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On the numerical treatment ofthenonlinear partial differentialequation of fractional order

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Abstract:*The existence and uniqueness solution of a nonlinear partial differential equation (**NPDE**)of fractional order are discussed and proved in a Banach space $\mathcal{H}$ due to Picard's method depending on the properties we expect a solution to possess. Moreover, some properties concerning the stability of solutions are obtained.A systemof nonlinear Volterra integral equations of the second kind (**SNVIEs**) is obtained. The modified Toeplitz matrix method (**MTMM**) is used, as a numerical method, to obtain a nonlinear systemof algebraic equations (**NAS**). Also, many important theorems related to the existence and uniqueness solution of the algebraic system are derived. Finally,numerical results are obtained and the error is calculated.*

Keywords: *nonlinear partial differential equation of fractional order,semigroup, system of nonlinear Volterra integral equations of the second kind,modified Toeplitz matrix method, nonlinear algebraic system.*

I. Introduction and Basic formulations

The semigroups methods play a special role for partial differential equations and in applications, for example they describe how densities of initial states evolve in time.Moreover, there are equations which generate semigroups. These equations appear in such diverse areas as astrophysics-fluctuations in the brightness of the Milky Way [1], population dynamics [2,3], and in the theory of jump processes.In [4], El-Borai studied the Cauchy problem in a Banach space E for a linear fractional evolution equation. In his paper, the existence and uniqueness of the solution of the Cauchy problem were discussed and proved. Also, the solution was obtained in terms of some probability densities. In [5], El-Borai discussed the existence and uniqueness solution of the nonlinear Cauchy problem. Also, some properties concerning the stability of solutions were obtained. In [6] Abdou et al., improved the work of El-Borai in[4]and obtainednumerically the solution of the Cauchy problem. More information for the **PDEs**and its solution can be found in [7-9].

Consider the following **NPDE** of fractional order:

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = A\,u(x,t) + F\big(x,t,B(t)u(x,t)\big), \tag{1.1}$$

with the initial condition: $u(x,0) = u_0(x)$, (1.2)

in a Banach space $\mathcal{H}$, where $u(x,t)$is an $\mathcal{H}$-valued function on $\mathcal{H}\times[0,T]$, $T<\infty$, A is a linear closed and bounded operator defined on a dense setS_1, $\{B(t), t\in[0,T]\}$is a family of linear closed and bounded operators defined on a dense set $S_2 \supset S_1$ in $\mathcal{H}$ into $\mathcal{H}$, F is a given abstract function defined on $\mathcal{H}\times[0,T]$with values in $\mathcal{H}$, $u_0(x)\in\mathcal{H}$ and $0<\alpha<1$.

It is assumed that A generates an analytic semigroup $Q(t)$.This condition implies:

$$\|Q(t)\| \le k \quad \text{for } t\ge 0 \ \ ,\text{and}\|AQ(t)\| \le \frac{k}{t} \ \text{ for } \ t>0 \ , \tag{1.3}$$

where$\|.\|$ is the norm in $\mathcal{H}$ and k is a positive constant (see Zaidman [7]).

Let us suppose that $B(t)g$ is uniformly Hölder continuous in $t\in[0,T]$, for every $g\in S_1$; that is

$$\|B(t_2)g - B(t_1)g\| \le k_1(t_2-t_1)^{\beta}, \tag{1.4}$$

for all $t_2>t_1, t_1,t_2\in[0,T]$, where k_1 and β are positive constants, $\beta\le 1$.

We suppose also that there exists a number $\gamma\in(0,1)$, such that

$$\|B(t_2)\,Q(t_1)\,h\| \le \frac{k_2}{t_1^{\gamma}}\|h\|\,, \tag{1.5}$$

where$t_1>0, t_2\in[0,T]$, $h\in\mathcal{H}$and $k_2$is a positive constant (see[4,8,9]).

(Notice that $Q(t)h \ \in S_1$for each $h\in\mathcal{H}$ and each $t>0$).

Also, it is assumed that, the function F satisfies the following conditions:

(i) F is uniformly Hölder continuous in $t\in[0,T]$; that is

$$\|F(x,t_2,W) - F(x,t_1,W)\| \le l(t_2-t_1)^{\beta}, \tag{1.6}$$

for all $t_2>t_1, t_1,t_2\in[0,T]$ and all $x,W\in\mathcal{H}$, where $l$and β are positive constants; $\beta\le 1, W=B(t)\,u(x,t)$ and$\|.\|$ is the norm in $\mathcal{H}$.

(ii) F satisfies Lipschitz condition

$$|F(x,t,W) - F(x,t,W^*)| \le N(x,t)|W-W^*|\,; \quad (\|N(x,t)\|\le l_1)\,, \tag{1.7}$$

for all $x, W, W^* \in \mathcal{H}$ and all $t \in [0,T]$, where l_1 is a positive constant.
(iii) $\|F(x,t,W)\| \le l^* \|W\|$, ($l^*$ is a constant) . (1.8)
Following Gelfand and Shilov (see [10],[11]), we can define the integral of order $\alpha > 0$ by

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\theta)^{\alpha-1} f(\theta) d\theta .$$

If $0 < \alpha < 1$, we can define the derivative of order α by

$$\frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\theta)}{(t-\theta)^\alpha} d\theta , \qquad \left(f'(\theta) = \frac{df(\theta)}{d\theta} \right),$$

where f is an abstract function with values in $\mathcal{H}$.
Now, it is suitable to rewrite the **NPDE**((1.1), (1.2)) in the form

$$u(x,t) = u_0(x) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\theta)^{\alpha-1} A\, u(x,\theta) d\theta$$
$$+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-\theta)^{\alpha-1} F\big(x,\theta,B(\theta)\, u(x,\theta)\big) d\theta . \tag{1.9}$$

Let $C_{\mathcal{H}}(\mathcal{H} \times [0,T])$ be the set of all continuous functions $u(x,t) \in \mathcal{H}$, and define on $C_{\mathcal{H}}(\mathcal{H} \times [0,T])$ a norm by

$$\|u(x,t)\|_{C_{\mathcal{H}}(\mathcal{H}\times[0,T])} = \max_{x,t} \|u(x,t)\|_{\mathcal{H}} , \quad \text{for all } t \in [0,T] , x \in \mathcal{H} .$$

