IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 12, Issue 6 Ver. II (Nov. - Dec.2016), PP 26-31
www.iosrjournals.org

Complementary Tree Paired Domination in Graphs

A. Meenakshi[1], J. Baskar Babujee[2]
[1](Department of Mathematics, SKR Engineering College, Chennai-600123, India)
[2](Department of Mathematics, Anna University, MIT Campus, Chennai-600 044,India)

Abstract: *A dominating set $S \subseteq V(G)$ is said to be a complementary tree dominating set if the induced sub graph $\langle V-S \rangle$ is a tree. The minimum cardinality of a complementary tree dominating set of G is called the complementary tree domination number of G and is denoted by $\gamma_{ctd}(G)$. In this paper we have introduced the new type of domination named complementary tree paired domination number for connected graphs and we study the theoretical properties of this parameter and many bounds are obtained in terms of order of G and its relationship with other domination parameters are also obtained. The relation between the complementary tree paired domination number and total domination number of a tree is also discussed.*
Keywords: *Graph, Domination, tree, Perfect Matching*

I. Introduction

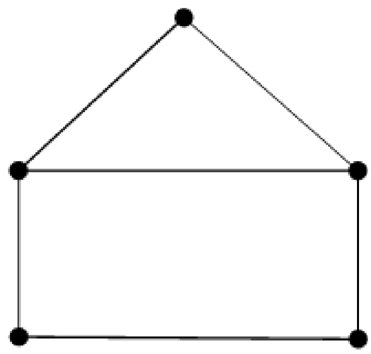
Let $G(p,q)$ be a simple undirected graph with p vertices and q edges. The set of vertices is denoted by $V(G)$; the set of edges by $E(G)$. Order of G is denoted as $|G|$ as a symbol for the cardinality of $V(G)$.The degree, Neighborhood and closed neighborhood of a vertex v in the graph G are denoted by $d(v)$, $N(v) and\ N[v] = N(v) \cup \{v\}$ respectively. For a subset S of $V(G)$, $N(S)$ denotes the set of all vertices adjacent to some vertex of S in G and $N[S] = N(S) \cup S$. Let $\langle S \rangle$ and $G - S$ denote the sub graphs of G induced by S and $V(G) - S$ respectively. The minimum degree and maximum degree of the graph G are denoted by $\delta(G)$ *and* $\Delta(G)$ respectively. A vertex of degree one is called a leaf and its neighbor is a support vertex. A double star is a tree with exactly two support vertices. The corona of a graph G is the graph formed from a copy of G by attaching for each $v \in V$, a new vertex v' and edge vv'. In general, the K-corona of a graph G is the graph of order $k\ |V(G)|$ obtained from G by adding a path of length k to each vertex of G so that the resulting paths are vertex disjoint. A set $S \subseteq V$ is a dominating set if for every vertex v in $V - S$, there exists a vertex u in S such that v is adjacent to u. The minimum cardinality of a dominating set in G is called the domination number of G and is denoted by $\gamma(G)$.We call a dominating set of minimum cardinality a $\gamma(G)$-set of G. A set $S \subseteq V$ is a total dominating set of abbreviated TDS, if every vertex in V is adjacent to a vertex in S. We call a total dominating set of minimum cardinality a $\gamma_t(G)$-set of G. A set $S \subseteq V$ is a paired dominating set if the induced sub graph $\langle S \rangle$ has a perfect matching. The minimum cardinality of a paired dominating set in G is called the paired domination number of G and is denoted by $\gamma_{pr}(G)$. We call a paired dominating set of minimum cardinality a $\gamma_{pr}(G)$-set of G.This concept was introduced by Haynes et al. [11]. A dominating set D of a connected graph G is a non split dominating set, if the induced sub graph $\langle V(G) - D \rangle$ is connected. The minimum cardinality of a non split dominating set of a graph G is called the non split domination number and is denoted by the $\gamma_{ns}(G)$. A dominating set D of a connected graph G is a split dominating set, if the induced sub graph $\langle V(G) - D \rangle$ is disconnected. The minimum cardinality of a split dominating set of a graph G is called the split domination number is denoted by the $\gamma_s(G)$. Both these concepts were introduced by Kulli et al.[4] . A dominating set $S \subseteq V(G)$ is said to be a complementary tree dominating set if the induced sub graph $\langle V - S \rangle$ is a tree. The minimum cardinality of a complementary tree dominating set of G is called the complementary tree domination number of G and is denoted by $\gamma_{ctd}(G)$.The complementary tree dominating set was introduced by S.Muthammai et al .[7] **.**

