the following ways.

Theorem 2.1 Let (X, G) be a complete G-metric space and let $T : X \to X$ be a mapping satisfying the following condition for all $, y, z \in X$: $G(Tx, Ty, Tz) \leq k\, G(x, y, z)$,
Where $k \in [0,1)$. Then T has a unique fixed point.

Theorem 2.2 Let (X, G) be a complete G-metric space and let $T : X \to X$ be a mapping satisfying the following condition for all $x, y \in X$:

$$G\,(Tx, Ty, Ty) \leq k\, G(x, y, y),$$

where k ∈ [0,1). Then T has a unique fixed point.

Theorem2.3 Let (X, G) be a G-metric space. Let $T : X \to X$ be a mapping suchthat

$$G(Tx, Ty, Tz) \leq a\, G(x, y, z) + b\, G(x, Tx, Tx) + c\, G(y, Ty, Ty) + d\, G(z, Tz, Tz)$$

for all x, y, z, where a, b, c, d are positive constants such that $k = a + b + c + d < 1$. Then there is a unique $x \in X$ such that $Tx = x$.

Theorem2.4 Let (X,G) be a G-metric space. Let $T : X \to X$ be a mapping such that

$$G(Tx, Ty, Tz) \leq k\, [\, G(x, Tx, Tx) + G(y, Ty, Ty) + G(z, Tz, Tz)]$$

$f$or all x, y, z, where $k \in [0, \frac{1}{3})$. Then there is a unique $x \in X$ such that $Tx = x$.

Theorem2.5 Let (X, G) be a G-metric space. Let $T : X \to X$ be a mapping such that

$$G(Tx, Ty, Tz) \leq a\, G(x, y, z) + b[G(x, Tx, Tx) + G(y, Ty, Ty) + G(z, Tz, Tz)]$$

$f$or all x, y, z, where a, b are positive constants such that $k = a + b < 1$. Then there is a unique $x \in X$ such that $Tx = x$.

Theorem2.6 Let (X, G) be a G-metric space. Let $T : X \to X$ be a mapping such that

$$G(Tx, Ty, Tz) \leq a\, G(x, y, z) + b\, max\{G(x, Tx, Tx), G(y, Ty, Ty), G(z, Tz, Tz)\}$$

for all x, y, z, where a, b are positive constants such that $k = a + b < 1$. Then there is a unique $x \in X$ such that $Tx = x$.

Theorem2.7 Let (X, G) be a G-metric space. Let $T : X \to X$ be a mapping such that

$$G(Tx, Ty, Tz) \leq k\, max\{G(x, y, z), G(x, Tx, Tx), G(y, Ty, Ty), G(z, Tz, Tz), \\ G(z, Tx, Tx), G(x, Ty, Ty), G(y, Tz, Tz)\}$$

for all x, y, z where $k \in \left[0, \frac{1}{2}\right)$. Then there is a unique $x \in X$ such that $Tx = x$.

Theorem 2.8 Let (X, G) be a complete G-metric space and let $T : X \to X$ be a given mapping satisfying

$$G(Tx, Ty, Tz) \leq G(x, y, z) - \phi(G(x, y, z))$$

for all $x, y \in X$, where $\phi : [0, \infty) \to [0, \infty)$ is continuous with $\phi - 1(\{0\}) = 0$. Then there is a unique $x \in X$ such that $Tx = x$

Definition 2.6 A quasi-metric on a nonempty set X is a mapping $p : X \times X \to [0,\infty)$ such that
$(p1)$ $x = y$ if and only if $p(x,y) = 0$,
$(p2)$ $p(x,y) \leq p(x,z) + p(z,y)$,
for all $x, y, z \in X$. A pair (X,p) is said to be a quasi-metric space.

Samet et al. and Jleli-Samet noticed that $p(x,y) = P_G(x,y) = G(x,y,y)$ is a quasimetric whenever $G : X \times X \times X \to [0,\infty)$ is a G-metric. It is well known that each quasi-metric induces a metric. Indeed, if (X,p) is a quasi-metric space, then the function defined by $d(x,y) = d_G(x,y) = max\{p(x,y), p(y,x)\}$ for all $x, y \in X$ is a metric on X.

Theorem 2.9 Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping with the property

$$d(Tx,Ty) \leq q\, max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}$$

for all $x \in X$, where q is a constant such that $q \in [0,1)$. Then T has a unique fixed point.

Proposition 2.3
(A) If (X,G) is a complete G-metric space, then (X,d) is a complete metric space.
(B) If (X,G) is a sequentially G-compact G-metric space, then (X,d) is a compact metric space.