By a solution of the **NPDE**((1.1),(1.2)), we mean an abstract function $u(x,t)$ such that the following conditions are satisfied:
(a) $u(x,t) \in C_{\mathcal{H}}(\mathcal{H} \times [0,T])$ and $u(x,t) \in S_1$ for all $t \in [0,T]$, $x \in \mathcal{H}$.
(b) $\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}$ exists and continuous on $\mathcal{H} \times [0,T]$, where $0 < \alpha < 1$.
(c) $u(x,t)$ satisfies Eq.(1.1) with the initial condition (1.2) on $\mathcal{H} \times [0,T]$.
Lemma 1(without proof): If $\lambda > 1$ and $0 < \delta < 1$, then

$$\int_0^t e^{\lambda\eta} (t-\eta)^{\delta-1} d\eta \le \left(\frac{1}{\lambda}\right)^\delta \left[1 + \frac{1}{\delta}\right] e^{\lambda t} . \tag{1.10}$$

II. The existence and uniqueness solution of NPDE of fractional order

Here, the existence and uniqueness solution of Eq. (1.9) and consequently its equivalent **NPDE (**(1.1),(1.2)), will be discussed and proved in a Banach space $\mathcal{H}$ by virtue of Picard's method.
The formula (1.9) is equivalent to the following integral equation (see[4])

$$u(x,t) = \int_0^\infty \xi_\alpha(\theta) Q(t^\alpha\theta) u_0(x) d\theta + \alpha \int_0^t \int_0^\infty \theta (t-\eta)^{\alpha-1} \xi_\alpha(\theta) Q((t-\eta)^\alpha\theta) \tilde{w}(x,\eta)\, d\theta\, d\eta , \tag{2.1}$$

where $\xi_\alpha(\theta)$ is a probability density function defined on $[0,\infty)$,

$$\tilde{w}(x,t) = F\big(x,t,B(t)u(x,t)\big) = F\big(x,t,W(x,t)\big) , \tag{2.2}$$

and

$$W(x,t) = \int_0^\infty \xi_\alpha(\theta) B(t) Q(t^\alpha\theta) u_0(x)\, d\theta$$
$$+\alpha \int_0^t \int_0^\infty \theta(t-\eta)^{\alpha-1} \xi_\alpha(\theta)\, B(t) Q((t-\eta)^\alpha\theta) \tilde{w}(x,\eta) d\theta d\eta. \tag{2.3}$$

Theorem1: The **NPDE**((1.1),(1.2)) has a unique solution in the Banach space $C_{\mathcal{H}}(\mathcal{H} \times [0,T])$.
The proof of this theorem comes as a result of the following lemmas.
Lemma 2: Under the conditions (1.5) and (1.7), the integral equation (2.1) has a solution in $C_{\mathcal{H}}(\mathcal{H} \times [0,T])$.
Proof: Using the method of successive approximations, the formulas (2.2) and (2.3), lead to

$$\tilde{w}_{n+1}(x,t) = F(x,t, \int_0^\infty \xi_\alpha(\theta)\, B(t)\, Q(t^\alpha\theta) u_0(x) d\theta$$
$$+\alpha \int_0^t \int_0^\infty \theta(t-\eta)^{\alpha-1} \xi_\alpha(\theta)\, B(t)\, Q((t-\eta)^\alpha\theta) \tilde{w}_n(x,\eta)\, d\theta\, d\eta \ ) .$$

Hence, in view of the condition (1.7), we get

$\|\tilde{w}_{n+1}(x,t)-\tilde{w}_n(x,t)\| \le$
$$\alpha\, l_1 \int_0^t \int_0^\infty \theta(t-\eta)^{\alpha-1}\xi_\alpha(\theta)\|B(t)Q((t-\eta)^\alpha\theta))\,[\tilde{w}_n(x,\eta)-\tilde{w}_{n-1}(x,\eta)]\|d\theta d\eta\,.$$
Using the condition (1.5), we obtain
$$\|\tilde{w}_{n+1}(x,t)-\tilde{w}_n(x,t)\| \le M^* \max_{x,t}\left[e^{-\lambda(t+x)}\|\tilde{w}_n(x,t)-\tilde{w}_{n-1}(x,t)\|\right]\int_0^t e^{\lambda(\eta+x)}(t-\eta)^{v-1}\,d\eta\,, \qquad (2.4)$$
where,
$$v=\alpha(1-\gamma)\ ,\ \ M^*=\alpha l_1\int_0^\infty \theta^{1-\gamma}\xi_\alpha(\theta)d\theta \qquad and\ \lambda>1\,. \qquad (2.5)$$
Introducing (1.10) in (2.4),we have
$$\max_{x,t}\left[e^{-\lambda(t+x)}\|\tilde{w}_{n+1}(x,t)-\tilde{w}_n(x,t)\|\right] \le M^*\left(\frac{1}{\lambda}\right)^v\left[1+\frac{1}{v}\right]\max_{x,t}\left[e^{-\lambda(t+x)}\|\tilde{w}_n(x,t)-\tilde{w}_{n-1}(x,t)\|\right].$$
We can choose λ sufficiently large such that
$$M^*\left(\frac{1}{\lambda}\right)^v\left[1+\frac{1}{v}\right]=\mu<1\ \ . \qquad (2.6)$$
Hence, the above inequality can be adapted in the form
$$\max_{x,t}\left[e^{-\lambda(t+x)}\|\tilde{w}_{n+1}(x,t)-\tilde{w}_n(x,t)\|\right] \le \mu\max_{x,t}\left[e^{-\lambda(t+x)}\|\tilde{w}_n(x,t)-\tilde{w}_{n-1}(x,t)\|\right].$$
By a successive application of the above inequality, we get
$$\begin{aligned}\max_{x,t}\left[e^{-\lambda(t+x)}\|\tilde{w}_{n+1}(x,t)-\tilde{w}_n(x,t)\|\right] &\le \mu\max_{x,t}\left[e^{-\lambda(t+x)}\|\tilde{w}_n(x,t)-\tilde{w}_{n-1}(x,t)\|\right]\\ &\le \mu^2\max_{x,t}\left[e^{-\lambda(t+x)}\|\tilde{w}_{n-1}(x,t)-\tilde{w}_{n-2}(x,t)\|\right]\le\cdots\\ &\le \mu^n\max_{x,t}\left[e^{-\lambda(t+x)}\|\tilde{w}_1(x,t)-\tilde{w}_0(x,t)\|\right]\ ,\end{aligned}$$
where$\tilde{w}_0(x,t)$ is the zero approximation which can be taken the zero element in thespace $\mathcal{H}$.

