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Vietoris Locale-Using Spectrum

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Abstract: *In the background of pointfree topology, if L is a locale, the Vietoris locale V (L) corresponding to L has been constructed using the spectrum of L. If L is a spatial, subfit locale, then V (L) is also a subfit locale. Also if L is second countable locale, then V(L) is second countable and hence seperable.*
Keywords: *Vietoris locale, second countable, Subfit, Seperable.*

I. Introduction

It is well known that a topological space is a lattice of open sets. The interrelation between topology and lattice was first studied by Marshall Stone. After his work on topological representation of Boolean algebra and distributive lattice, the relation between topology and lattice theory was studied in detail by Banaschewski[4],John Isabell[5], Picado,Pultr[1], Johnstone[2] etc. Johnstone, in his paper 'The point of pointless topology' expressed the complete lattice satisfying infinite distributive law as pointless topology. Afterwards most of the topological ideas have been studied in the locale background. Relations In this paper we have defined Vietoris locale V(L),as in classical topology, using the spectrum of L. It is observed that the Vietoris locale constructed using spectrum is in fact the locale of open subsets of a topological space and hence is always spatial. If L is spatial, subfit locale, then V (L) is also a subfit locale. Also if L is second countable locale, then Vietoris locale V (L) is second countable and hence seperable.

II. Preliminaries

2.1 Definition: [2] A frame (or a locale) is a is a complete lattice L satisfying the infinite distributivity law $a\wedge\bigvee B=\{a\wedge b : b\in B\}$ for all $a\in L$ and $B\subseteq L$.
Given the frames L, M, a frame homomorphism is a map h: L →M preserving all finite meets (including the top 1) and all joins (including the bottom 0). The category of frames is denoted by **Frm**. The opposite of category Frm is the category **Loc** of locales.
2.2 Example: [2] The lattice of open subsets of topological space is a locale.
2.3 Defintion: [2] A subset F of locale L is said to be a filter if
i)F is a sub-meet-semi lattice of L;that is $1 \in F$ and $a \in F, b \in F$ imply $a \wedge b \in F$.
ii)F is an upper set; that is $a \in F$ and $a \leq b$ imply $b \in F$.
2.4 Defintion: [2] A filter F is proper if $F \neq L$,that is if 0 is not an element of F.
2.5 Defintion: [2] A proper filter F in a locale L is prime if $a_1 \vee a_2 \in F$ implies that $a_1 \in F$ or $a_2 \in F$.
2.6 Defintion: [2] A proper filter F in a locale L is a completely prime filter if for any J and
$a_i\in L, i \in J$,$\bigvee a_i \in F$ implies that $\exists\, i\in J$ such that $a_i \in F$. Completely prime filters are denoted by c.p filters.
Example: $U(x)=\{V\in \Omega (X); x \in V\}$ is a completely prime filter in the locale $\Omega(X)$.
For $a \in L$, set $\sum_a = \{ F \subseteq L; F\neq\Phi$; F is c:p filters ; $a \in F \}$. Thus we have $\Sigma_0 = \Phi$, $\Sigma_{\bigvee ai}=\bigcup \Sigma_{ai}$, $\Sigma_{a\wedge b}= \Sigma_a\cap \Sigma_b$ and $\Sigma_1=\{$ all c.p filters$\}$.
From 2.6 Definition, if $a \leq b$,then $\Sigma_a\subseteq \Sigma_b$. But $\Sigma_a\subseteq \Sigma_b$ need not imply $a \leq b$.
2.7 Defintion: [2] The spectrum of a locale is defined as follows.
$Sp(L)=(\{$all c:p filters$\}, \{ \Sigma_a : a \in L\})$ .Then Sp(L)is a topological space with the
topology $\Omega(Sp(L)) = \{ \Sigma_a : a \in L\}$.
2.8 Defintion: [2] A locale L is said to be spatial if it is isomorphic to $\Omega(X)$ of some topologicalspace X.
2.9 Proposition: [2]The following statements on a locale are equivalent.
(1)L is spatial.
(2)L is isomorphic to $\Omega(Sp(L))$
As in classical topology,the point free topology have seperation axioms. Subfit and fit correspond to T1 axiom of classical topology.
2.10 Defintion: [2] A locale is said to be sub fit if for $a,b \in L$; $a \nleq b$,then $\exists\, c$,such that $a\vee c = 1$ and $b \vee c \neq 1$
2.11 Fact: [2] A topological space X is subfit if and only if for each open set U and each $x\in U$,there is a $y\in cl(\{x\})$ such that $cl(\{y\})\subseteq U$.
2.12 Defintion: [6] A collection *B* of elements of a locale L is said to be L-base for the locale L if for every $a(\neq0) \in L$,there exists a non empty sub-collection $\{bi: bi\in B, i \in \Delta\}$ such that