If $G_1(p_1,q_1)$ *and* $G_2(p_2,q_2)$ are two connected graphs, $G_1\hat{o}G_2$ is obtained by merging any selected vertex of G_2 on any selected vertex of G_1. If we arbitrarily choose the vertices of G_1 and G_2 then $G_1\hat{o}G_2$ is a class of graphs consisting of $p_1 + p_2 - 1$ vertices and $q_1 + q_2$ edges. In general, we construct $p_1 p_2$ possible combination of graphs from G_1 and G_2. This concept was introduced by Jayapal Baskar Babujee et al .[3]. The concept of complement of a graph was defined by Bondy et al. [1]**.** Many of the authors introduced the concept related with complements. In particular, complementary perfect dominating set was introduced by Paulraj Joseph.et al.[9].The concept Complementary nil domination number of a graph was introduced by T.Tamizhchelvam et al.[10]. Likewise, In this paper we have introduced the new parameter named complementary tree paired domination number and it is denoted by $\gamma_{ctpd}(G)$. We obtain the upper bounds and lower bounds of complementary tree paired domination number in terms of maximum degree ,size and order of the graph G.

II. Complementary Tree Paired Domination Number And Its Lower And Upper Bounds

Definition 2.1: A paired dominating set $S \subseteq V(G)$ is said to be a complementary tree paired dominating set if the induced sub graph $\langle V-S \rangle$ is a tree. The minimum cardinality of a complementary tree paired dominating set (CTPD set) of G is called the complementary tree paired domination number of G and is denoted by $\gamma_{ctpd}(G)$ and the corresponding set is $\gamma_{ctpd} - set\ of\ G.$ The existence of such a set is guaranteed only for connected graphs.

Example: Let G be a graph mentioned in figure (i)



Fig(i) $\gamma_{ctpd}(G) = \gamma_{pr}(G) = \gamma(G) = 2$ Some observations are made.

Observation 2.2:

1. For the path graph P_n , $\gamma_{ctpd}(P_n) = \begin{cases} n-2 & if\ n \equiv 0 (\text{mod } 2) \\ n-1 & if\ n \equiv 1 (\text{mod } 2) \end{cases}$
2. For the cycle graph C_n , $\gamma_{ctpd}(C_n) = \begin{cases} n-2 & if\ n \equiv 0 (\text{mod } 2) \\ n-1 & if\ n \equiv 1 (\text{mod } 2) \end{cases}$

   For the complete graph K_n, $\gamma_{ctpd}(K_n) = \begin{cases} n-2 & if\ n \equiv 0 (\text{mod } 2) \\ n-1 & if\ n \equiv 1 (\text{mod } 2) \end{cases}$
3. For any connected graph G , $\gamma(G) \le \gamma_{ctpd}(G)$
4. For any connected graph G , $\gamma_{ctpd}(G) \le \begin{cases} n-2 & if\ n \equiv 0 (\text{mod } 2) \\ n-1 & if\ n \equiv 1 (\text{mod } 2) \end{cases}$

The inequality holds good for C_n and K_n .

Observation 2.3[6]: A paired dominating set(PDS) S of a graph G is a minimal PDS if and only if any two vertices $x, y \in S$ satisfy one of the following conditions:

(i) $G[S-\{x,y\}]$ does not contain a perfect matching

(ii) Without loss of generality, x is an end vertex in $G[S]$ adjacent to y.