III. Main Result

Theorem-3.1- Let (X,G) be a complete G-metric space and let $f : X \to X$ be a given mapping satisfy

$$G(fx, f^2x, fy) \leq \varphi\big(G(x, fx, y)\big) \quad (1)$$

for all $x, y \in X$, where $\varphi: [0,1] \to [0,1]$ is continuous function s.t. $\varphi(s) < s$, and $\varphi(0) = 0$ then there is a unique $x \in X$ s.t. $fx = x$.
Proof:- We first show that if the fixed point of the operator f exist , then it is unique .
Suppose, on contrary ,that x and y are two fixed point of f, such that $x \neq y$, Hence $G(x, x, y) \neq 0$
From equation (1), we get

$$G(fx, f^2x, fy) \leq \varphi\big(G(x, fx, y)\big)$$

Which is equivalent to,

$$G(x, x, y) \leq \varphi\,\big(G(x,x,y)\big) < G(x,x,y)$$

Which is a contradiction, hence f has a unique fixed point.
Let $x_0 \epsilon X$, we define a sequence $\{x_n\}$ by $x_n = fx_{n-1}, n \in N$.
If $x_{n_0} = x_{n_0+1}$, for some $n_0 \in N$, then trivially f has a fixed point.
Taking $x = x_n$, y = z = x_{n+1}
Now from equation (1), we have

$$\begin{aligned} G(x_{n+1}, x_{n+2}, x_{n+2}) &= G(fx_n, f^2x_n, fx_{n+1}) \\ &= G\big(fx_n, fx_{n+1} fx_{n+1}\big) \\ &\leq \varphi\big(G(x_n, fx_n, x_{n+1})\big) \\ &= \varphi\big(G(x_n, x_{n+1}, x_{n+1})\big) \\ &< G(x_n, x_{n+1}, x_{n+1}) \end{aligned} \quad (2)$$

This shows that $\{G(x_n, x_{n+1}, x_{n+1})\}$ is monotone positive decreasing sequence , thus the sequence $\{G(x_n, x_{n+1}, x_{n+1})\}$ converges to $s \geq 0$ .We shall show that $s = 0$.
Suppose, on contrary that $s > 0$,
Letting $n \to \infty$,in equation (2)
We get $s \leq \emptyset(s) < s$
It is a contradiction, Hence conclude that $lim_{n\to\infty} G\{(x_n, x_{n+1}, x_{n+1})\} = 0$
By lemma [2.1],

$$lim_{n\to\infty} G\{(x_n, x_n, x_{n+1})\} = 0 \quad (3)$$

Now next we show that the $\{x_n\}$is G-Cauchy, on contrary let $\{x_n\}$ is not G-Cauchy sequence, so there exist $\epsilon > 0$ and subsequence $\{x_{n_k}\}$ *and* $\{x_{m_k}\}$ *of* $\{x_n\}$ with $n(k) > m(k) > k$.
Such that $G\big(x_{n_k}, x_{m_k}, x_{m_k}\big) \geq \epsilon$, *for all* $k \in N$ (4)
Moreover, corresponding to m_k , we can choose n_k, such that it is the smallest integer with $n_k > m_k$ Satisfying equation (4).
Then that $G\big(x_{n_{k-1}}, x_{m_k}, x_{m_k}\big) < \epsilon \quad \forall\, k \in N$
(5)
Then we have by triangular inequality,

$$\epsilon \leq G(x_{n_k}, x_{m_k}, x_{m_k})$$
$$\leq G(x_{n_k}, x_{n_{k-1}}, x_{n_{k-1}}) + G(x_{n_{k-1}}, x_{m_k}, x_{m_k}) \quad (6)$$

Setting $k \to \infty$ and using equation (3),
$$lim_{n\to\infty} G(x_{n_k}, x_{m_k}, x_{m_k}) = \epsilon \quad (7).$$
Now ,
$$G(x_{n_{k+1}}, x_{m_{k+1}}, x_{m_{k+1}}) \leq G(x_{n_{k+1}}, x_{n_k}, x_{n_k}) + G(x_{n_k}, x_{m_k}, x_{m_k}) + G(x_{m_k}, x_{m_{k+1}}, x_{m_{k+1}})$$
(8)
and
$$G(x_{n_k}, x_{m_k}, x_{m_k}) \leq G(x_{n_k}, x_{n_{k+1}}, x_{n_{k+1}}) + G(x_{n_k+1}, x_{m_{k+1}}, x_{m_{k+1}}) + G(x_{m_{k+1}}, x_{m_k}, x_{m_k}) \quad (9)$$
Letting $k \to \infty$ in above inequality and using (3) and (5)
$$lim\ n \to \infty, G(x_{n_{k+1}}, x_{m_{k+1}}, x_{m_{k+1}}) = \epsilon \quad (10)$$
Further we have
$$G(x_{n_k}, x_{m_k}, x_{m_k}) \leq G(x_{n_k}, x_{m_k}, x_{m_{k+1}}) \leq (x_{n_k}, x_{m_k}, x_{m_k}) + (x_{m_k}, x_{m_k}, x_{m_{k+1}}) \quad (11)$$

