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Estimating the Parameters in the Two-Parameter Weibull Model Using Simulation Study and Real-Life Data

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Abstract: *A numerical algorithm from Maximum Likelihood (ML) and Improved Analytical (IA) was developed. This was used to estimate the parameters of the two-parameter Weibull distribution, namely scale and shape parameters. Since the Maximum Likelihood Estimators of the Weibull distribution do not have closed form solutions, the profile likelihood of the two-parameter Weibull distribution was considered for the purpose.Real life data were used in the study and data were also simulated for the distribution with sample sizes (10, 100, 1000 and 10000) and numerical approach was adopted to obtained the estimate of the distribution. The standard errors were computed and a 5% wald-confidence interval was constructed for the estimates of the distribution. The results of this study show that the profile likelihood constructed by the combination of MLE and Newton Raphson's method provides an efficient means of estimating parameters from intractable probability distribution model (two-parameter Weibull distribution) in both simulation and real life data.*
Keywords: *Two-parameter Weibull Model, Profile Likelihood, Maximum Likelihood Estimation,Newton Raphson Method*

I. Introduction

The Weibull distribution is an important distribution in reliability and maintenance analysis, variables such as wind speed can easily and effectively be modeled using the Weibull distribution. It is of great importance as it fully characterizes with only two parameters, the shape and moments of the distribution of the wind speed. This distribution is broadly used in the wind energy sector to produce maps of wind energy potential. There are several methods which can be used for the estimation of the parameters of Weibull distribution, for example, the method of least square, method of moment, and maximum likelihood estimation (MLE) method.In this study, we focus on the application of maximum likelihood method. Maximum-likelihood method (MLE) is used extensively for the estimation of the parameters of a statistical model, but with more complicated models, maximum likelihood alone may not result in a closed form solution, as it does not perform well in complex models. In the MLE-based methods, since the basic estimating equationsin complex models are not in closed form, the equations can only be solved numerically. There are several typical MLE-based methods for solving such equations, these include the secant method, the bisection method and the Newton-Raphson method. However, in both the secant and bisection methods, the convergence rates are very low. TheNewton-Raphson method however converges very fast even as it computes both the basic estimating function and its derivative at each iterative step. This study used both MLE and Newton Raphson Methods jointly due to the inability of the maximum likelihood method alone to obtain parameter estimates from two-parameter Weibull distribution.

II. Review of Related Literature

The likelihood function tells us how likely the observed sample is a function of the possible parameter values. Thus, maximizing the likelihood function for the data gives the parameter values for which the observed sample is most likely to have been generated, that is, the parameter values that ''agree most closely'' with the observed data (Fisher,1920).

Modern applied statistics deals with many settings in which the point wise evaluation of the likelihood function is impossible or computationally difficult. Examples of such are common in the areas of financial modelling, genetics, geostatistics, neurophysiology and stochastic dynamical systems (Pritchard *et al*., 1999). It is then consequently difficult to perform any inference (classical or Bayesian) about the parameters of the model.

Various approaches to overcome this difficulty have been developed and used by several authors; Cox and Reid (2004) used composite likelihood methods for approximating the likelihood function, and Pritchard *et al.,* 1999*;* Beaumont *et al., 2002,* applied Approximate Bayesian Computational methods for approximating the posterior distribution for obtaining estimates of the parameters.However, the ABC produces approximation of the posterior distribution in which there exist a deterministic error in addition to Monte Carlo variability(Beaumont *et al.*, 2002). The quality of the approximation to the posterior and theoretical properties of the estimators obtained with ABC have been studied in Wilkinson (2008); Marin *et al*., (2011); Dean *et al*.,

(2011) and Fearnhead and Prangle (2012), where Didelot*et al*., (2011) and Robert *et al*., (2011)used the ABC method for model comparisons. Using the sample approximation to characterize the mode of the posterior would in principle allow (approximate) maximum a posteriori (MAP) estimation. Furthermore, using a uniform prior distributionfor the parameters of interest over any set which contains the MLE will lead to a MAP estimate which coincides with the MLE. In low-dimensional problems, samples from the posterior distribution of the parameterscan be used to estimate its mode by using either nonparametric estimators of the density or another mode seeking technique such as the mean-shift algorithm (Fukunaga and Hostetler, 1975). Although Marjoram *et al*., (2003) noted that (ABC) can also be used in frequentist applications for maximum-likelihood estimation, this approach did not receive much attention. Alternative nonparametric density estimators within the AMLE context have been proposed (Cule*et al*., 2010; Jing *et al*., 2012).Cheng and Amin (1983) suggest the maximum product of spacing (MPS) method which can be applied to any univariate distribution. However, Cheng and Traylor (1995) point out that the drawbacks of the MPS method is the effects of tied observations in ordering when explanatory variables are involved in the model. Atkinson, Pericchi*et al.*, (1991) apply the grouped-data likelihood approach to the shifted power transformation model of Box and Cox (1964).