Thus , the series $\sum_{n=0}^{\infty}\|\tilde{w}_{n+1}(x,t)-\tilde{w}_n(x,t)\|$ converges uniformly in $\mathcal{H}\times[0,T]$.

Since $\tilde{w}_{n+1}(x,t)=\sum_{i=0}^{n}\left(\tilde{w}_{i+1}(x,t)-\tilde{w}_i(x,t)\right)$, it follows that the sequence $\{\tilde{w}_n(x,t)\}$converges uniformly in the space $C_{\mathcal{H}}(\mathcal{H}\times[0,T])$ to a continuous function $F\left(x,t,W(x,t)\right)$ which satisfies Eq.(2.1) for all$(x,t)\in\mathcal{H}\times[0,T]$. Consequently, $u(x,t)\in C_{\mathcal{H}}(\mathcal{H}\times[0,T])$, where
$$u(x,t)=\int_0^\infty \xi(\theta)\,Q(t^\alpha\theta)u_0(x)d\theta+\alpha\int_0^t\int_0^\infty \theta(t-\eta)^{\alpha-1}\xi_\alpha(\theta)Q((t-\eta)^\alpha\theta)F(x,\eta,W(x,\eta))d\theta d\eta\ \ .$$
Lemma 3: Under the conditions (1.5) and (1.7), the integral equation (2.3) has a uniquesolution in $C_{\mathcal{H}}(\mathcal{H}\times[0,T])$.

Proof: Let $u_1(x,t)$ and $u_2(x,t)$ be two solutions of Eq.(2.1), then from the formulas (2.2) and (2.3) with the aid of condition (1.7) , we have
$$\|\tilde{w}_2(x,t)-\tilde{w}_1(x,t)\| \le \alpha\, l_1\int_0^t\int_0^\infty \theta(t-\eta)^{\alpha-1}\xi_\alpha(\theta)\|B(t)\,Q((t-\eta)^\alpha\theta)[\tilde{w}_2(x,\eta)-\tilde{w}_1(x,\eta)]\|d\theta d\eta\,.$$
Using the same argument of lemma (2), we get
$$\max_{x,t}\left[e^{-\lambda(t+x)}\|\tilde{w}_2(x,t)-\tilde{w}_1(x,t)\|_{\mathcal{H}}\right]\le\mu\rho\,,$$
where $\rho=\max_{x,t}\left[e^{-\lambda(t+x)}\|\tilde{w}_2(x,t)-\tilde{w}_1(x,t)\|_{\mathcal{H}}\right]$.

Thus, from (2.6) we have
$$\rho=\max_{x,t}\left[e^{-\lambda(t+x)}\|\tilde{w}_2(x,t)-\tilde{w}_1(x,t)\|_{\mathcal{H}}\right]=0\,.$$
This completes the proof of the lemma.

Lemma 4(without proof): Under the conditions (1.4), (1.5) and (1.6), the solution $u(x,t)$ of Eq.(2.1) satisfies a uniform Hölder condition.(see [4])

Proof of Theorem 1: By virtue oflemmas (2), (3)and (4), we deduce that the solution $u(x,t)$ of Eq.(2.1) represents the unique solution of the **NPDE**((1.1),(1.2)) in the Banach space $C_{\mathcal{H}}(\mathcal{H}\times[0,T])$, and $u(x,t)\in S_1$.

Now, we will prove the stability of the solutions of the **NPDE**((1.1),(1.2)).In other words, we will show that the **NPDE**((1.1), (1.2))is correctly formulated.

Theorem 2 : Let $\{u_n(x,t)\}$ be a sequence of functions, each of which is a solution of Eq.(1.1) with the initial condition $u_n(x,0)=g_n(x)$, where $g_n(x)\in S_1(n=1,2,\dots)$. If the sequence $\{g_n(x)\}$ converges to an element

$u_0(x) \in S_1$, the sequence $\{Ag_n(x)\}$ converges and the sequence $\{B(t)g_n(x)\}$ converges uniformly on $\mathcal{H} \times [0,T]$. Then, the sequence of solutions $\{u_n(x,t)\}$ converges uniformly on $\mathcal{H} \times [0,T]$ to a limit function $u(x,t)$, which is the solution of the **NPDE**((1.1),(1.2)).
Proof: Consider the sequences $\{f_n(x,t)\}$ and $\{u_n^*(x,t)\}$, where

$$\frac{\partial^\alpha u_n^*(x,t)}{\partial t^\alpha} - Au_n^*(x,t) = f_n(x,t)\,,$$

$$u_n^*(x,t) = u_n(x,t) - g_n(x)\ \ ,\ \ u_n(x,0) = g_n(x)\,,$$

$$u_n^*(x,t) = \alpha \int_0^t \int_0^\infty \theta(t-\eta)^{\alpha-1}\xi_\alpha(\theta)Q((t-\eta)^\alpha\theta)f_n(x,\eta)d\theta d\eta\,,$$

and

$$f_n(x,t) = F\big(x,t,B(t)u_n^*(x,t) + B(t)g_n(x)\big) + Ag_n(x)\,.$$

Thus, we get

$$\|f_n(x,t) - f_m(x,t)\| \le \|Ag_n(x) - Ag_m(x)\| + \big\|F\big(x,t,B(t)u_n^*(x,t) + B(t)g_n(x)\big) - F\big(x,t,B(t)u_m^*(x,t) + B(t)g_m(x)\big)\big\|\,.$$

Using the condition (1.7), we obtain

$$\|f_n(x,t) - f_m(x,t)\| \le l_1\|B(t)[u_n^*(x,t) - u_m^*(x,t)]\| + l_1\|B(t)g_n(x) - B(t)g_m(x)\| + \|Ag_n(x) - Ag_m(x)\|\,.$$

Consequently,

$$\|f_n(x,t) - f_m(x,t)\| \le \alpha l_1 \int_0^t \int_0^\infty \theta(t-\eta)^{\alpha-1}\xi_\alpha(\theta)\|B(t)Q((t-\eta)^\alpha\theta)[f_n(x,\eta) - f_m(x,\eta)]\|d\theta d\eta$$
$$+\, l_1\|B(t)g_n(x) - B(t)g_m(x)\| + \|Ag_n(x) - Ag_m(x)\|.$$

In view of the conditions (1.5) and (2.5), the above inequality becomes

$$\|f_n(x,t) - f_m(x,t)\| \le M^* \int_0^t (t-\eta)^{v-1}\|f_n(x,\eta) - f_m(x,\eta)\|\,d\eta$$
$$+\, l_1\|B(t)g_n(x) - B(t)g_m(x)\| + \|A\,g_n(x) - A\,g_m(x)\|\,.$$