By G-3 and the triangular inequality, Letting $k \to \infty\ in$ (11) and using (3) and (7)
We conclude that
$$\lim_{n\to\infty} G(x_{n_k}, x_{m_k}, x_{m_{k+1}}) = \epsilon$$
Analogously, we have
$$G(x_{n_{k+1}}, x_{m_{k+1}}, x_{m_{k+1}}) \leq G(x_{n_{k+1}}, x_{m_{k+2}}, x_{m_{k+1}})$$
$$\leq G(x_{n_{k+1}}, x_{m_{k+1}}, x_{m_{k+1}}) + G(x_{m_{k+1}}, x_{m_{k+2}}, x_{m_{k+1}})$$

By G-3 and the triangular inequality, Letting $k \to \infty$ in (11),
$$G(x_{n_{k+1}}, x_{m_{k+2}}, x_{m_{k+1}}) = \epsilon$$
Now again from equation (1) and (4), we have
$$\epsilon \leq G(fx_{mk}, f^2x_{m_k}, fx_{n_k})$$
$$= G(x_{m_{k+1}}, x_{m_{k+2}}, x_{n_{k+1}})$$
$$= G(x_{n_{k+1}}, x_{m_{k+2}}, x_{m_{k+1}})$$
$$\leq \varphi(G(x_{n_k}, fx_{m_k}, x_{m_{k+1}})$$
$$= \varphi(G(x_{n_k}, x_{m_k}, x_{m_{k+1}})) < G(x_{n_k}, x_{m_k}, x_{m_{k+1}})$$

Letting $k \to \infty$, we hav , $\epsilon \leq \emptyset(\epsilon) < \epsilon$, Which is a contradiction.
This shows that $\{x_n\}$ is G-cauchy sequence in X. Since X is complete G-metric space.
So there exists z ϵX, such that $lim_{n\to\infty} x_n \to z$,
Now we claim that f z = z.
Consider
$$G(x_{n+1}, x_{n+2}, fz) = G(fx_n, f^2x_n, fz)$$
$$\leq \varphi(G(x_n, fx_n, z))$$
$$= \varphi(G(x_n, x_{n+1}, z))$$
Let $k \to \infty$, we get
$$G(z, z, fz) \leq \varphi(G(z, z, z)) = \varphi(0) = 0$$

Hence $G(fz, z, z) = 0, i.e, fz = z$. Hence z is a unique fixed point.

Theorem 3.2:- Let (X, G) be a G-metric space .Let $f: X \to X$ be a mapping such that

$$G(fx, fy, fz) \leq k\ M(x, y, z) \quad (1)$$
$for\ all\ x, y, z \in X\ and\ k \in$ [0,1) and
$$M(x, y, z) = max\{G(x, y, z), G(f^2x, fy, fz), G(z, fx, fy), G(y, f^2x, fy), G(x, fx, fx)$$
$$G(y, fy, fy), G(z, fz, fz), G(fx, f^2x, fz), G(z, f^2x, fz), G(fx, f^2x, fy)\}$$
Then there is a unique $x\epsilon X\ such\ that\ fx = x$.
Proof: $Let\ x_0\epsilon X$, $We\ define\ \{x_n\}\ in\ the\ following\ fx_n = x_{n+1}$, $n \in N$
(2)
$Taking\ x = x_n, y = z = x_{n+1,}$ we get from eq.(1)

$$G(fx_n, fx_{n+1}, fx_{n+1}) \leq k\, M(x_n, x_{n+1}, x_{n+1})$$

(3)

