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Equitable Coloring of Some Path Related Graphs

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Abstract: *A proper vertex coloring of a graph G is equitable if the size of color classes differ by at most 1. The equitable chromatic number, denoted by* $\chi_e(G)$*, is the smallest integer* k *for which the graph G is equitably colorable. We investigate the equitable chromatic number of some path related graphs.*
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I. Introduction

All graphs considered here are simple, finite, connected and undirected. The vertex set, the edge set, the maximum degree, the open neighborhood of a vertex v and size of a color class V_i are denoted by $V(G)$, $E(G)$, $\Delta(G)$, $N(v)$ and $|V_i|$ respectively. For any undefined term we refer to Balakrishnan and Ranganathan [1]. A proper k-coloring of a graph G is a function $c: V(G) \to \{1, 2, ..., k\}$ such that $c(u) \neq c(v)$, for all $uv \in E(G)$. The chromatic number $\chi(G)$ is the minimum integer k for which the graph G admits a proper coloring. The graph G is called equitably k-colorable if the vertex set of G can be partitioned into k non empty independent sets $V_1, V_2, ..., V_k$ such that $\left||V_i| - |V_j|\right| \leq 1$, for every i and j. The smallest integer k for which the graph G is equitably k-colorable is called the equitable chromatic number of G and is denoted by $\chi_e(G)$.

The concept of equitable coloring was introduced by Meyer [2] in 1973 in which he stated the equitable coloring conjecture (ECC) states that if the connected graph G is neither a complete graph nor an odd cycle then $\chi_e(G) \leq \Delta(G)$. The equitable coloring of bipartite graphs is discussed in Lih and Wu [3] while Chen and Lih [4] contributed some new results on equitable coloring of trees. The equitable colorings of line graphs and complete r-partite graphs are investigated by Wang and Zhang [5].

The present paper is aimed to investigate the equitable chromatic number of middle graph, shadow graph and splitting graph of path.

Proposition 1.1 *[6]* $\chi_e(G) \geq \chi(G)$

Proposition 1.2 *[7]* If G contains a clique of order n, then $\chi(G) \geq n$.

II. Main Results

Definition 2.1 A middle graph $M(G)$ of a graph G is the graph whose vertex set is $V(G) \cup E(G)$ and in which two vertices are adjacent whenever either they are adjacent edges of G or one is vertex of G and other is an edge incident with it.

Theorem 2.2 $\chi_e(M(P_n)) = 3$, for all $n \geq 3$.

Proof: Let $V(P_n) = \{v_1, v_2, ..., v_n\}$ and $E(P_n) = \{e_1, e_2, ..., e_{n-1}\}$ where $e_i = v_i v_{i+1}$; $1 \leq i \leq n-1$.

By the definition of middle graph, $V(M(P_n)) = V(P_n) \cup E(P_n)$ and

$E(M(P_n)) = \{v_i e_i; 1 \leq i \leq n-1\} \cup \{e_i e_{i+1}; 1 \leq i \leq n-1\} \cup \{e_i v_{i+1}; 1 \leq i \leq n\}$. Thus $|V(M(P_n))| = 2n-1$ and $|E(M(P_n))| = 3n-4$.

As $M(P_n)$ contains a clique of order 3, $\chi(M(P_n)) \geq 3$. Hence by Proposition 1.1, $\chi_e(M(P_n)) \geq 3$.
………………………………(1)

Now define the color function $c: V(M(P_n)) \to \mathbb{N}$ as

$$c(v_{3k-2}) = 1; k = 1, 2, ..., \frac{n+2}{3},$$

$$c(e_{3k-2}) = 2;\ c(e_{3k-1}) = 1; c(v_{3k-1}) = 3; c(v_{3k}) = 2; c(e_{3k}) = 3; k = 1, 2, ..., \frac{n-1}{3}$$

Now, partition the vertex set $V(M(P_n))$ as

$$V_1 = \{v_1, v_4, ..., v_n, e_2, e_5, ..., e_{n-2}\},$$
$$V_2 = \{v_3, v_6, ..., v_{n-1}, e_1, e_4, ..., e_{n-3}\},$$
$$V_3 = \{v_2, v_5, ..., v_{n-2}, e_3, e_6, ..., e_{n-1}\}$$

Clearly, V_1, V_2 and V_3 are the independent sets of $M(P_n)$.