Given $\varepsilon > 0$, we can find a positive integer $N = N(\varepsilon)$ such that

$$\|f_n(x,t) - f_m(x,t)\| \le M^* \int_0^t (t-\eta)^{v-1}\|f_n(x,\eta) - f_m(x,\eta)\|\,d\eta + (1-\mu_2)\,\varepsilon,$$

for all $n\,, m \ge N$ and $(x,t) \in \mathcal{H} \times [0,T]$.
Using (1.10), the above inequality takes the form

$$(1-\mu)e^{-\lambda(t+x)}\|f_n(x,t) - f_m(x,t)\| \le (1-\mu)e^{-\lambda(t+x)}\,\varepsilon\,.$$

Thus, for sufficiently large λ, we get

$$\max_{x,t}\big[e^{-\lambda(t+x)}\|f_n(x,t) - f_m(x,t)\|\big] \le \varepsilon\,.$$

Since $\mathcal{H}$ is a complete space, it follows that the sequence$\{f_n(x,t)\}$ converges uniformly on $\mathcal{H} \times [0,T]$ to a continuous function $f(x,t)$, so the sequence $\{u_n^*(x,t)\}$ converges uniformly on $\mathcal{H} \times [0,T]$ to a continuous function$u^*(x,t)$. It can be proved that $f(x,t)$satisfies a uniform Hölder condition on$[0,T]$, thus $u^*(x,t) \in S_1$.
Corollary 1: The integral equation (2.1) has a unique solution in the Banach space $C_{\mathfrak{R}}(\mathfrak{R} \times [0,T])$.

III. The numerical solution of the NPDEof fractional order

In this section, we will usethe (MTMM)to obtain numerically, the solution of the **NPDE**((1.1),(1.2))in the Banach space $C_{\mathfrak{R}}(\mathfrak{R} \times [0,T])$, where

$$\|u(x,t)\|_{C_{\mathfrak{R}}(\mathfrak{R}\times[0,T])} = \max_{x,t}|u(x,t)|\,,\ \forall t \in [0,T]\,, -\infty < x < \infty.$$

For this, we write Eq.(2.2) in the form

$$u(x,t) = f^*(x,t) + \alpha \int_0^t p(t,\eta)Q^*(t,\eta)F\big(x,\eta,B(\eta)u(x,\eta)\big)d\eta\,, \tag{3.1}$$

where,

$$f^*(x,t) = \int_0^\infty \xi_\alpha(\theta)\,Q(t^\alpha\theta)u_0(x)\,d\theta\,, \qquad Q^*(t,\eta) = \int_0^\infty \theta\xi_\alpha(\theta)Q((t-\eta)^\alpha\theta)d\theta,$$

and the bad kernel

$$p(t,\eta) = (t-\eta)^{\alpha-1} \quad , \quad (0 < \alpha < 1, 0 \le \eta \le t \le T\ ;\ T < \infty)\,.$$

Here, the unknown function $u(x,t) \in C_{\Re}(\Re \times [0,T])$, while $f^*(x,t), Q^*(t,\eta)$ and $p(t,\eta)$are known functions and satisfy the following conditions:
($1'$) $f^*(x,t)$ is a continuous function in $(\Re \times [0,T])$.
($2'$) $Q^*(t,\eta)$ with its partial derivatives are continuous function in $[0,T]$, hence there exists a constant c_1 such that:$|Q^*(t,\eta)| \leq c_1$.
($3'$) $p(t,\eta)$is a badly behaved function of its arguments such that:
(a) for each continuous function $u(x,t)$and $0 \leq t_1 \leq t_2 \leq t$, the integrals

$$\int_{t_1}^{t_2} p(t,\eta)Q^*(t,\eta)F\big(x,\eta,B(\eta)u(x,\eta)\big)d\eta \text{ , and} \int_0^t p(t,\eta)Q^*(t,\eta)\, F\big(x,\eta,B(\eta)u(x,\eta)\big)d\eta$$

are continuous functions in $(\Re \times [0,T])$.
(b) $p(t,\eta)$is absolutely integrable with respect to $\eta$for all$0 \leq t \leq T$, thus there exists a constant c_2 ,such that: $\int_0^t |p(t,\eta)|d\eta \leq c_2$.
($4'$)The given function$F\big(x,t,B(t)\,u(x,t)\big)$ is continuous in$\Re \times [0,T]$, and satisfies Lipschitz condition

$$\big|F\big(x,t,B(t)u(x,t)\big) - F\big(x,t,B(t)v(x,t)\big)\big| \\ \leq N^*(x,t)\big|B(t)\big(u(x,t)-v(x,t)\big)\big| \ , \qquad \Big(\max_{x,t}|N^*(x,t)| \leq L^*\Big),$$

for all $u(x,t), v(x,t) \in C_{\Re}(\Re \times [0,T])$, whereL^* is a positive constant.
Putting $x = x_i$, $x_i = ih$, $h = x_{i+1} - x_i$, and using the following notations

$$u\,(t,x_i) = u_i(t) \ , \ f^*(t\,,x_i) = f_i^*(t\,),$$
$$F\,\big(x_i\,,\eta\,,B\,(\eta)\,u\,(x_i\,,\eta)\big) = F_i\big(\eta\,,B\,(\eta)u_i(\eta)\big)\,,$$

the integral equation (3.1) can be transformed to the following (SNVIEs)

$$u_i(t) = f_i^*(t) + \alpha \int_0^t p^*(t\,,\eta)F_i\big(\eta\,,B\,(\eta)u_i(\eta)\big)d\,\eta \quad , \tag{3.2}$$

where,$p^*(t\,,\eta) = p\,(t\,,\eta)\,Q^*(t\,,\eta)$.
Remark 1: let Ξ be the set of all continuous functions
$U\,(t) = \{u_1(t)\,,u_2(t),\dots,u_i(t)\,,\dots\}$, where $u_i(t) \epsilon$ C $[0,T]$, for all , and define on Ξ the norm:

$$\|U\,(t)\|_{\Xi} = \sup_i \max_{0 \leq t \leq T}|u_i(t)| = \sup_i \|u_i(t)\|_{C\,[0\,,T]} \ ,\forall\, i\,.$$

Then Ξ is a Banach space.