III. Materials and Methods

This research work employed the use of MLE and Numerical method (Newton Raphson method) jointly to obtain the estimates, profile-likelihood, standard errors and Wald interval of the two-parameter Weibull distributionusing simulation studies and real life data.

3.1 Model Specification

Weibull Probability Distribution

$$f(x;\propto,\beta)=\beta\alpha^{-1}\left(\frac{x}{\alpha}\right)^{\beta-1}e^{-}\left(\frac{x}{\alpha}\right)^{\beta}\qquad x\geq 0,\lambda>0,\beta>0 \qquad 3.1$$

IV. Implementing the Iterative Method for theWeibull Probability Distibution

Consider the Weibull Distribution with PDF

$$f(x;\propto,\beta)=\beta\alpha^{-1}\left(\frac{x}{\alpha}\right)^{\beta-1}e^{-}\left(\frac{x}{\alpha}\right)^{\beta} \qquad (3.2)$$

$$L(x;\propto,\beta)=\prod_{1}^{n}\beta\alpha^{-1}\left(\frac{x}{\alpha}\right)^{\beta-1}e^{-\left(\frac{x}{\alpha}\right)^{\beta}} \qquad (3.3)$$

$$L(x;\propto,\beta)=nlog(\beta)-n\beta\log(\propto)+(\beta-1)\sum_{1}^{n}\log(xi)-\sum_{1}^{n}\left(\frac{xi}{\alpha}\right)^{\beta} \qquad (3.4)$$

$$\frac{\partial L(\propto,\beta)}{\partial\propto}=-\frac{n\beta}{\propto}+\frac{\beta}{\propto}\sum_{1}^{n}\left(\frac{xi}{\alpha}\right)^{\beta} \qquad (3.6)$$

$$\frac{\partial L(\propto,\beta)}{\partial\beta}=\frac{n}{\beta}-nlog(\propto)+\sum_{1}^{n}\log(xi)-\sum_{1}^{n}\left(\frac{xi}{\alpha}\right)^{\beta}\log(\frac{xi}{\propto}) \qquad (3.7)$$

equating the derivatives to zero and solving the equations is again difficult, making direct analytically solutions intractable.

$$\frac{n}{\beta}-nlog(\propto)+\sum_{1}^{n}\log(xi)-`\sum_{1}^{n}\left(\frac{xi}{\alpha}\right)^{\beta}\log(\frac{xi}{\propto})=0 \qquad (3.8)$$

4.2 The Numerical Method (Simulation Study)

4.2.1 Weibull Probability Distribution

The log-likelihood is given in equation (3.4) while the score function is given in equation (3.6) and (3.7) .Differentiating further is given as:

$$\frac{\partial^{2}}{\partial\propto^{2}}L(\alpha,\beta)=\frac{n\beta}{\alpha^{2}}+\frac{\beta(\beta+1)}{\alpha^{2}}\sum_{i=1}^{n}(x_i/\alpha)^{\beta} \qquad (3.9)$$

$$\frac{\partial^{2}}{\partial\propto\beta}L(\alpha,\beta)=\frac{n}{\alpha}-\frac{1}{\alpha}\sum_{i=1}^{n}(x_i/\alpha)^{\beta}\log(x_i/\alpha)^{2} \qquad (3.10)$$

$$\frac{\partial^{2}}{\partial\beta^{2}}L(\alpha,\beta)=\frac{n}{\beta^{2}}+\sum_{i=1}^{n}(x_i/\alpha)^{\beta}\,[\log(x_i/\alpha)]^{2} \qquad (3.11)$$

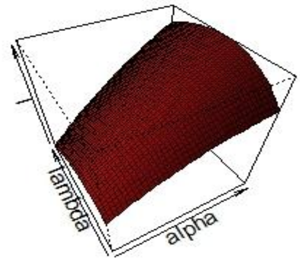

Figure 4.1: Loglikelihood graph of a weibull distribution.