(iii) There exists a vertex $u \in V-S$ *such that* $N(u) \cap S \subseteq \{x,y\}$

Observation 2.4: *A CTPD set S of a graph G is a minimal CTPD set if and only if any two vertices $x, y \in S$ satisfy one of the following conditions:*

(i) $G[S-\{x,y\}]$ does not contain a perfect matching

(ii) There exists a vertex $u \in V-S$ such that $N(u) \cap S \subseteq \{x,y\}$

Proof: Suppose S is a minimal complementary tree paired dominating set of G. Then for any two vertices x *and* y, $S-\{x,y\}$ is not a complementary tree paired dominating set of G. Therefore $G[S-\{x,y\}]$ does not contain a perfect matching. Let $u \in V-S$ is not dominated by $S-\{x,y\}$, but it is dominated by S then u is adjacent to either x *or* y or both.

Conversely, suppose S is a minimal complementary tree paired dominating set of G. Then for any two vertices $x, y \in S$, one of the two stated conditions holds. Now prove that S is a minimal complementary tree paired dominating set of G. Suppose S is not a minimal complementary tree paired dominating set,then there exists two vertices say $x, y \in S$ such that $S-\{x,y\}$ is a complementary tree paired dominating set. Therefore condition (i) does not hold. If $S-\{x,y\}$ is a complementary tree paired dominating set then every vertex in $V-S$ is adjacent to at least one vertex in $S-\{x,y\}$. Therefore for any two vertices $x, y \in S$, condition (ii) does not hold. Hence neither condition (i) nor (ii) holds, which is a contradiction.

Next let us see some lower bounds of CTPD number.

Theorem 2.5: For any connected graph, $\gamma(G) \le \gamma_{pr}(G) \le \gamma_{ctpd}(G)$

Proof: Let G be any connected graph. Any complementary tree paired dominating set of G is also a paired dominating set of G .Thus $\gamma_{pr}(G) \le \gamma_{ctpd}(G)$. Furthermore, any paired dominating set of G is also a dominating set of G. Hence $\gamma(G) \le \gamma_{pr}(G)$. Therefore we conclude that $\gamma(G) \le \gamma_{pr}(G) \le \gamma_{ctpd}(G)$.

H.B.Walikar et al.[12] already proved that

For any graph G , $\left\lceil \frac{p}{\Delta(G)+1} \right\rceil \leqslant \gamma(G)$

Theorem 2.6: For any connected graph $G(p,q)$ with $p \ge 2$, $\left\lceil \frac{p}{\Delta+1} \right\rceil \leqslant \gamma_{ctpd}(G) \leqslant 2q - p + 1$

Proof: Since $\left\lceil \frac{p}{\Delta+1} \right\rceil \leqslant \gamma(G)$ and By theorem 2.5, we have $\gamma(G) \le \gamma_{ctpd}(G)$

$$\left\lceil \frac{p}{\Delta+1} \right\rceil \leqslant \gamma_{ctpd}(G) \quad \text{……………………(i)}$$

Since $\langle V-D \rangle$ is a tree, $\gamma_{ctpd}(G) \leqslant p-1$

Hence $\gamma_{ctpd}(G) \leqslant p-1 = 2(p-1)-(p-1)$

$$\gamma_{ctpd}(G) \le 2q - p + 1 \text{……………………(ii)}$$

From (i) and (ii) we have $\left\lceil \frac{p}{\Delta+1} \right\rceil \leqslant \gamma_{ctpd}(G) \leqslant 2q - p + 1$

Theorem 2.7: If $G(p,q)$ is a connected graph with $\delta(G) \ge 2$ then $\gamma_{ctpd}(G) \ge 3p - 2q - 2$

Proof: Let D be a γ_{ctpd} – set of G . Let t be the number of edges in G having one vertex in D and other vertex in $V-D$.

The number of vertices in $\langle V-D\rangle$ is $|V-D|$ = $p-\gamma_{ctpd}(G)$ and the number of edges in $\langle V-D\rangle$ is $(p-\gamma_{ctpd}(G))-1$, We know that

$$2q=\sum_{v_i\in D}d(v_i)+\sum_{v_i\in V-D}d(v_i)+t$$

Since $\sum_{v_i\in V-D}d(v_i)=2[(p-\gamma_{ctpd}(G))-1]$

$$\sum_{v_i\in D}d(v_i)+t=2q-\sum_{v_i\in V-D}d(v_i)$$

$$\sum_{v_i\in D}d(v_i)+t=2\{q-((p-\gamma_{ctpd}(G))-1)\}$$

Since $|V(G)-D|=p-\gamma_{ctpd}(G)$ there are at least $p-\gamma_{ctpd}(G)$ edges from $V(G)-D$ to D