Also, $|V_1| = \frac{n+2}{3} + \frac{n-1}{3} = \frac{2n+1}{3}$, $|V_2| = \frac{n-1}{3} + \frac{n-1}{3} = \frac{2(n-1)}{3}$, $|V_3| = \frac{n-1}{3} + \frac{n-1}{3} = \frac{2(n-1)}{3}$.

It holds the inequality $\left||V_i| - |V_j|\right| \le 1$, for every i and j.

Thus, $\chi_e(M(P_n)) \le 3$. ……………………..(2)

Hence from (1) and (2) we get $\chi_e(M(P_n))$=3, for all $n \ge 3$.

Illustration 2.3 $M(P_{11})$ and its equitable coloring is shown in Fig. 1 for which $\chi_e(M(P_7))$ =3.

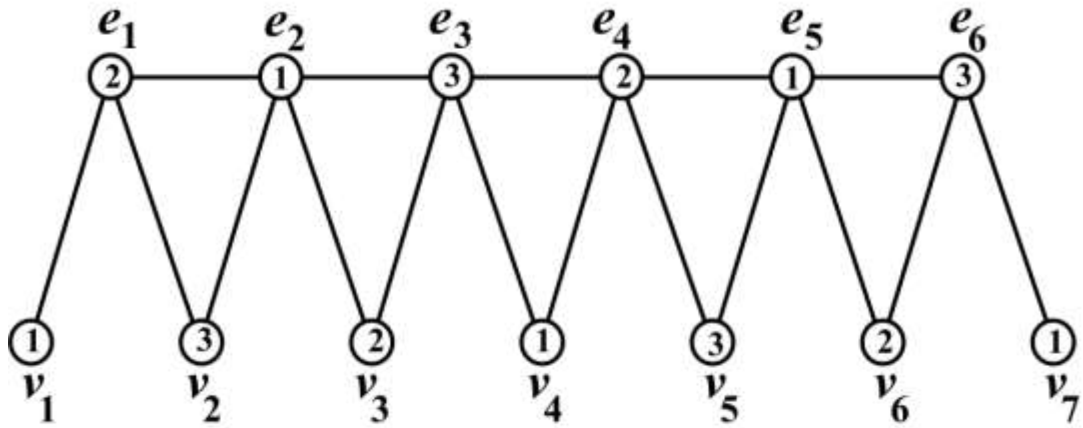

Figure 1

Definition 2.4 The Shadow graph $D_2(G)$ of a connected graph G is constructed by taking two copies of G, say G' and G''. Join each vertex u' in G' to the neighbors of the corresponding vertex u'' in G''.

Theorem 2.5 $\chi_e(D_2(P_n)) = \begin{cases} 2, & n \text{ is even} \\ 3, & n \text{ is odd} \end{cases}$

***Proof*:** Let $V(P_n) = \{v_1, v_2, ..., v_n\}$. The graph $D_2(P_n)$ has two copies, say P_n' and P_n'', with $V(D_2(P_n)) = \{v_i', v_i''; 1 \le i \le n\}$ and join each vertex v_i' in P_n' to the neighbors of the corresponding vertex v_i'' in P_n''. Thus $|V(D_2(P_n))| = 2n.$

We consider the following two cases:

Case 1: When n *is even*:

Define the color function $c: V(D_2(P_n)) \rightarrow$ as

$c(v'_{2k-1}) = c(v''_{2k-1}) = 1$ and $c(v'_{2k}) = c(v''_{2k}) = 2$.

Now, Partition the vertex set of $D_2(P_n)$ as

$|V_1| = \{v'_1, v''_1, v'_3, v''_3, ..., v'_{2n-1}, v''_{2n-1}\}$,

$|V_2| = \{v'_2, v''_2, v'_4, v''_4, ..., v'_{2n}, v''_{2n}\}$

Clearly, V_1 and V_2 are independent sets of $D_2(P_n)$ and $|V_1| = |V_2|$.

It holds the inequality $||V_i| - |V_j|| \leq 1$, for every i and j. Thus $\chi_e(D_2(P_n))$ =2.