$\bigvee_{i\in\Delta} b_i \leq a$ and $\bigvee \{bi: bi\in B,\}=1$

2.13 Remark: [6] If B_L is a L-base for the frame (locale) L iff $\forall$ p$\in$ L there exists some bp ϵ B L such that bp$\leq$ p

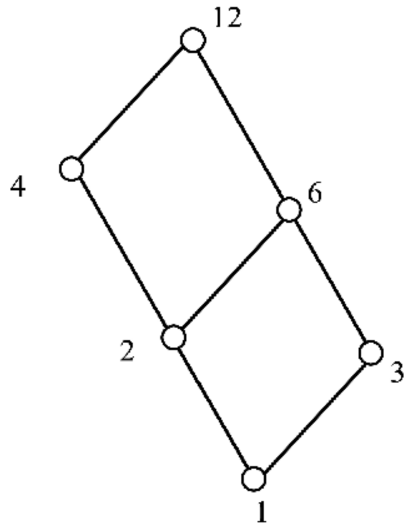
2.14 Defintion: [6] A frame L is said to be a $B_L{}^2$ frame if it has a countable L- base. And this property is called 2-countability or second countability.

2.15 Theorem: [6] Every $B_L{}^2$ frame is separable.

III. Vietoris locale

3.1 Definition: Let L be a locale and Sp(L) denotes the spectrum associated with the locale L. Let H denotes the collection of all non empty closed subsets of Sp(L).For $\Sigma_{ai} \in \Omega(Sp(L))$, consider $[\Sigma_{a1,} \Sigma_{a2,}..... \Sigma_{an}]=\{C\in H: C\subseteq \Sigma_{\vee ai}$ and $C\cap \Sigma_{ai}\neq \Phi \}$.These sets generates a topology $\Omega(H)$ on H. The locale $\Omega(H)$ is called the Vietoris locale corresponding to the locale L and is denoted by V(L).

3.2 Example: Let the locale L be given as follows.



Then the completely prime filters of L are given by

$F_1= \{2,4,6,12\}$ $F_2= \{3,6,12\}$ $F_3= \{4,12\}$.

$\Sigma_1 =\Phi$, $\Sigma_2=\{F_1\}$, $\Sigma_3=\{F_2\}$, $\Sigma_4=\{F_1,F_3\}$, $\Sigma_6=\{F_1,F_2\}$, $\Sigma_{12}=\{F_1,F_2,F_3\}$.

The spectrum Sp(L) is given by

$Sp(L)= \{F_1,F_2,F_3\}$ with $\Omega(Sp(L))=\{ \Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4, \Sigma_6, \Sigma_{12} \}$.

The non empty closed subsets of Sp(L) is given by $H=\{ \Sigma_{12}, \Sigma_4, \Sigma_3, \{F_2,F_3\}, \{F_3\} \}$

$[\Sigma_2]=\{ C\in H: C\subseteq \Sigma_2, C\cap \Sigma_2\neq \Phi \}=\Phi$

$[\Sigma_3]= [\Sigma_6]=\{ \Sigma_3 \}$

$[\Sigma_4]=\{ \{ F_3\}, \Sigma_4 \}$

$[\Sigma_{12} \}= H$

$[\Sigma_2, \Sigma_3]= [\Sigma_2, \Sigma_6]=[\Sigma_2, \Sigma_3, \Sigma_6]= \Phi$,

$[\Sigma_2, \Sigma_4]=\{ \Sigma_4 \}$

$[\Sigma_2, \Sigma_{12}]=[\Sigma_2, \Sigma_6, \Sigma_4]= [\Sigma_2, \Sigma_{12}, \Sigma_4]= [\Sigma_2, \Sigma_6, \Sigma_{12}]= \{ \Sigma_{12}, \Sigma_4 \}$

