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On the Convergence of a Polynomial

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Abstract: *We have tested the convergence of a newly definedpolynomialon the interval* $[0,1+\frac{r}{n}]$ *for Lebesgue integral in* L_1 *norm as*

$$U^{\alpha}_{nr}(f,x)=(n+r+1)\sum_{k=0}^{n+r}\left\{\int_{k/(n+r+1)}^{(k+1)/(n+r+1)}f(t)dt\right\}q_{nr,k}(x;\alpha)$$

where

$q_{nr,k}(x;\alpha)=\binom{n+r}{k}\frac{x(x+k\alpha)^{k-1}(1-x)(1-x+(n+r-k)\alpha)^{n+r-k-1}}{(1+(n+r)\alpha)^{n+r-1}}$ *andsothe results of Bernstein has been extended in the given interval .*

Keywords: *Bernstein Polynomials, Convergence, Bernstein typePolynomial,* L_1 *norm,Lebesgueintegrable function*

I. Introduction and Results

If $f(x)$ is a function defined [0,1], the Bernstein polynomial $B^{f}_{n}(x)$ of f is given as

$B^{f}_{n}(x)=\sum_{k=0}^{n}f(k/n)\,p_{n,k}(x)$ ……… (1.1)

where

$p_{n,k}(x)=\binom{n}{k}x^{k}(1-x)^{n-k}$ ………. (1.2)

Schurer [6] introduced an operator

S_{nr}: c $[0, 1+\frac{r}{n}]$ → c [0, 1]

defined by

$S_{nr}(f,x)=\sum_{k=0}^{n+r}f(\frac{k}{n+r})\,p_{nr,k}(x)$ --------- (1.3)

where

$p_{nr,k}(x)=\binom{n+r}{k}x^{k}(1-x)^{n+r-k}$ --------- (1.4)

and r is a non-negative integer. In case $r=0$, this reduces to the well-known Bernstein operator.

Bernstein (1912-13) proved that if $f(x)$ is continuous in closed interval [0 , 1] , then $B^{f}_{n}(x)\rightarrow f(x)$ uniformly as $n\rightarrow\infty$. This yields a simple constructive proof of Weierstrass's approximation theorem .

A slight modification of Bernstein polynomials due to Kantorovich[8] makes it possible to approximate Lebesgueintegrable function in L_1-norm by the modified polynomials

$P^{f}_{n}(x)=(n+1)\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)}f(t)dt\right\}p_{n,k}(x)$……………………………. (1.5)

where $p_{n,k}(x)$ is defined by (1.2)

By Abel's formula (see Jensen [7]) defined on [0,1] is given by

$(x+y)(x+y+n\alpha)^{n-1}=\sum_{k=0}^{n}\binom{n}{k}\,x(x+k\alpha)^{k-1}y(y+(n-k)\alpha)^{n-k-1}$…. .. (1.6)

Whichcan be defined on $[0, 1+\frac{r}{n}]$as

$(x+y)(x+y+(n+r)\alpha)^{n+r-1}=\sum_{k=0}^{n+r}\binom{n+r}{k}\,x(x+k\alpha)^{k-1}y(y+(n+r-k)\alpha)^{n+r-k-1}$…. (1.7)

If we put $y=1-x$, we obtain (see Cheney and Sharma [5])

$1=\sum_{k=0}^{n+r}\binom{n+r}{k}\frac{x(x+k\alpha)^{k-1}(1-x)(1-x+(n+r-k)\alpha)^{n+r-k-1}}{(1+(n+r)\alpha)^{n+r-1}}$……………………………. (1.8)

Thus defining

$q_{nr,k}(x;\alpha)=\binom{n+r}{k}\frac{x(x+k\alpha)^{k-1}(1-x)(1-x+(n+r-k)\alpha)^{n+r-k-1}}{(1+(n+r)\alpha)^{n+r-1}}$……………………….. (1.9)

we have
$\sum_{k=0}^{n+r} q_{nr,k}(x;\alpha) = 1$………………………………………………………………. (1.10)

For a finite interval [0, 1+$\frac{r}{n}$] ,we now defined a Bernstein type Polynomials (see Anwar Habib [2])
U_{nr}: c [0,1+$\frac{r}{n}$] $\rightarrow$ c[0,1]

by

$$U_{nr}^{\alpha}(f,x) = (n+r+1)\sum_{k=0}^{n+r}\left\{\int_{\frac{k}{n+r+1}}^{\frac{k+1}{n+r+1}} f(t)dt\right\} q_{nr,k}(x;\alpha) \text{------------} \quad (1.11)$$

where$q_{nr,k}(x;\alpha)$same as (1.9) and r isis a non-negative integer. When r =0 &$\alpha = 0$, then (1.9) & (1.11) reduces to (1.2) & (1.5) respectively.
In this paper we shall test the convergence of the polynomial (1.11) for Lebesgueintegrable function in L_1 norm. In fact our result is as follows