Case 2: When n *is odd*:

As any vertex (other than $v'_1, v''_1,$ v'_n and v''_n) is adjacent to four vertices, one of the partition gets $\frac{|V(D_2(P_n))|}{2}+1$ vertices and the other $\frac{|V(D_2(P_n))|}{2}-1$. Thus $|V_1| = n+1$ and $|V_2| = n-1$. Hence $||V_1| - |V_2|| = 2$. That is, it does not hold the inequality $||V_i| - |V_j|| \leq 1$ for every pair (i, j). Thus, $\chi_e(D_2(P_n)) \neq 2$.

Define the color function $c: V(D_2(P_n)) \rightarrow \{1, 2, 3\}$ as

$c(v'_{3k-2}) = c(v''_{3k-2}) = 1$, $c(v'_{3k-1}) = c(v''_{3k-1}) = 2$, $c(v'_{3k}) = c(v''_{3k}) = 3$.

Now, partition the vertex set of $D_2(P_n)$ as

$V_1 = \{v'_1, v''_1, v'_4, v''_4, ..., v'_{3k-2}, v''_{3k-2}\}$,

$V_2 = \{v'_2, v''_2, v'_5, v''_5, ..., v'_{3k-1}, v''_{3k-1}\}$,

$V_3 = \{v'_3, v''_3, v'_6, v''_6, ..., v'_{3k}, v''_{3k}\}$.

Clearly, V_1, V_2 and V_3 are independent sets of $D_2(P_n)$ and $|V_1| = 2k,$ $|V_2| = 2k$ while $|V_3| = 2k$ or $2(k-1)$ depending on $n \equiv 0 \,(mod\, 3)$ or otherwise respectively. It holds the inequality $||V_i| - |V_j|| \leq 1$ for i and j. Thus $\chi_e(D_2(P_n))$ =3.

Illustration 2.5 $D_2(P_{11})$ and its equitable coloring is shown in Fig. 2 for which $\chi_e(D_2(P_{11}))$ =3.

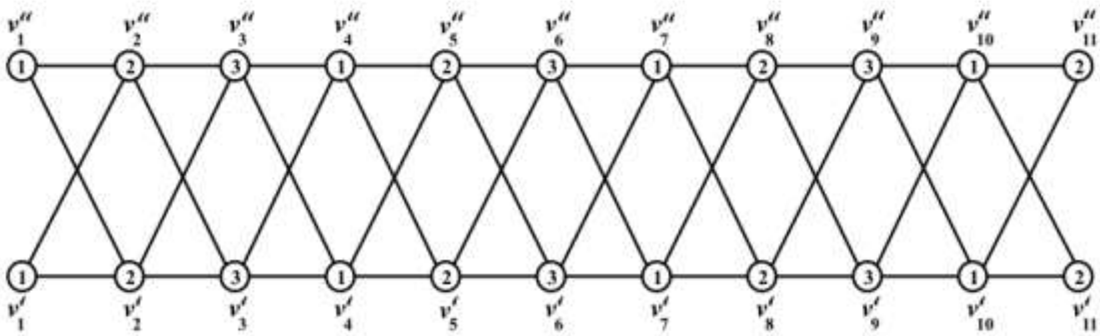

Figure 2

Definition 2.6 The splitting graph $S'(G)$ of a connected graph G is obtained by adding new vertex v' corresponding to each vertex v of G such that $N(v) = N(v')$ where $N(v)$ and $N(v')$ are the neighborhood sets of v and v' respectively.

Theorem 2.7 $\chi_e(S'(P_n)) = \begin{cases} 2, & n \text{ is even} \\ 3, & n \text{ is odd} \end{cases}$

Proof: Let $V(P_n)=\{v_1,v_2,...,v_n\}$. The splitting graph of P_n, $S'(P_n)$, is obtained by adding new vertices $v_1', v_2',...,v_n'$ corresponding to each vertex of P_n. Clearly, $N(v_i)=N(v_i'); i=1,2,...,n$ and $|V(S'(P_n))|=2n.$

We consider the following two cases:

Case 1: When n *is even*:

Define the color function $c:V(S'(P_n))\to\mathbb{N}$ as

$c(v_{2k-1})=c(v_{2k-1}')=1$ and $c(v_{2k})=c(v_{2k}')=2$.