4.2.2 Weibull Probability Distribution

Table 4.1: Weibull Probability Distribution

N	$\hat{\alpha}\,mle$	$\hat{\beta}\,mle$	S.E($\hat{\alpha}$)	S.E($\hat{\beta}$)	Wald C.I for $\hat{\alpha}$	Wald C.I for $\hat{\beta}$
10	1.9840354	0.3097679	0.52335695	0.05180851	1.460678, 2.035844	-0.2135891, 0.3615764
100	1.9840325	0.3097673	0.16549983	0.001638327	1.818533, 2.000416	0.1442674, 0.3261505
1000	1.9840375	0.3097673	0.052335643	0.005180843	1.931697, 1.9899213	0.2574316, 0.319481
10000	1.9840325	0.309767	0.016549983	1.985671	1.967483, 1.985671	0.2932173, 0.3114056

4.2.3 Profile Likelihood for Weibull Distribution

From equation (3.6)

$$\frac{\beta}{\propto}[-n+\sum(\frac{x}{\propto})^{\beta}]=0$$

$$-n+\sum(\frac{x}{\propto})^{\beta}=0$$

$$n=\sum(\frac{x}{\propto})^{\beta}$$

$$n=\frac{\sum x^{\beta}}{\propto^{\beta}}$$

$$\propto^{\beta}=[\frac{1}{n}\sum(\frac{x}{\propto})^{\beta}]$$

$$\hat{\propto}(\beta)=[\frac{1}{n}\sum(\frac{x}{\propto})^{\beta}]^{\frac{1}{\beta}} \quad (3.12)$$

Substituting (3.12) in (3.4) gives

$$l_{\beta}(\beta)=nlog\beta-nlog[\frac{1}{n}\sum x^{\beta}]+(\beta+1)\sum\log(x)-\frac{\sum x^{\beta}}{\frac{1}{n}\sum x^{\beta}} \quad (3.13)$$

$$l_{\beta}(\beta)=nlog\beta-nlog[\frac{1}{n}\sum x^{\beta}]+(\beta+1)\sum\log(x)-n \quad (3.14)$$

Differentiating equation (3.13) with respect to β to obtain the score function

$$\frac{\partial}{\partial\beta}L(\beta)=\frac{n}{\beta}-n\frac{\sum x^{\beta}\log(x)}{\sum x^{\beta}+\sum\log(x)} \quad (3.15)$$

Differentiating equation (3.15) gives

$$\frac{\partial^2}{\partial\beta^2}L(\beta)=\frac{n}{\beta^2}-n\frac{\sum x^{\beta}\log(x)^2-(\sum x^{\beta}\log(x))^2}{(\sum x^{\beta})^2} \quad (3.16)$$

4.2.4 Pro le Likelihood of Weibull Distribution

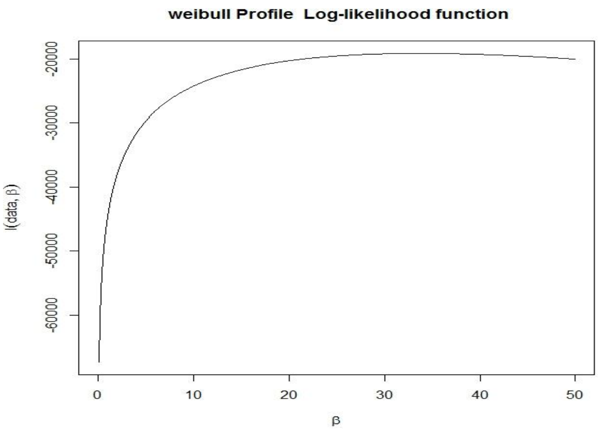

Figure 4.2: Profile Log-likelihood graph of a Weibull distribution.

Table 4.2: Profile Likelihood of Weibull Distribution Result Table

N	$\hat{\beta}$ *mle*	S.E($\hat{\beta}$)	Wald C.I for $\hat{\beta}$
10	1.984034	0.523429	1.460605, 2.502463
100	1.984034	0.1655228	1.818512, 2.149557
1000	1.984034	0.05234289	1.931691, 2.036377
10000	1.984034	0.01655228	1.967482, 2.000587

4.3 The Numerical Method (Real Life Data)

4.3.1 Weibull Probability Distribution

Figure 4.3: Loglikelihood graph of a Weibull distribution

Table 4.3: Weibull Probability Distribution

$\hat{\alpha}$ *mle*	$S.E(\hat{\alpha})$	Wald CI	$\hat{\beta}mle$	$S.E(\hat{\beta})$	Wald CI
0.9218841	0.176140	0.7457409, 1.0980280	17.2019258	4.951544	17.02579, 22.15347