3.1. The existence of a unique solution of aSNVIEs:

In order to guarantee the existence of a unique solution of the SNVIEs (3.2) in the Banach space Ξ , we write this system in the integral operator form

$$\bar{\mathcal{L}}u_i(t) = f_i^*(t) + \mathcal{L}u_i(t) \quad , \qquad \forall\, i \tag{3.3}$$

where,

$$\mathcal{L}\,u_i(t) = \alpha \int_0^t p^*(t\,,\eta)F_i\big(\eta\,,B\,(\eta)u_i(\eta)\big)d\,\eta \quad . \tag{3.4}$$

Then assume the following conditions:
1^*) $\sup_i \max_{0 \leq t \leq T}|f_i^*(t)| = \|f^*(t)\|_{\Xi} \leq D$, (D is constant) .
2^*) The known continuous functions $F_i\big(t\,,B\,(t)u_i(t)\big)$ for all i, satisfy for the constants A_1^* , A_2^* and $A^* \geq \max\{A_1^*\,,A_2^*\}$, the following conditions:
(a_1) $\sup_i \max_{0 \leq t \leq T}\big|F_i\big(t\,,B\,(t)u_i(t)\big)\big| \leq A_1^*\|U\,(t)\|_{\Xi}$,
(a_2) $\Big|F_i\Big(t\,,B(t)u_i^{(1)}(t)\Big) - F_i\Big(t\,,B(t)u_i^{(2)}(t)\Big)\Big| \leq A_2^*\big|u_i^{(1)}(t) - u_i^{(2)}(t)\big|$.
3^*) The kernel$p^*(t\,,\eta)$ is a discontinuous function which satisfies:
(b_1)foreachcontinuous function $F_i\big(t\,,B\,(t)u_i(\,t)\big)$and $0 \leq t_1 \leq t_2 \leq t$, the integrals:
$\int_{t_1}^{t_2} p^*(t\,,\eta)F_i\big(\eta\,,B\,(\eta)u_i(\eta)\big)d\,\eta$, and $\int_0^t p^*(t\,,\eta)F_i\big(\eta\,,B\,(\eta)u_i(\eta)\big)d\,\eta$
are continuous in $[0\,,T]$.
(b_2)$p^*(t\,,\eta)$ is absolutely integrable with respect to η for all $0 \leq t \leq T$, thus there exists a constant c^*, such that:$\int_0^t |p^*(t\,,\eta)|\ d\,\eta \ \leq \ c^*$.
Theorem 3 (without proof): The formula (3.2) has a unique solution in the space Ξ under the following condition:$\delta^* = \alpha\,A^*c^* < 1$.

3.2.The modified Toeplitz matrix method (MTMM):

Here, we present the MTMM to obtain the numerical solution of a NVIE of the second kind with singular kernel. So, we assume the NVIE:

$$u\,(t) = f^*(t) + \alpha \int_0^t p^*(t,\eta)\;\; F\left(\eta, B(\eta)\, u\,(\eta)\right) d\eta\,. \tag{3.5}$$

Following the same way of Abdou etal.,(see [12],[13]), we can apply the MTMM for Volterra term to obtain the following equation

$$u\,(t) - \alpha \sum_{n=0}^{N} D_n\,(t)\, F\left(nh^*\,, B\,(nh^*)\, u(nh^*)\right) = f^*(t). \tag{3.6}$$

Putting $t = mh^*\,, h^* = \frac{T}{N}$, in (3.6) and using the following notations

$$u(mh^*) = u_m \quad , D_n(mh^*) = D_{mn} \quad , f^*(mh^*) = f_m^* \quad ,$$
$$F\left(mh^*\,, B\,(mh^*)\, u(mh^*)\right) = F_m(B_m\, u_m)\,, \tag{3.7}$$

we get the following NAS

$$u_m - \alpha \sum_{n=0}^{N} D_{mn}\; F_n(B_n u_n) = f_m^* \quad , \qquad 0 \le n \le m \le N\,, \tag{3.8}$$

where,

$$D_{mn} = \begin{cases} A_0(mh^*) & , n = 0 \\ A_n(mh^*) + B_{n-1}^*(mh^*) & , 0 < n < N \\ B_{N-1}^*(mh^*) & , n = N\;, \end{cases} \tag{3.9}$$

$$A_n(t) = \frac{F\left(nh^* + h^*\,, (nh^* + h^*)\, B(nh^* + h^*)\right) I(t) - F(nh^* + h^*, B(nh^* + h^*)\, J(t)}{h_1^*}\,, \tag{3.10}$$

$$B_n^*(t) = \frac{F\left(nh^*, B\,(nh^*)\right) J(t) - F\left(nh^*, nh^*\, B\,(nh^*)\right)\; I(t)}{h_1^*}\;, \tag{3.11}$$

where,

$$\begin{aligned} h_1^* = \; & \left[F\left(nh^*\,, B(nh^*)\right) F\,(nh^* + h^*\,, (nh^* + h^*) B(nh^* + h^*)) \right. \\ & \left. - F\,(nh^*, nh^*\, B(nh^*))\, F(nh^* + h^*\,, B(nh^* + h^*)\right]\,, \end{aligned} \tag{3.12}$$

and

$$I(t) = \int_{nh^*}^{nh^*+h^*} P^*(t,\eta) F\left(\eta, B(\eta)\right) d\eta\,, \qquad J(t) = \int_{nh^*}^{nh^*+h^*} p^*(t,\eta) F\left(\eta, \eta B(\eta)\right) d\eta\;. \tag{3.13}$$

The matrix D_{mn} can be written in the Toeplitz matrix form:

$$D_{mn} = G_{mn} - E_{mn}\;.$$

Here, the matrix

$$G_{mn} = A_n(mh^*) + B_{n-1}^*(mh) \quad , \;\; 0 \le n \le m \le N\,, \tag{3.14}$$

is called the Toeplitz matrix of order $(\mathrm{N}+1)$and

$$E_{mn} = \begin{cases} B_{-1}^*(mh^*) & , n = 0 \\ 0 & 0 < n < N \\ A_N(mh^*) & , n = N\;, \end{cases} \tag{3.15}$$

represents a matrix of order $(N + 1)$ whose elements are zeros except the first and the last rows (columns).