Where

$$M(x_n, x_{n+1}, x_{n+1}) = max\{G(x_n, x_{n+1}, x_{n+1}), G(f^2x_n, fx_{n+1}, fx_{n+1}), G(x_{n+1}, fx_n, fx_{n+1}),$$
$$G(fx_{n+1}, f^2x_n, fx_{n+1})\ G(x_n, fx_n, fx_n), G(x_{n+1}, fx_{n+1}, fx_{n+1}),$$
$$G(x_{n+1}, fx_{n+1}, fx_{n+1}), G(fx_n, f^2x_n, fx_{n+1}), G(x_{n+1}, f^2x_n, fx_{n+1}), G(fx_n, f^2x_n, fx_{n+1})\}$$
$$= max\{G(x_n, x_{n+1}, x_{n+1}), G(x_{n+2}, x_{n+2}, x_{n+2}), G(x_{n+1}, x_{n+1}, x_{n+2})$$
$$G(x_{n+2}, x_{n+2}, x_{n+2}), G(x_n, x_{n+1}, x_{n+1}), G(x_{n+1}, x_{n+2}, x_{n+2}),$$
$$G(x_{n+1}, x_{n+2}, x_{n+2}) G(x_{n+1}, x_{n+2}, x_{n+2}), G(x_{n+1}, x_{n+2}, x_{n+2}), G(x_{n+1}, x_{n+2}, x_{n+2})\}$$
$$= max\ \{G(x_n, x_{n+1}, x_{n+1}), G(x_{n+1}, x_{n+2}, x_{n+2}), G(x_{n+1}, x_{n+1}, x_{n+2})\} \quad (4)$$

Case (i)- First let $M(x_n, x_{n+1}, x_{n+1}) = G(x_{n+1}, x_{n+1}, x_{n+2})$

By G_5, we get from above

$$G(x_{n+1}, x_{n+2}, x_{n+2}) = G(fx_n, fx_{n+1}, fx_{n+1})$$
$$\leq kM(x_n, x_{n+1}, x_{n+1})$$
$$= kG(x_{n+1}, x_{n+1}, x_{n+2})$$
$$\leq k[G(x_{n+1}, x_{n+2}, x_{n+2}) + G(x_{n+2}, x_{n+1}, x_{n+2})\}$$

(5)

Which is a contradiction, since $0 \leq k < 1$.

Case-(ii)- If $M(x_n, x_{n+1}, x_{n+1}) = G(x_{n+1}, x_{n+2}, x_{n+2})$

Then we get $G(x_{n+1}, x_{n+2}, x_{n+2}) = G(fx_n, fx_{n+1}, fx_{n+1}),$

$$\leq kM(x_n, x_{n+1}, x_{n+1})$$
$$= kG(x_{n+1}, x_{n+2}, x_{n+2}) \quad (6)$$

This is a contradiction, since $0 \leq k < 1$.

Case (iii)- If $M(x_n, x_{n+1}, x_{n+1}) = G(x_n, x_{n+1}, x_{n+1})$

Then we get, $G(x_{n+2}, x_{n+2}, x_{n+1}) \leq kG(x_{n+1}, x_{n+1}, x_n)$ (7)

Continuing in this way, we get

$$G(x_{n+2}, x_{n+2}, x_{n+1}) \leq k^{n+1}G(x_1, x_1, x_0) \quad (8)$$

Again,

$$G(x_m, x_m x_n) \leq G(x_{n+1}, x_{n+1}, x_n) + G(x_{n+2}, x_{n+2}, x_{n+1}) + \cdots \ldots \ldots G(x_{m-1}, x_{m-1}, x_{m-2}) + G(x_m, x_m, x_{m-1})$$
$$\leq k^n G(x_1, x_1, x_0) + k^{n+1}G(x_1, x_1, x_0) + \cdots \ldots \ldots \ldots \ldots \ldots \ldots \ldots + k^{m-1}G(x_1, x_1, x_0)$$

Let $n, m \to \infty\ we\ get$, $G(x_m, x_m x_n) \to 0$. (9)

Hence $\{x_n\}$ is a Cauchy sequence in X. Since (X, G) is G-complete, then there exist $z \in X$ s.t. $\{x_n\}$ is G-converges to z. Let on contrary that $z \neq fz$. for this let $x_{n+1} = fx_n$

$$G(x_{n+1}, fz, fz) = G(fx_n, fz, fz)$$
$$\leq kM(x_n, z, z) \quad (10)$$

Where

$$M(x_n, z, z) = max\{(x_n, z, z), G(fz, f^2x_n, fz), G(z, fx_n, fz), G(z, f^2x_n, fz),$$
$$G(x_n, fx_n, fx_n), (z, fz, fz), G(x_n, fz, fz), G(fx_n, f^2x_n, fz),$$
$$(z, f^2x_n, fz), (fx_n, f^2x_n, fz)\}$$
$$= max\{(x_n, z, z), G(fz, x_{n+2}, fz), G(z, x_{n+1}, fz), G(z, x_{n+2}, fz),$$
$$G(x_n, x_{n+1}, x_{n+1}), \quad (z, fz, fz), G(x_n, fz, fz), G(fx_n, x_{n+2}, fz),$$
$$(z, x_{n+2}, fz), (x_{n+1}, x_{n+2}, fz)\}$$