Also deg $(v_i)\geq\delta(G)$

Hence, $2\{q-(p-\gamma_{ctpd}(G)-1)\}\geq\delta(G)\gamma_{ctpd}(G)+p-\gamma_{ctpd}(G)$

Since $\delta(G)\geq 2$, we have

$$2\{q-p+\gamma_{ctpd}G)+1\}\geq 2\gamma_{ctpd}(G)+p-\gamma_{ctpd}(G)$$

$$2q-2p+2\gamma_{ctpd}(G)+2\geq 2\gamma_{ctpd}(G)+p-\gamma_{ctpd}(G)$$

Hence $\gamma_{ctpd}(G)\geq 3p-2q-2$

Theorem 2.8:** **If** $G(p,q),\delta(G)\geq 2$ **is a connected graph, then there exists a graph** G', $\delta(G')\geq 2$ **from the class** $G\,\hat{o}\,C_n$ **such that $\gamma_{ctpd}(G')\geq 3p-2q+n-4+k, where\, k=n(\mathrm{mod}\,2)$

Proof: Let $G(p,q)$ be a connected graph with $\delta(G)\geq 2$ and C_n be a cycle graph

Form the graph $G\hat{0}C_n$ and let $G'=G\hat{0}C_n$

$|V(G\hat{0}C_n)|=p+n-1$ and $|E(G\hat{0}C_n)|=q+n$

By theorem 2.7, $\gamma_{ctpd}(G)\geq 3p-2q-2$ and $\gamma_{ctpd}(C_n)=n-2$ *if n is even*

Hence $\gamma_{ctpd}(G\hat{0}C_n)\geq(3p-2q-2)+(n-2)$

Therefore $\gamma_{ctpd}(G')\geq 3p-2q+n-4$

Also $\gamma_{ctpd}(C_n)=n\text{-}1$ *if n is odd*

Therefore $\gamma_{ctpd}(G\hat{0}C_n)\geq(3p-2q-2)+(n-1)$

$\geq 3p-2q+n-3$

Hence $\gamma_{ctpd}(G')\geq 3p-2q+n-4+k, where\, k=n(\mathrm{mod}\,2)$

Theorem 2.9:** **If** $G_1(p_1,q_1)$; $\delta(G_1)\geq 2$ **and** $G_2(p_2,q_2)$; $\delta(G_2)\geq 2$ **are connected graphs, then there exists a graph** G **from the class** $G_1\,\hat{o}\,G_2$ **such that $\gamma_{ctpd}(G)\geq 2p-q-4$

Proof: Let $G=G_1\hat{0}G_2$

Let $G_1(p_1,q_1)$ be a connected graph and $G_2(p_2,q_2)$ be a connected graph.

Form the graph $G_1\hat{0}G_2$ with p vertices and q edges, where $p=p_1+p_2-1$ and $q=q_1+q_2$

Let D be a $\gamma_{ctpd}-$ set of G. Let t be the number of edges in G having one

vertex in D and other vertex in $V-D$.

The number of vertices in $\langle V-D\rangle$ is $|V-D|$ = $p-\gamma_{ctpd}(G)$ and the number of edges in $\langle V-D\rangle$ is $(p-\gamma_{ctpd}(G\ ))-1$ then we have,

$$\sum_{v_i\in D}\deg(v_i)+t=2\{q-(p-\gamma_{ctpd}(G)-1)\}$$

Since $G=G_1\widehat{0}G_2$ and G have p_1+p_2-1 vertices, there are at least $2(p-1-\gamma_{ctpd}(G))$ edges from $V(G)-D$ to D Also deg $(v_i)\geq\delta(G)$

Hence, $2\{q-(p-\gamma_{ctpd}(G)-1)\}\geq\delta(G\ )\gamma_{ctpd}(G)+(2(p-1-\gamma_{ctpd}(G))$

Since $\delta(G)\geq 2$, we have

$$2\{q-p+\gamma_{ctpd}G)+1\}\geq 2\gamma_{ctpd}(G)+2p-2-2\gamma_{ctpd}(G)$$

$$2q-2p+2\gamma_{ctpd}(G)+2\geq 2p-2$$

$$\gamma_{ctpd}(G)\geq 2p-q-4$$

III. Complementary Tree Paired Domination Number And Total Domination Number of a Tree

The following upper bounds are known to us

Cockayne et al.[2] proved that, For any connected graph G of order $n\geq 3$, $\gamma_t(G)\leq 2n/3$

Haynes et al.[11] proved that, For any graph G without isolated vertices, $\gamma_{pr}(G)\leq 2\gamma(G)$

Before to prove our main result, first to prove the following theorem.