Theorem:If $f(x)$ is continuous Lebesgueintegrable function [0, 1+$\frac{r}{n}$] then$\alpha = \alpha_{nr} = o(\frac{1}{n+r})$

$$\lim_{(n+r)\to\infty} U_{nr}^{\alpha}(f,x) = f(x)$$

holdsuniformly on[0, $1+\frac{r}{n}$]

II. Lemmas

In order to prove our result we need the following lemmas (see AnwarHabib [2])
Lemma 2.1: For all values of x

$$\sum_{k=0}^{n+r} kq_{nr,k}(x;\alpha) \leq \frac{1+(n+r)\alpha}{1+\alpha}(n+r)x - \frac{(n+r)(n+r-1)x\alpha}{1+2\alpha}$$

Lemma 2.2: For all values of x

$$\sum_{k=0}^{n+r} k(k-1)q_{nr,k}(x;\alpha) \leq (n+r)(n+r-1)(x+2\alpha)\{\frac{1+(n+r)\alpha}{(1+2\alpha)^2} - \frac{(n+r-2)\alpha}{(1+3\alpha)^3}$$

$$+(n+r-2)\alpha^2(\frac{1+(n+r)\alpha}{(1+3\alpha)^3} - \frac{(n+r-3)\alpha}{(1+4\alpha)^4})\}$$

Lemma 2.3: For all values of x of [0, 1+$\frac{r}{n}$] and for $\alpha = \alpha_{nr} = 0\left(\frac{1}{n+r}\right)$, we have

$$(n+r+1)\sum_{k=0}^{n+r}\left\{\int_{k/(n+r+1)}^{(k+1)/(n+r+1)} ((t-x)^2)dt\right\} q_{nr,k}(x;\alpha) \leq \frac{x(1-x)}{n+r}$$

III. Proof of the theorem

$$\left|U_{nr}^{\alpha}(f,x) - f(x)\right| \leq (n+r+1)\sum_{k=0}^{n+r}\left\{\int_{k/(n+r+1)}^{(k+1)/(n+r+1)} |f(t)-f(x)|dt\right\} q_{nr,k}(x;\alpha)$$

Now splitting above inequality into two parts corresponding to those values of t for which $|t-x| < \delta$ and those for which $|t-x| \geq \delta$, we get

$$\leq (n+r+1)\sum_{|t-x|<\delta}\left(\int_{k/(n+r+1)}^{(k+1)/(n+r+1)} |f(t)-f(x)|dt\right)q_{nr,k}(x;\alpha)$$

$$+ (n+r+1)\sum_{|t-x|\geq\delta}\left(\int_{k/(n+r+1)}^{(k+1)/(n+r+1)} |f(t)-f(x)|dt\right)q_{nr,k}(x;\alpha)$$

= I_1+I_2 (say) -- (3.1)

If the function $f(x)$ is bounded say $|f(x)| \leq M$ in $0 \leq x \leq (1+\frac{r}{n})$ &x is a point of continuity,
for a given ε> 0 , ∃ a number δ> 0, ∋: $|x_2 - x_1| < \delta$ implies $|f(x_2) - f(x_1)| < \varepsilon$
and therefore

$$I_1 \leq \frac{\varepsilon}{2}(n+r+1)\sum_{k=0}^{n+r}\left\{\int_{\frac{k}{n+r+1}}^{\frac{k+1}{n+r+1}} dt\right\} q_{nr,k}(x;\alpha) = \frac{\varepsilon}{2}$$

&

$$I_2 \leq 2M(n+r+1) \sum_{|t-x|\geq\delta} \left(\int_{k/(n+r+1)}^{(k+1)/(n+r+1)} dt\right) q_{nr,k}(x;\alpha)$$

$$\leq \frac{2M}{\delta^2}(n+r+1)\sum_{k=0}^{n+r}\left\{\int_{k/(n+r+1)}^{(k+1)/(n+r+1)} (t-x)^2 dt\right\} q_{nr,k}(x;\alpha)$$

$\leq \frac{2M}{4(n+r)\delta^2}$ by lemma 2.3 and the fact $x(1-x) \leq \frac{1}{4}$ on $[0,1+\frac{r}{n}]$ for large n

On substituting the values of I_1 & I_2 in (3.1) we get

$$\left|U_{nr}^{\alpha}(f,x) - f(x)\right| \leq \frac{\varepsilon}{2} + \frac{2M}{4(n+r)\delta^2}$$

For $\delta = (\frac{M}{n+r)\varepsilon})^{1/2}$, we get

$$\left|U_{nr}^{\alpha}(f,x) - f(x)\right| < \varepsilon$$

Hence the proof of the theorem.

IV. Conclusion

The result of Bernstein has been extended for Lebesgueintegrable function in L_1-norm by our newly defined Polynomials $U_{nr}^{\alpha}(f,x)$on the interval [0, 1+$\frac{r}{n}$].

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