Letting n→∞, since G is continuous, we get

$$G(z, fz, fz) \leq kG(z, fz, fz)$$

Or

$$G(z, fz, fz) \leq kG(z, z, fz)$$
$$\leq k[G(z, fz, fz) + G(fz, z, fz)]$$
$$= k[2G(z, fz, fz)]$$

so

$$G(z, fz, fz) \leq 2kG(z, fz, fz), \text{ Since } 0 \leq k < 1.$$

This is a contradiction. $G(z, fz, fz) = 0$. So $fz = z$.

Uniqueness - Next we show that uniqueness of z of f. suppose on contrary, there exist another common fixed point $u \in X$ with $z \neq u$.

We get $\quad G(z,z,u) = G(fz,fz,fu)$

$\leq kM(z,z,u)$

We get a contradiction, since $0 \leq k < 1$. Thus $z = u$ is a unique fixed point of f.

Theorem 3.3 - Let (X, G) be a G-metric space .Let $f: X \to X$ be a mapping such

$$G(fx,fy,fz) \leq kM(x,y,z), for\ all\ x,y,z \in X\ and\ k \epsilon \left[0,\frac{1}{2}\right)\ and$$

$$M(x,y,z) = max\{G(x,y,z), G(f^2x,fy,fz), G(z,fx,fy), G(y,f^2x,fy), G(x,fx,fx)$$
$$G(y,fy,fy), G(z,fz,fz), G(fx,f^2x,fz), G(z,f^2x,fz), G(fx,f^2x,fy)\}$$

Then there is a unique $x \in X\ such\ that\ fx = x$.

Proof – Proof of the theorem is same as above.

Example:-Let $X = [\,0,\infty), G: X \times X \times X \to R\ be\ defined\ by$

$$G(x,y,z) = \begin{cases} 0, & if\ x = y = z \\ max\{x,y,z\}, & otherwise \end{cases}$$

Then (X, G) is a complete G-metric space

Let $f: X \to X$ be defined by

$$\begin{cases} \frac{1}{3}x, & if\ 0 \leq x < \frac{1}{2} \\ \frac{1}{6}x^3, & if\ \frac{1}{2} \leq x \leq 1 \end{cases}$$

And $\varphi(t) = \frac{2}{3}\ t\ , for\ all\ t \in [0,\infty)$

Solution:- First we examine the following cases:

Let $0 \leq x, y < \frac{1}{2}$, then

$$G(fx,f^2x,fy) = max\left\{\frac{1}{3}x, \frac{1}{9}x, \frac{1}{3}y\right\}$$
$$\leq \frac{1}{3}max\left\{x, \frac{1}{3}x, y\right\}$$

Let $\frac{1}{2} \leq x, y < 1$, then

$$G(fx,f^2x,fy) = max\{\frac{1}{6}x^3, \frac{1}{36}x^9, \frac{1}{6}y^3\}$$
$$\leq \frac{1}{3}max\{x, \frac{1}{6}x^3, y\}$$

Let $0 \leq x < \frac{1}{2} \leq y < 1$, then

$$G(fx,f^2x,fy) = max\left\{\frac{1}{3}x, \frac{1}{9}x\frac{1}{6}y^3\right\}$$
$$\leq \frac{1}{3}max\{x, \frac{1}{3}x, y\}$$

Let $0 \leq y < \frac{1}{2} \leq x < 1$, then

$$G(fx,f^2x,fy) = max\left\{\frac{1}{6}x^3, \frac{1}{36}x^9, \frac{1}{3}y\right\}$$
$$\leq \frac{1}{3}max\{x, \frac{1}{6}x^3, y\}$$

Above cases hold the condition –

$$G(fx,f^2x,fy) = \varphi(G(x,fx,y))$$

Hence f has a unique fixed point.

Here (0, 0, 0) is a fixed point.

Theorem -3.4 Let (X, G) be a G-metric space and let f and g be self mappings on X satisfying the followings –

(1) $g(X) \subseteq f(X)$

(2) $f(X)$ or $g(X)$ is complete subspace of X.

(3)$G(gx,gy,gz) \leq \varphi(G(fx,fy,fz))$

where $\varphi: [0,1] \to [0,1]$ is continuous function s.t. $\varphi(s) < s$, and $\varphi(0) = 0$.