Definition 1: The Toeplitz matrix method is said to be convergent of order r in the interval $[0, T]$, if and only if for sufficiently large N, there exists a constant $d^* > 0$independent on N such that

$$\|u\,(t) - \;u_N(t)\| \le d^* N^{-r}\;. \tag{3.16}$$

Definition 2: The estimate local error R_N takes the form

$$R_N = \left|\int_0^t P^*(t,\eta) F\,(\eta\,, B\,(\eta)\, u\,(\eta) d\eta - \sum_{n=0}^{N} D_{mn} F_n\,(B_n u_n)\right|\,. \tag{3.17}$$

Lemma 5: If the kernel$p^*(t,\eta)$of Eq. (3.5) satisfies condition (3^*) and the following condition

$$\lim_{\acute{t} \to t} \int_0^t |p^*(\acute{t},\eta) - p^*(t,\eta)|\, d\eta = 0 \qquad ;\; \acute{t}, t \in [0, T], \tag{3.18}$$

then $\sup\limits_N \sum\limits_{n=0}^{N} |D_{mn}|$ exists , and $\lim\limits_{\acute{m} \to m} \sup\limits_N \sum\limits_{n=0}^{N} |D_{\acute{m}n} - D_{mn}| = 0$.

Proof: From the formulas (3.10) and (3.13), we get

$$|A_n(t)| \le \frac{1}{|h_1|}\left[\left|F\big(nh^*+h^*,(nh^*+h^*)B(nh^*+h^*)\big)\right| \int_{nh^*}^{nh^*+h^*} |p^*(t,\eta)||F(\eta,B(\eta)|d\eta \right.$$

$$\left. +|F(nh^*+h^*,B(nh^*+h^*)| \int_{nh^*}^{nh^*+h^*} |p^*(t,\eta)|\,|F(\eta,\eta\, B(\eta)|d\eta \right].$$

Summing from$n=0$to $n=N$, then taking in account the continuity of the function $F\big(t,B(t)\,u(t)\big)$ in the interval $[0,T]$ and finally using the condition (3^*),there exists a small constant M_1 ,

$$\sum_{n=0}^{N}|A_n(t)| \le M_1\ ,\forall N\,. \qquad \left(M_1=\frac{2L_1c^*}{|h_1|}\ ;\ |F(t,B(t)u(t)| \le L_1\right)$$

Since , each term of $\sum_{n=0}^{N}|A_n(t)|$ is bounded above, hence for $t=mh^*$, we deduce

$$\sup_N \sum_{n=0}^{N}|A_n(mh^*)| \le M_1\ . \tag{3.19}$$

Similarly, from the formulas (3.11) and (3.13), we have

$$\sup_N \sum_{n=0}^{N}|B_n^*(mh^*)| \le M_1\,. \tag{3.20}$$

In the light of (3.9), and with the help of (3.19) and (3.20), there exists a small constant M_2, such that

$$\sup_N \sum_{n=0}^{N}|D_{mn}| \le \sup_N \sum_{n=0}^{N}|A_n(mh^*)| + \sup_N \sum_{n=0}^{N}|B_n^*(mh^*)| \le M_2\ ,\ \ (M_2=2M_1)\,.$$

Hence , $\sup_N \sum_{n=0}^{N}|D_{mn}|$ exists .

By virtue of the formulas (3.10) and (3.13), we have for $t\,,t^{'}\epsilon[0,T]$

$$|A_n(t^{'})-A_n(t)| \le \frac{1}{|h_1^*|}\Big[\left|F\big(nh^*+h^*,(nh^*+h^*)B(nh^*+h^*)\big)\right|$$

$$\times \int_{nh^*}^{nh^*+h^*} |p^*(t^{'},\eta)-p^*(t,\eta)|\left|F\big(\eta,\eta\, B(\eta)\big)\right|d\eta +$$

$$\left.\left|F\big(nh^*+h^*,B(nh^*+h^*)\big)\right| \int_{nh^*}^{nh^*+h^*} |p^*(t^{'},\eta)-p^*(t,\eta)||F(\eta,\eta\, B(\eta))|d\eta\right].$$

Summing from $n=0$ to $n=N$, and taking in account the continuity of the function F, the above inequality can be adapted in the form

$$\sup_{\mathrm{N}} \sum_{n=0}^{N}|A_n(t^{'})-A_n(t)| \le \frac{2(L_1)^2}{|h_1|}\int_0^t |P^*(t^{'},\eta)-P^*(t,\eta)|\,d\eta.$$

Putting $t=mh^*, t^{'}=m'h^*$, then using the condition (3.19), we get

$$\lim_{m'\to m} \sup_N \sum_{n=0}^{N}|A_n(m'h^*)-A_n(mh^*)| = 0\,. \tag{3.21}$$

Similarly, in view of the formulas (3.11) and (3.13), we can prove that

$$\lim_{m'\to m} \sup_N \sum_{n=0}^{N}|B_n^*(m'h^*)-B_n^*(mh^*)| = 0. \tag{3.22}$$

Finally, with the aid of (3.9), (3.21) and (3.22), we have

$$\lim_{m'\to m} \sup_N \sum_{n=0}^{N}|D_{m'n}-D_{mn}| = 0\,.$$

IV. The existence of a unique solution of NAS

The SNVIEs(3.2) after using MTMM takes the form

$$u_{i,m} = f^*_{i,m} + \alpha \sum_{n=0}^{N} D^{[i]}_{mn} F_{i,n}\big(B_{i,n} u_{i,n}\big) . \tag{4.1}$$

Lemma 6 (without proof): If the kernel $p^*(t,\eta)$ of Eq. (3.2) satisfies the conditions (3^*) and (3.18), then we have

$$\sup_{i,N} \sum_{n=0}^{N} \left|D^{[i]}_{mn}\right| \quad \text{exists}, \quad \text{and} \lim_{m' \to m} \sum_{n=0}^{N} \left|D^{[i]}_{m'n} - D^{[i]}_{mn}\right| = 0 .$$

According to lemma (5) and in order to guarantee the existence of a unique solution of the NAS (4.1) in the Banach space ℓ^∞, we write this system in the following operator form

$$\bar{V} u_{i,m} = f^*_{i,m} + V u_{i,m} \qquad , (\forall i, m) \tag{4.2}$$

where,

$$V u_{i,m} = \alpha \sum_{n=0}^{N} D^{[i]}_{mn} F_{i,n}\big(B_{i,n} u_{i,n}\big) . \tag{4.3}$$

Then we assume the following conditions:

I) $\sup_{i,m} \left|f^*_{i,m}\right| \le H$, (H is a constant). II) $\sup_{i,N} \sum_{n=0}^{N} \left|D^{[i]}_{mn}\right| \le e$, (e is a constant).