***Theorem 3.1:** If v is a support vertex of a connected graph G, then v is in every $\gamma_{ctpd}-$set and in every γ_t-set.*

Proof: Let v be a support vertex of a connected graph $G.$ Suppose v is not a member of the complementary tree paired dominating set then the pendent vertex adjacent to v is not dominated by any other vertex of $G.$ So v must be present in every $\gamma_{ctpd}-set\ of\ G$ and is also in every $\gamma_t-set\ of\ G.$

Observation 3.2:[5] For any connected graph G with diameter at least three, there exists a $\gamma_t(G)$-set that contains no leaves of G For a vertex u in a rooted tree T, we denote by T_u the sub tree of T induced by its descendents, s denote support vertices and l denote leaves .We now present the upper bounds.

***Theorem 3.3:** If T is a non path tree of order at least five with equal number of support vertices and leaves then $\gamma_t(T)\leq\gamma_{ctpd}(T)\leq\gamma_t(T)+s+1$.*

Proof: By the definition of CTPD- set the lower bound follows. To prove the upper bound, we proceed this theorem by induction on T which is of order at least five with s support vertices and l leaves.

It is obvious that the inequality holds for $p=5.$ Let diam (T) $\geq$ 5. Assume that for any tree T' of order $6\leq p' < p$ having s' support vertices, $\gamma_{ctpd}(T')\leq\gamma_t(T')+s'+1.$ Let T be a tree of order p with s support vertices, and let $S\ and\ D$ be a $\gamma_{ctpd}(T)$ set and a $\gamma_t(T)$ set respectively.

Root the tree at a vertex r of maximum eccentricity $diam(T)\geq 5.$ Let u be a support vertex at maximum distance from r and v be a parent of u in the rooted tree. So deg(u) = 2. Let w be the parent of v and x be the parent of $w.$ By our choice of u, every child of v is either a leaf or a support vertex of degree two. Now three cases arises.

Case (i): Let v has a child besides u say y that is a support vertex. By theorem 3.1, u and y are in S .Now either (i) u may be paired with its leaf u' and y may be paired with its leaf y' or (ii) at most one of u or y is paired with $v.$ In particular let u be paired with its leaf u' and y may be paired with its leaf $y'.$ Let

$T^{'} = T - T_u$. Since $y \in S$, $S \cap T'$ is a complementary tree paired dominating set of T', and so $\gamma_{ctpd}(T') \le \gamma_{ctpd}(T) - 2$. On the other hand, every $\gamma_{ctpd}(T')$ -set can be extended to a complementary tree paired dominating set of T by adding u and u'. Thus $\gamma_{ctpd}(T) \le \gamma_{ctpd}(T') + 2$. By observations 3.2 , there exists a $\gamma_t(T')$ set D' containing v *and* y. Hence $D' \cup \{u\}$ is a TDS of T and $\gamma_t(T) \le \gamma_t(T') + 1$. Since every $\gamma_t(T)$ set contains v, y and u, it follows that $D \cap V(T')$ is a TDS of T', and so $\gamma_t(T') \le \gamma_t(T) - 1$. Thus $\gamma_t(T') = \gamma_t(T) - 1$ and $s' = s - 1$. By induction hypothesis to T', We have $\gamma_{ctpd}(T') \le \gamma_t(T') + s' + 1$. Therefore $\gamma_{ctpd}(T) - 2 \le \gamma_t(T) - 1 + (s-1) + 1$ and hence $\gamma_{ctpd}(T) \le \gamma_t(T) + s + 1$. We proceed the same procedure for (ii).