Then, f and g have a point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof –Let $x_0 \in X$, from eq.(1) ,we can construct a sequence $\{x_n\}$ and $\{y_n\}$ in X,

$$y_n = fx_{n+1} = gx_n \quad, n = 0,1,2,3 \ldots\ldots\ldots$$

From eq. (3),we have

$G(y_n, y_{n+1}, y_{n+1})$=$G(gx_n, gx_{n+1}, gx_{n+1})) \leq \varphi(G(fx_n, fx_{n+1}, fx_{n+1}))$

$$= \varphi(G(y_{n-1}, y_n, y_n))$$

$$< G(y_{n-1}, y_n, y_n) \quad (4)$$

Since φ is non-decreasing, therefore we have

$$G(y_n, y_{n+1}, y_{n+1}) \leq G(y_{n-1}, y_n, y_n)$$

$let\ s_n = G(y_n, y_{n+1}, y_{n+1})$,then $0 \leq s_n \leq s_{n-1}$, for all $n > 0$.

$it\ follows\ that\ the\ sequence\ \{s_n\}$ is monotonically decreasing and bounded below .So there exist some $r \geq 0$.

Such that $lim_{n\to\infty} G((y_{n+1}, y_n, y_n) = lim_{n\to\infty} s_n = r$ (5)

From eq.(4) and (5), and letting $n \to \infty$, we have that $lim_{n\to\infty} G\{(y_{n+1}, y_n, y_n)\} = 0$ (6)

Now next we show that the $\{y_n\}$ is G-Cauchy sequence, on contrary let $\{y_n\}$ is not G-cauchy sequence then so there exist $\epsilon > 0$ and subsequence $\{y_{n_k}\}\ and\ \{y_{m_k}\}$ of $\{y_n\}$

with $n(k) > m(k) > k$.

Such that $G(y_{n_k}, y_{m_k}, y_{m_k}) \geq \epsilon\ , for\ all\ k \in N$ (7)

More over, corresponding to m_k , we can choose n_k, such that it is the smallest integer with $n_k > m_k$ Satisfying equation (4).

Then that $G(y_{n_{k-1}}, y_{m_k}, y_{m_k}) < \epsilon$ (8)

Then we have ,

$$\begin{aligned} \epsilon &\leq G(y_{n_k}, y_{m_k}, y_{m_k}) \\ &\leq G(y_{n_k}, y_{n_{k-1}}, y_{n_{k-1}}) + G(y_{n_{k-1}}, y_{m_k}, y_{m_k}) \\ &< \epsilon + G(y_{n_k}, y_{n_{k-1}}, y_{n_{k-1}}) \end{aligned} \quad (9)$$

Setting $k \to \infty$ and using equation (6),$lim_{k\to\infty} G(y_{n_k}, y_{n_{k-1}}, y_{n_{k-1}}) = 0$

Then from (8), $\lim_{k\to\infty} G(y_{n_k}, y_{m_k}, y_{m_k}) = \epsilon$ (10)

Moreover we have,

$$G(y_{n_k}, y_{m_k}, y_{m_k}) \leq G(y_{n_k}, y_{n_{k-1}}, y_{n_{k-1}}) + G(y_{n_{k-1}}, y_{m_{k-1}}, y_{m_{k-1}}) + G(y_{m_{k-1}}, y_{m_k}, y_{m_k})$$

$$G(y_{n_k-1}, y_{m_{k-1}}, y_{m_{k-1}}) \leq G(y_{n_k-1}, y_{nk}, y_{nk}) + G(y_{n_k}, y_{m_k}, y_{m_k}) + G(y_{m_k}, y_{mk-1}, y_{mk-1})$$

Now letting $k \to \infty$ in the above inequality and using (6)-(10), we get

$lim_{k\to\infty} G(y_{n_k-1}, y_{m_{k-1}}, y_{m_{k-1}}) = \epsilon$ (11)

Taking $x = x_{nk}, y = x_{mk}\ in$ (3), we get,

$$G(y_{n_k}, y_{m_k}, y_{m_k}) = G(gx_{n_k}, gx_{m_k}, gx_{m_k}) \leq \varphi\left(G(fx_{n_k}, fx_{m_k}, fx_{m_k})\right)$$

$$= \varphi(G(y_{n_{k-1}}, y_{m_{k-1}}, y_{m_{k-1}})) < G(y_{n_{k-1}}, y_{m_{k-1}}, y_{m_{k-1}})$$

letting $k \to \infty$ in the above inequality (11), we get

$\epsilon \leq \varphi(\epsilon) < \epsilon$, which is a contradiction, since $\epsilon > 0$.