III) The known continuous functions $F_{i,n}\big(B_{i,n} u_{i,n}\big)$ satisfy ($\forall i, n$) and for the constants q_1, q_2 and $q \ge \max\{q_1, q_2\}$ the following conditions:

(a_1') $\sup_{i,n} \left|F_{i,n}\big(B_{i,n} u_{i,n}\big)\right| \le q_1 \|u_{i,n}\|_{\ell^\infty}$,

(b_1') $\left|F_{i,n}\big(B_{i,n} u^{(1)}_{i,n}\big) - F_{i,n}\big(B_{i,n} u^{(2)}_{i,n}\big)\right| \le q_2 \left|u^{(1)}_{i,n} - u^{(2)}_{i,n}\right|$.

Theorem 4 (without proof): The formula (4.2) has a unique solution in the Banach space ℓ^∞ under the following condition: $\sigma^* = \alpha\, q\, e < 1$.

Definition 3: The estimate total error R_j is determined by the relation

$$R_j = \left| \int_0^t p^*(t,\eta)\, F\big(x,\eta,B(\eta)u(x,\eta)\big)\, d\eta - \sum_{n=0}^{N} D^{[i]}_{mn} F_{i,n}\big(B_{i,n} u_{i,n}\big) \right| .$$

when $j = \max\{N, i\} \to \infty$, then the sum

$$\sum_{n=0}^{N} D^{[i]}_{mn} F_{i,n}\big(B_{i,n} u_{i,n}\big), \text{tends to} \int_0^t P^*(t,\eta)\, F\big(x,\eta,B(\eta)u(x,\eta)\big)\, d\eta .$$

Theorem 5: If the sequence of continuous functions $\{f^*_n(x,t)\}$ converges uniformly to the function $f^*(x,t)$, and the functions $Q^*(t,\eta)$, $p(t,\eta)$ and $F\big(x,t,B(t)u(x,t)\big)$ satisfy, respectively, the conditions $(2')$, $(3' - b)$ and $(4')$. Then, the sequence of approximate solutions $\{u_n(x,t)\}$ converges uniformly to the exact solution of Eq.(3.1) in the Banach space $C_{\Re}(\Re \times [0,T])$.

Proof: The formula (3.1) with its approximate solution give

$$\max_{x,t} |u(x,t) - u_n(x,t)| \le \max_{x,t} |f^*(x,t) - f^*_n(x,t)|$$

$$+\alpha \int_0^t |p(t,\eta)||Q^*(t,\eta)| \max_{x,\eta} \left|F\big(t,\eta,B(\eta)\, u(x,\eta)\big) - F\big(t,\eta,B(\eta) u_n(x,\eta)\big)\right| d\eta ,$$

$$\forall\, 0 \le \eta \le t \le T, -\infty < x < \infty .$$

Since B is a bounded operator, there exists a positive constant M, such that

$$\|B(t)u(x,t)\| \le M\, \|u(x,t)\| . \tag{4.4}$$

In view of the conditions (4.4), $(2')$, $(3' - b)$, and $(4')$, the above inequality can be adapted in the form

$$\|u(x,t) - u_n(x,t)\|_{C_{\Re}(\Re\times[0,T])} \le \frac{1}{(1 - D^*)} \|f^*(x,t) - f^*_n(x,t)\|_{C_{\Re}(\Re\times[0,T])} ;\ (D^* = \alpha c_1 c_2 M L^*).$$

Since $\|f^*(x,t) - f^*_n(x,t)\|_{C_{\Re}(\Re\times[0,T])} \to 0$ as $n \to \infty$, so that

$$\|u(x,t) - u_n(x,t)\|_{C_{\Re}(\Re\times[0,T])} \to 0.$$

Theorem 6 : The total error R_j satisfies $\lim_{j\to\infty} R_j = 0$.

Proof: From the definition of R_j, we have

$$\left|R_j\right| \le \sup_{i,j}\left|u_i(mh^*) - \left(u_i(mh^*)\right)_j\right| + \sup_{i,N}\sum_{n=0}^{N}\left|D_{mn}^{[i]}\right|\sup_{i,j}\left|F_i\left(nh^*, B_i(nh^*)u_i(nh^*)\right) - \left(F_i\left(nh^*, B_i(nh^*)u_i(nh^*)\right)\right)_j\right|.$$

Using the conditions (I) and (III - $\acute{b_1}$) , we get

$$\left|R_j\right| \le (1+eq)\left\|u_i(mh^*) - \left(u_i(mh^*)\right)_j\right\|_{\ell^\infty} \quad ,\forall j.$$

Since each term of $\left|R_j\right|$ is bounded above, hence for $t = mh^*$, we deduce

$$\sup_j\left|R_j\right| \le (1+eq)\sup_i\max_t\left|u_i(t) - \left(u_i(t)\right)_j\right| = (1+\ eq)\left\|u_i(t) - \left(u_i(t)\right)_j\right\|_{\Xi}.$$

The above inequality can be adapted in the form

$$\left\|R_j\right\|_{\ell^\infty} \le (1+eq)\left\|u(x,t) - \left(u(x,t)\right)_j\right\|_{C_{\Re}(\Re\times[0,T])}.$$

Since $\left\|u(x,t) - \left(u(x,t)\right)_j\right\|_{C_{\Re}(\Re\times[0,T])} \to 0$ as $j \to \infty$, then $\left\|R_j\right\|_{\ell^\infty} \to 0$, and consequently $R_j \to 0$.

V. Application

In Eq. (3.1) let: $0 < \alpha < 1$, $Q^*(t,\eta) = 1$, $F\left(x,t,B(t)\,u(x,t)\right) = (x^2+t^2)^3$,
thus, we get a NVIE of the second kind with Abel kernel

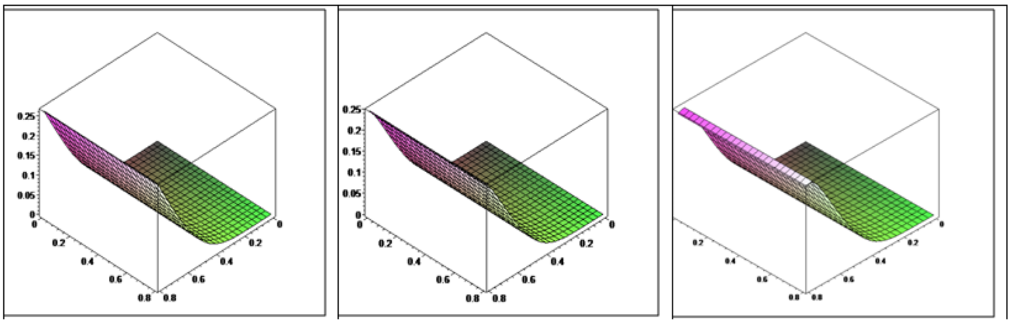
$$u(x,t) = f^*(x,t) + \alpha\int_0^t (t-\eta)^{\alpha-1}\,(x^2+\eta^2)^3\,d\eta \ , \qquad (5.1)$$

where the exact solution: $u(x,t) = x^2 + t^2$.