Now, Partition the vertex set of $S'(P_n)$ as

$|V_1|=\{v_1,v_1',v_3,v_3',...,v_{2n-1},v_{2n-1}'\}$,

$|V_2|=\{v_2,v_2',v_4,v_4',...,v_{2n},v_{2n}'\}$.

Clearly, V_1 and V_2 are independent sets of $S'(P_n)$ and $|V_1|=|V_2|$.

It holds the inequality $\left||V_i|-|V_j|\right|\le 1$, for every i and j. Thus $\chi_e(S'(P_n))$ =2.

Case 2: When n *is odd*:

As one vertex, say v_2, is adjacent to four vertices v_1, v_1', v_3 and v_3', these vertices receive the same color , say color 1, and v_2 and v_2' receives color 2. Consequently, v_4 and v_4' receive color 2. Hence, if we partition the vertex set of $S'(P_n)$ into two independent sets, one of the partition gets $\frac{|V(S'(P_n))|}{2}+1$ vertices and the other $\frac{|V(S'(P_n))|}{2}-1$. Thus $|V_1|=n+1$ and $|V_2|=n-1$. Hence $\left||V_1|-|V_2|\right|=2$. That is, it does not hold the inequality $\left||V_i|-|V_j|\right|\le 1$ for every pair (i,j). Thus, $\chi_e(S'(P_n))\ne 2$.

Define the color function $c:V(S'(P_n))\to\{1,2,3\}$ as

$c(v_{3k-2})=c(v_{3k-2}')=1$, $c(v_{3k-1})=c(v_{3k-1}')=2$, $c(v_{3k})=c(v_{3k}')=3$.

Now, partition the vertex set of $S'(P_n)$ as

$V_1=\{v_1,v_1',v_4,v_4',...,v_{3k-2},v_{3k-2}'\}$,

$V_2=\{v_2,v_2',v_5,v_5',...,v_{3k-1},v_{3k-1}'\}$,

$V_3=\{v_3,v_3',v_6,v_6',...,v_{3k},v_{3k}'\}$.

Clearly, V_1, V_2 and V_3 are independent sets of $S'(P_n)$ and $|V_1|=2k,$ $|V_2|=2k$ while $|V_3|=2k$ or $2(k-1)$ depending on $n\equiv 0(mod\,3)$ or otherwise respectively. It holds the inequality $\left||V_i|-|V_j|\right|\le 1$ for i and j. Thus $\chi_e(S'(P_n))$ =3.

Illustration 2.8 $S'(P_{11})$ and its equitable coloring is shown in Fig. 3 for which $\chi_e(S'(P_{11}))$=3.

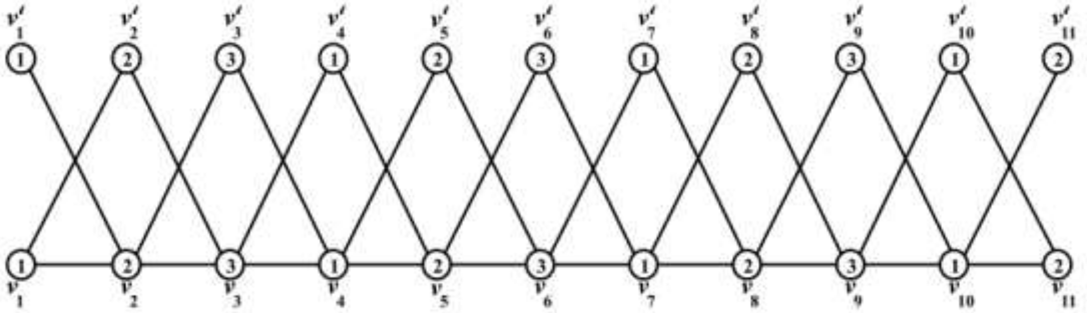

Figure 3

III. Concluding Remarks

The equitable coloring is a variant of proper coloring where almost equal distribution of colors among vertices is emphasized. Some results on equitable coloring of path are available in the literature while we investigate the equitable coloring of larger graphs obtained from path. To investigate similar results for other graph families and in the context of some other variants of proper coloring are an open area of research.

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