4.3.2 Pro□le Likelihood of Weibull Distribution

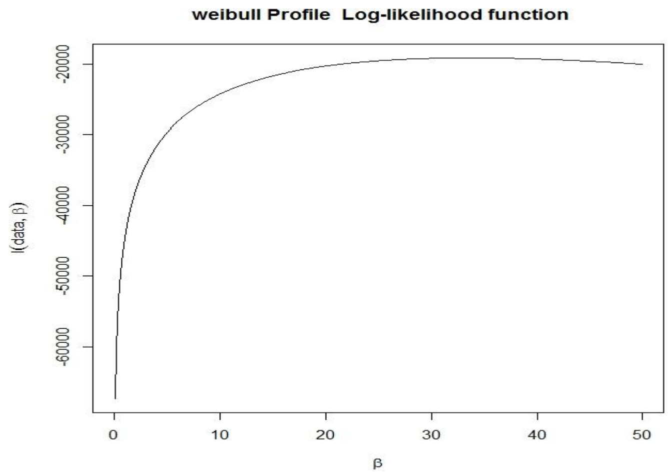

Figure 4.4: Profile Log-likelihood graph of a Weibull distribution.

Table 4.4: Profile Likelihood of Weibull Distribution Result Table

$\hat{\beta}mle$	$S.E(\hat{\beta})$	Wald C.I for $\hat{\beta}$
0.9218845	0.1761436	0.7457409,1.0980280

V. Discussion

From the iterations obtained by applying Newton Raphson method in obtaining maximum likelihood estimates for Weibull probability distribution, a total of 10 iterations were performed. The solution converge at the 9thiteration returning -5487.188 as the value of the log-likelihood and the values of the estimate which maximizes the function was 1.9840325 and 0.3097673 with gradient 6.234337×10^{-5} and -1.095032×10^{-6} respectively. The hessian matrix which was the value of the second derivative is

$$= \begin{bmatrix} 4013.616 & -12187.75 \\ -12187.75 & 409571.80 \end{bmatrix}$$

Table 4.2 shows the estimate which maximizes the likelihood function of the Weibull probability distribution for different sizes which ranges between (1.9840325 and 1.9840325) and (0.3097673 and 0.3097679) with standard error decreasing as the sample size increases (0.52335695 to 0.016549983) and (0.05180851 to 0.001638326).

In the application of Newton Raphson method to obtain the maximum likelihood estimates for profile Weibull probability likelihood function, 9 iterations were performed to obtain the maximum likelihood estimate. Convergence was achieved at the 8thiteration returning -5487.088 as the value of the log-likelihood and the value of the estimate which maximizes the function is 1.984034 with gradient 8.067959×10^{-5}. The variance which was the value of the second derivative is 3.649.93.

Table 4.6 shows the estimate which maximizes the likelihood function of profile Weibull likelihood function for different sample sizes which was between 1.984034 and 1.984034 with standard error reducing as the sample size increases (0.523429 to 0.001655228).

Applying the Newton Raphson method on the real life data, 43 iterations were performed to obtain the maximum likelihood estimates for Weibull probability distribution. The solution converged at the 40^{th}iteration returning 62.09617 as the value of the log-likelihood and the value of the estimate which maximizes the function was 0.9218841 and 17.2019258 with gradient -3.417711$\times 10^{-8}$ and -1.095032×10^{-6}. The hessian matrix which was the value of the second derivative is

$$= \begin{bmatrix} 36.3005931 & -0.43232478 \\ -0.43232478 & 0.04593551 \end{bmatrix}$$

Table 4.8 shows the estimate which maximizes the likelihood function of Weibull probability distribution was (0.9218841 and 1.72019258) with standard error of (0.176140 and 4.951544).
Using the Newton Raphson method to obtain the maximum likelihood estimates for the profile Weibull probability likelihood function, 5 iterations were performed and convergence was reached at the 4^{th}iteration returning 62.09617and the value of the estimate which maximizes the function is 0.9218845 with gradient 2.714273×10^{-06}. The variance which is the value of the second derivative was 0.1761436.

VI. Conclusion

Parameter estimates from intractable likelihood functions can easily be obtained using the Maximum Likelihood Estimation jointly with numerical (Newton Raphson) methods. Comparing Tables 4.1 and 4.2 it can be concluded that the estimate values of β that maximizes weibull probability distribution and profile likelihood weilbull function are not significantly different. Also comparing Tables 4.3 and 4.4,we can conclude that the estimated values of β that maximizes weibull probability distribution and profile likelihood weilbull function are also not significantly different. It was observed that as the sizes of the sample increases, the standard error reduces which obey the law of large numbers.It can also be concluded that the profile likelihood constructed by the combination of MLE and Newton Raphson's method provides an efficient means of estimating parameters from intractable probability distribution model (two-parameter Weibull distribution) in both simulation and real life data.

VII. Recommendation

Based on the results drawn from this study, the following recommendation was made:

- Parameter estimates from the two-parameter Weibull distribution should be obtained using Maximum Likelihood Estimation jointly with Newton Raphson Method

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