Using Maple 12, the results are obtained numerically for $x = t \in [0,0.8]$, with $\alpha = 0.98$, 0.8 , 0.6 , 0.4 , 0.1 and 0.02.Theinterval$[0,0.8]$is divided into $N = 40$ unites. The following tables and diagrams are selected from a large amount of data to compare betweentheexactsolutionof Eq. (5.1) (Exact.sol.) and its numerical solution (Approx.sol.).

$x = t$	u_E	$\alpha = 0.98$		$\alpha = 0.8$		$\alpha = 0.4$	
		u_T	E_T	u_T	E_T	u_T	E_T
0	0	0	0	0	0	0	0
0.08	0.0128	0.012798366	1.63363E-06	1.27982E-02	1.77133E-06	1.27979E-02	2.13568E-06
0.16	0.0512	0.051099467	0.000100533	5.10885E-02	0.000111494	5.10627E-02	1.37250E-04
0.24	0.1152	0.114122314	0.001077686	0.113991204	0.001208796	0.11370819	0.00149181
0.32	0.2048	0.199327485	0.005472515	0.198664247	0.006135753	0.197359549	0.007440451
0.40	0.32	0.302160901	0.0178391	0.30023762	0.01976238	0.296880041	0.023119959
0.48	0.4608	0.417910805	0.042889195	0.414242829	0.046557171	0.40874595	0.05205405
0.56	0.6272	0.544459139	0.082740861	0.539576412	0.087623588	0.533658437	0.093541563
0.64	0.8192	0.683408633	0.135791367	0.679324851	0.139875149	0.676370856	0.142829144
0.72	1.0368	0.839208791	0.197591209	0.840044456	0.196755544	0.845757589	0.191042411
0.80	1.28	1.017679793	0.262320207	1.034279351	0.245720649	1.075850445	0.204149555

Table (1)

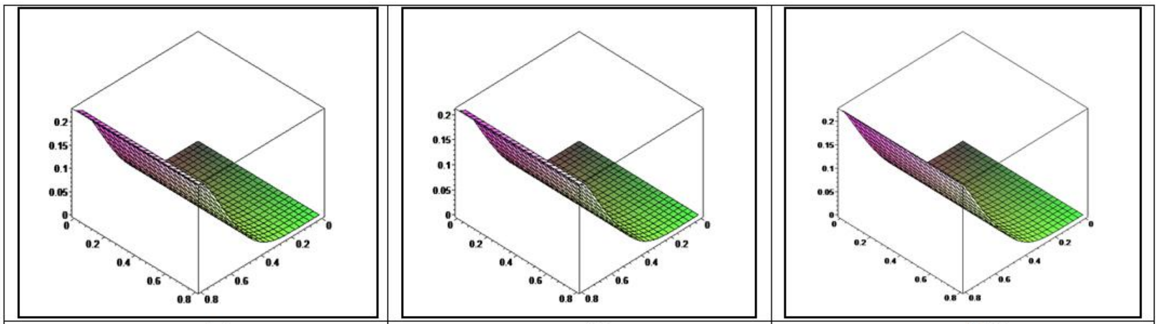


Dig. (1)

$x = t$	u_E	$\alpha = 0.6$		$\alpha = 0.1$		$\alpha = 0.02$	
		u_T	E_T	u_T	E_T	u_T	E_T
0	0	0	0	0	0	0	0
0.08	0.0128	1.27981E-02	1.94784E-06	1.27978E-02	2.24925E-06	1.27979E-02	2.14171E-06
0.16	0.0512	5.10753E-02	1.24660E-04	5.10578E-02	1.42172E-04	5.10650E-02	1.34994E-04
0.24	0.1152	0.113841226	0.001358774	0.113675557	0.001524443	0.113753121	0.001446879
0.32	0.2048	0.197944613	0.006855387	0.197355258	0.007444742	0.197749012	0.007050988
0.40	0.32	0.298282951	0.021717049	0.297504399	0.022495601	0.298769511	0.021230489

0.48	0.4608	0.41079253	0.05000747	0.411522694	0.049277306	0.414411032	0.046388968
0.56	0.6272	0.535390987	0.091809013	0.540332414	0.086867586	0.545310163	0.081889837
0.64	0.8192	0.676367701	0.142832299	0.687175892	0.132024108	0.693690718	0.125509283
0.72	1.0368	0.842100807	0.194699193	0.856922144	0.179877856	0.862027837	0.174772163
0.80	1.28	1.055921233	0.224078767	1.071585681	0.208414319	1.056464672	0.223535328

Table (2)



$\alpha = 0.6$	$\alpha = 0.1$	$\alpha = 0.02$

Dig. (2)

VI. Conclusions

From this paper, we can conclude the following points:

1) The **NPDE (**(1.1),(1.2)) of fractional order is equivalent to the following **NVIE**:

$$u(x,t) = u_0(x) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\theta)^{\alpha-1} A\, u(x,\theta)d\theta + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\theta)^{\alpha-1} F\big(x,\theta,B(\theta)\, u(x,\theta)\big)d\theta\ .$$

From the above integral equation, we can discuss many cases for the nonlinear integro- differential equation if for example $F\big(x,t,B(t)\, u(x,t)\big) = f\ (u(x\,,t),u_t\,(x\,,t),u_{tt}\,(x\,,t))$.

2) The **MTMM**, as a best method to solve the singular integral equations, is used to obtain a **NAS,** and many theorems are derived to prove the existence and uniqueness of the**NAS**.

3) From the numerical results, we establish the following:

a) For fixed values of α , the error values are increasing with the increase values of x and t .

b) For fixed values of x and t,the error values are slowly increasing with the decrease values of α.So , the change in the values of α is lightly effective in the numerical calculations.

c) The error is 0 at $x = t = 0$ for all cases we have studied.

d) The maximum value of the error at $\alpha = 0.98$ is 0.262320207, while the maximum value of the error at $\alpha = 0.02$ is 0.223535328, for $x = t = 0.8$.

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