Thus $\{y_n\}$ is a G-cauchy sequence.

Since $f(X)$ is complete subspace of X, so there exist a point $u \in f(X)$, such that

$lim_{n\to\infty} y_n = lim_{n\to\infty} fx_{n+1} = u$ (12)

Now we show that u is a common fixed point of $f\ and\ g$.

Since $u \in f(X)$, so there exist a point $p \in X$, such that $fp = u$.

From eq.(3),

$$G(fp, gp, gp) = \lim_{n\to\infty} G(gx_n, gp, gp) \leq \lim_{n\to\infty} \varphi(G(fx_n, gp, gp))$$

Using (12) and the property of φ, we have

$G(fp, gp, gp) \leq \varphi(0) = 0$, hence $fp = gp = u$.

$hence\ u$ is the coincidence point of f and g.

Since $fp = gp$ and f , g are weakly compatible, we have $fu = fgp = gfp = gu$.

Now we claim that $fu = gu = u$.

Let if possible, $gu \neq u$, from eq. (3),we get

$$G(gu, u, u) = G(gu, gp, gp) \leq \varphi G(fu, fp, fp) = \varphi G(gu, u, u) < G(gu, u, u)$$

Which is a contradiction, hence $gu = u = fu.so\ u$ is a common fixed point of f and g

Uniqueness – let v be another common fixed point of f and g.so that $fv = gv = v$.

We claim that,$u = v$. let if possible $u \neq v$.

From eq. (3),

$$G(u, v, v) = G(gu, gv, gv) \leq \varphi G(fu, fv, fv) < G(fu, fv, fv) = G(u, v, v)$$

Which is a contradiction. we get, $u = v$

Hence u is the common fixed point of f and g.