$[\Sigma_3, \Sigma_4]= [\Sigma_3, \Sigma_6, \Sigma_4]= [\Sigma_3, \Sigma_4, \Sigma_{12}]= [\Sigma_6, \Sigma_3, \Sigma_4, \Sigma_{12}]= \{ \Sigma_{12} ,\{F_2, F_3\} \}$,

$[\Sigma_3, \Sigma_6]=\{ \Sigma_3, \{F_2,F_3\} \}$

$[\Sigma_3, \Sigma_{12}]=[\Sigma_3, \Sigma_6, \Sigma_{12}]= \{ \Sigma_{12} ,\{F_2, F_3\} ,\Sigma_3\}$

$[\Sigma_4, \Sigma_6]= [\Sigma_4, \Sigma_6, \Sigma_{12}]= \{ \Sigma_{12} ,\{F_2, F_3\} ,\Sigma_4\}$

$[\Sigma_4, \Sigma_{12}]= \{ \Sigma_{12}, \Sigma_4, \{F_2,F_3\}, \{F_3\} \}$

$[\Sigma_{12}, \Sigma_6]= \{ \Sigma_{12}, \Sigma_4, \Sigma_3, \{F_2,F_3\} \}$

$[\Sigma_2, \Sigma_3, \Sigma_4]= [\Sigma_2, \Sigma_3, \Sigma_{12}]= [\Sigma_2, \Sigma_3, \Sigma_4, \Sigma_6]= [\Sigma_2, \Sigma_3, \Sigma_4, \Sigma_{12}]= [\Sigma_2, \Sigma_3, \Sigma_6, \Sigma_{12}]=$
$[\Sigma_2, \Sigma_3, \Sigma_4, \Sigma_6, \Sigma_{12}]= \{ \Sigma_{12} \}$

Then the subsets

$\{ \{ \Sigma_3 \}, \Phi, \{ \{ F_3\}, \Sigma_4 \}$, H, $\{ \Sigma_4 \}$, $\{ \Sigma_{12}, \Sigma_4 \}$, $\{ \Sigma_3, \{F_2,F_3\} \}$, $\{ \Sigma_{12} ,\{F_2, F_3\} ,\Sigma_3\}$, $\{ \Sigma_{12} ,\{F_2, F_3\} ,\Sigma_4\}$, $\{ \Sigma_{12}, \Sigma_4, \{F_2,F_3\}, \{F_3\} \}$, $\{ \Sigma_{12}, \Sigma_4, \Sigma_3, \{F_2,F_3\} \}$, $\{ \Sigma_{12} \} \}$ is a base for a unique topology Ω (H) on H.

Thus the Vietoris locale V(L) is given by

$V(L) = \{ \Phi,H, \{ \Sigma_3 \}, \{ \Sigma_4 \},\{ \Sigma_{12} \}$, $\{ \Sigma_3, \Sigma_4 \}$,$\{ \Sigma_{12}, \Sigma_3 \}$, $\{ \Sigma_3, \{F_2,F_3\} \}$, $\{ \{ F_3\}, \Sigma_4 \}$,

$\{ \Sigma_{12}, \Sigma_4 \}, \{ \Sigma_4, \{F_2,F_3\} \}, \{ \Sigma_{12}, \{F_2,F_3\} \}, \{ \Sigma_3, \Sigma_4, \{F_3\} \},\{ \Sigma_{12}, \Sigma_4, \Sigma_3 \},$
$\{ \Sigma_{12}, \Sigma_3, \{F_2,F_3\} \},\{ \Sigma_{12}, \Sigma_4, \{F_3\} \},\{ \Sigma_{12}, \Sigma_4, \{F_2,F_3\} \},\{ \Sigma_3, \Sigma_4, \{F_2,F_3\} \},$
$\{ \Sigma_4, \{F_2,F_3\}, \{F_3\} \}, \{ \Sigma_{12}, \Sigma_4, \Sigma_3, \{F_2,F_3\} \},\{ \Sigma_{12}, \Sigma_4, \{F_2,F_3\}, \{F_3\} \},$
$\{ \Sigma_3, \Sigma_4, \{F_2,F_3\}, \{F_3\} \},\{ \Sigma_{12}, \Sigma_4, \Sigma_3 , \{F_3\} \} \}$