Case (ii): Let v is a support vertex and has no child besides u of degree two. Let $T' = T - T_v$. Assume that T' has order at least three. Then $\gamma_{ctpd}(T) \le \gamma_{ctpdt}(T') + 2$ and $s - 2 \le s' \le s - 1$. By theorem 3.1, u is in D. If $(N(v) - \{u\}) \cap D \ne \phi$, then $D - \{u\}$ is a TDS of T'. If $(N(v) - \{u\}) \cap D = \phi$ then $(D \cup \{w\}) - \{u\}$ is a TDS of T'. Thus $\gamma_t(T') \le \gamma_t(T) - 1$. By induction hypothesis to T', We have $\gamma_{ctpd}(T') \le \gamma_t(T') + s' + 1$. Therefore $\gamma_{ctpd}(T) - 2 \le \gamma_t(T) - 1 + (s-1) + 1$ and hence $\gamma_{ctpd}(T) \le \gamma_t(T) + s + 1$.

Case (iii) **:** Let v has no child besides u, that is, $\deg(v) = 2$. Suppose $\deg(w) \ge 3$, and $T' = T - T_v$. Then $s' = s - 1$. Any $\gamma_{ctpd}(T')$ set can be extended to CTPD set of T by adding u and $u^{'}$ and so, $\gamma_{ctpd}(T) \le \gamma_{ctpdt}(T') + 2$. Furthermore, by theorem 3.1, we may assume that u and $v \in D$. If $(N(w) - \{v\}) \cap D \ne \phi$, then $D - \{u, v\}$ is a TDS of T'. If $(N(w) - \{v\}) \cap D = \phi$ then $(D \cup \{x\}) - \{u, v\}$ is a TDS of T'. Thus $\gamma_t(T') \le \gamma_t(T) - 1$. By induction hypothesis to T', We obtain the desired inequality.

This theorem shows the relationship between total domination number and complementary tree paired domination number of a tree. Also it shows that the upper and lower bound exists for complementary tree paired domination number of a tree in terms of total domination number and support vertices of T.

IV. Conclusion

This paper signifies the relationship between total domination number and complementary tree paired domination number of a tree. Also it shows the upper and lower bound exists for complementary tree paired domination number of a tree in terms of total domination number and support vertices of T. In future we have planned to work for vertex critical and edge critical of the determined parameter and also we find the relation between them.

References

[1] J.A. Bondy Murthy, *Graph Theory with Applications* (Elsevier science publishing co., Inc, Newyork, N.Y.10017, 1976).

[2] E.J. Cockayne, R.M. Dawes and S.T. Hedetniemi(1980), Total domination in graphs, *Networks, 10*, 1980, 211-219.

[3] Jayapal Baskar Babujee, Babitha Suresh, New Constructions of Edge Bimagic Graphs from Magic Graphs, *Applied Mathematics 2,* 2011, 1393-1396.

[4] V.R. Kulli and B. Janakiram, The split domination number of a graph, *Graph Theory Notes of New York, New York Academy of sciences XXXXII*, (1997), 16-19.

[5] Mustapha Chellai and Teresa W. Haynes , Total and paired domination numbers of a tree, *AKCE J. Gaphs Combin., 1(2),* (2004), 69-75.

[6] Mustapha Chellai and Teresa W. Haynes , On paired and double domination in graphs, *Utilitas Math., 67,* (2005, 161-171.

[7] S. Muthammai, M. Bhanumathi and Vidya, Complementary tree domination number of a graph, *International Mathematical forum, 6(26),* (2011), 1273-1282.

[8] O. Ore, Theory of Graphs, *Amer. Math. Soc. Colloq. Publ., 38 Providence,* (1962).

[9] J. Paulraj Joseph and G. Mahadevan, On complementary perfect domination number of a graph, *Acta Ciencia Indica, XXXIM(211),* (2006), 847-853.

[10] T. Tamizhchelvam and Robinson chellathurai, Complementary nil domination number of a graph, *Tamkang Journal of Mathematics, 40(2),* (2009), 165-172.

[11] Teresa W. Haynes, Stephen T. Hedetnimi, Peter J. Slater, (Fundamentals of Domination in Graphs, Marcel Decker, 1998).