Reference

[1]. Karapinar,E, Agarwal,R.P : Further fixed point results on G-metric spaces, Fixed point theory and application 2013,2013:154
[2]. Samet, B, Vetro, C, Vetro, F: Remarks on G-metric spaces. Int. J. Anal. 2013, Article ID 917158 (2013)
[3]. Jleli, M, Samet, B: Remarks on G-metric spaces and fixed point theorems. Fixed Point Theory Appl. 2012, Article ID 210 (2012)
[4]. Mustafa, Z, Sims, B: A new approach to generalized metric spaces. J. Nonlinear Convex Anal. 7(2), 289-297 (2006)
[5]. Banach, S: Sur les operations dans les ensembles abstraits et leur application aux equations integrales. Fundam.Math. 3, 133-181 (1922)
[6]. Abbas, M, Sintunavarat, W, Kumam, P: Coupled fixed point of generalized contractive mappings on partially ordered G-metric spaces. Fixed Point Theory Appl. 2012, Article ID 31 (2012)
[7]. Abbas, M, Nazir, T, Vetro, P: Common fixed point results for three maps in G-metric spaces. Filomat 25(4), 1-17 (2011)
[8]. Agarwal, R, Karapınar, E: Remarks on some coupled fixed point theorems in G-metric spaces. Fixed Point Theory Appl.2013, Article ID 2 (2013)
[9]. Aydi, H, Shatanawi, W, Vetro, C: On generalized weak G-contraction mapping in G-metric spaces. Comput. Math.Appl. 62, 4223-4229 (2011)
[10]. Aydi, H, Karapinar, E, Shatnawi, W: Tripled fixed point results in generalized metric spaces. J. Appl. Math. 2012, ArticleID 314279 (2012)
[11]. Aydi, H, Karapinar, E, Mustafa, Z: On common fixed points in G-metric spaces using (E.A) property. Comput. Math.Appl. 64(6), 1944-1956 (2012)
[12]. Aydi, H, Postolache, M, Shatanawi, W: Coupled fixed point results for (ψ,φ)-weakly contractive mappings in ordered G-metric spaces. Comput. Math. Appl. 63(1), 298-309 (2012)
[13]. Aydi, H, Karapınar, E, Shatanawi, W: Tripled fixed point results in generalized metric spaces. J. Appl. Math. 2012, ArticleID 314279 (2012)
[14]. Aydi, H, Damjanovi´c, B, Samet, B, Shatanawi, W: Coupled fixed point theorems for nonlinear contractions in partially ordered G-metric spaces. Math. Comput. Model. 54, 2443-2450 (2011)
[15]. Berinde, V: Generalized coupled fixed point theorems for mixed monotone mappings in partially ordered metricspaces. Nonlinear Anal. 74, 7347-7355 (2011)
[16]. Berinde, V: Coupled fixed point theorems for _-contractive mixed monotone mappings in partially ordered metric spaces. Nonlinear Anal. 75, 3218-3228 (2012)
[17]. Berinde, V: Coupled coincidence point theorems for mixed monotone nonlinear operators. Comput. Math. Appl.(2012). doi:10.1016/j.camwa.2012.02.012
[18]. Cho, YJ, Rhoades, BE, Saadati, R, Samet, B, Shatanawi, W: Nonlinear coupled fixed point theorems in ordered generalized metric spaces with integral type. Fixed Point Theory Appl. 2012, Article ID 8 (2012)
[19]. Choudhury, BS, Kundu, A: A coupled coincidence point result in partially ordered metric spaces for compatible mappings. Nonlinear Anal. 73, 2524-2531 (2010)
[20]. Choudhury, BS, Maity, P: Coupled fixed point results in generalized metric spaces. Math. Comput. Model. 54, 73-79(2011)
[21]. ´ Ciri´c, L: A generalization of Banach's contraction principle. Proc. Am. Math. Soc. 45(2), 267-273 (1974)
[22]. ´ Ciri´c, L, Agarwal, RP, Samet, B: Mixed monotone-generalized contractions in partially ordered probabilistic metric spaces. Fixed Point Theory Appl. 2011, Article ID 56 (2011)
[23]. Ding, H-S, Karapınar, E: A note on some coupled fixed point theorems on G-metric space. J. Inequal. Appl. 2012, Article ID 170 (2012)
[24]. Mustafa, Z, Aydi, H, Karapınar, E: On common fixed points in image-metric spaces using (E.A) property. Comput.Math. Appl. 64(6), 1944-1956 (2012)
[25]. Mustafa, Z: A new structure for generalized metric spaces with applications to fixed point theory. Ph.D. thesis, TheUniversity of Newcastle, Australia (2005)
[26]. Mustafa, Z, Obiedat, H, Awawdeh, F: Some fixed point theorem for mapping on complete G-metric spaces. Fixed Point Theory Appl. 2008, Article ID 189870 (2008)
[27]. Mustafa, Z, Khandaqji, M, Shatanawi, W: Fixed point results on complete G-metric spaces. Studia Sci. Math. Hung. 48,304-319 (2011)
[28]. Mustafa, Z, Sims, B: Fixed point theorems for contractive mappings in complete G-metric spaces. Fixed Point Theory Appl. 2009, Article ID 917175 (2009)
[29]. Mustafa, Z, Shatanawi, W, Bataineh, M: Existence of fixed point results in G-metric spaces. Int. J. Math. Math. Sci. 2009, Article ID 283028 (2009)
[30]. Mustafa, Z, Obiedat, H: A fixed point theorem of Reich in G-metric spaces. CUBO 12(1), 83-93 (2010)
[31]. Nashine, HK: Coupled common fixed point results in ordered G-metric spaces. J. Nonlinear Sci. Appl. 1, 1-13 (2012)
[32]. Rasouli, SH, Bahrampour, M: A remark on the coupled fixed point theorems for mixed monotone operators in partially ordered metric spaces. J. Math. Comput Sci. 3(2), 246-261 (2011)
[33]. Samet, B: Coupled fixed point theorems for a generalized Meir-Keeler contraction in partially ordered metric spaces. Nonlinear Anal. 74, 4508-4517 (2010)
[34]. Shatanawi, W: Coupled fixed point theorems in generalized metric spaces. Hacet. J. Math. Stat. 40(3), 441-447 (2011)
[35]. Shatanawi, W: Fixed point theory for contractive mappings satisfying _-maps in G-metric spaces. Fixed Point Theory Appl. 2010, Article ID 181650 (2010)
[36]. Shatanawi, W: Some fixed point theorems in ordered G-metric spaces and applications. Abstr. Appl. Anal. 2011, Article ID 126205 (2011)
[37]. Shatanawi, W: Coupled fixed point theorems in generalized metric spaces. Hacet. J. Math. Stat. 40(3), 441-447 (2011)
[38]. Shatanawi, W, Abbas, M, Nazir, T: Common coupled coincidence and coupled fixed point results in two generalized metric spaces. Fixed Point Theory Appl. 2011, Article ID 80 (2011)
[39]. Tahat, N, Aydi, H, Karapınar, E, Shatanawi, W: Common fixed points for single-valued and multi-valued maps satisfying a generalized contraction in G-metric spaces. Fixed Point Theory Appl. 2012, Article ID 48 (2012)