3.3 Proposition: Let L be a spatial locale. If L is a subfit locale, then the Vietoris locale V(L) is also a subfit locale.
Proof: Suppose the spatial locale L is subfit.
Since L is spatial by 2.9 Proposition L is isomorphic to $\Omega(Sp(L))$ and $\Omega(Sp(L))$ is also a subfit locale as L is a subfit locale. Then by 2.11,for every open set $\Sigma_a \in \Omega(Sp(L))$ and for every completely prime filter $F_i \in \Sigma_a$, there exist a completely prime filter F_j such that $F_j \in cl(\{F_i\})$ and $cl(\{F_j\}) \subseteq \Sigma_a$
Now let $U \in V(L)$ and let $C \in U$.
$C \in U$ implies there exist $\Sigma_a \in \Omega(Sp(L))$ such that $C \cap \Sigma_a \neq \Phi$.
Let $F_1 \in C \cap \Sigma_a$. Then $F_1 \in \Sigma_a$. Since $\Omega(Sp(L))$ is subfit locale, there exist F_2 such that
$F_2 \in cl(\{F_1\})$ and $cl(\{F_2\}) \subseteq \Sigma_a$.
Since $F_2 \in cl(\{F_1\})$ and $\{F_1\} \subseteq C$,we have $F_2 \in cl(C)$.
Since $cl(\{F_2\}) \subseteq \Sigma_a$, $cl(\{F_2\}) \cap \Sigma_a \neq \Phi$. Hence $cl(\{F_2\}) \subseteq U$ by construction.
Thus corresponding to $U \in V(L)$ and $C \in U$, there exist $F_2 \in cl(C)$ such that $cl(\{F_2\}) \subseteq U$. Hence by 2.11, V(L) is a subfit locale.
3.4 Lemma: If the locale L is second countable, then $\Omega(Sp(L))$ is second countable.
Proof: Suppose L is second countable. Then L has a countable L-base.
Let B= {bi: bi∈L, i ∈ N} be the countable L-base for L.
Consider the collection $\Gamma = \{\Sigma_{bi} : i \in N\}$. This is a sub collection of $\Omega(Sp(L))$.
Since {bi: bi∈L, i ∈ N} is countable,Γ is also countable. Now we have to show that Γ is a base for $\Omega(Sp(L))$.
Let $\Sigma_a \in \Omega(Sp(L))$. Then a ∈ L.Since B is a L-base for L, there exist a non empty sub collection
{bi: bi∈B, i ∈ Δ} of B such that $\bigvee_{i\in\Delta} b_i \leq a$ and $\bigvee B=\mathbf{1}$.
Since $\bigvee_{i\in\Delta} b_i \leq a$, $\Sigma_{\vee bi} \subseteq \Sigma_a$ and $\bigvee\Gamma = \Sigma_1$.
Thus Γ is a countable base for $\Omega(Sp(L))$.
3.5 Proposition: If L is a second countable locale, then V(L) is second countable.
Proof: Let B= {bi: bi∈L, i ∈ N} be the countable L-base for L.
Consider the collection $\mathcal{H}=\{ [\Sigma_{bi}] ; b_i \in B \}$.Clearly $\mathcal{H}$ is countable.
Let U be any generator of V(L).Then U is of the form
$U=[\Sigma_{a1}, \Sigma_{a2},\ldots \Sigma_{an}]= \{C \in H: C \subseteq \Sigma_{\vee ai} \text{ and } C \cap \Sigma_{ai} \neq \Phi \}$.
By lemma 3.4 $\Gamma = \{\Sigma_{bi} : i \in N\}$ is a base for $\Omega(Sp(L))$. Then by 2.13, there exist Σ_{bj} such that $\Sigma_{bj} \subseteq \Sigma_{\wedge ai}$.
Now we will show that $[\Sigma_{bj}] \subseteq U$.
$D \in [\Sigma_{bj}]$ implies $D \subseteq \Sigma_{bj} \subseteq \Sigma_{\wedge ai} \subseteq \Sigma_{\vee ai}$.
Also $D \cap \Sigma_{bj} \neq \Phi$ implies $D \cap \Sigma_{ai} \neq \Phi$.Hence $D \in U$.
Thus $[\Sigma_{bj}] \subseteq U$. Thus by 2.13, $\mathcal{H}$ is a countable base for V (L).
3.6 Proposition: If L is second countable, V (L) is separable.
Proof: Proof follows from 3.5 Proposition and 2.15 Theorem

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