IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 12, Issue 5 Ver. VI (Sep. - Oct.2016), PP 75-78
www.iosrjournals.org

Prime Graph vs. Zero Divisor Graph

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Abstract: *In this paper we consider prime graph of R (denoted by $PG(R)$) of an associative ring R (introduced by Satyanarayana, Syam Prasad and Nagaraju [22]). We also consider zero divisor graph of a finite associative ring R (denoted by $ZDG(R)$). It is proved that every prime graph is a subgraph of the zero divisor graph but the converse need not be true. An example of a ring for which PG(R) ≠ ZDG(R) was presented.*
Keywords: *Prime graph, Associative ring,zero divisor graph.*
Mathematics subject classification*: 05C20, 05C76, 05C99, 13E15, 68R10.*

I. Introduction

Let G = (V, E) be a graph consist of a finite non-empty set V of vertices and finite set E of edges such that each edge e_k is identified as an unordered pair of vertices $\{v_i, v_j\}$, where v_i, v_j are called end points of e_k. The edge e_k is also denoted by either $v_i v_j$ or $\overline{v_i v_j}$. We also write $G(V, E)$for the graph. Vertex set and edge set of G are also denoted by $V(G)$ and $E(G)$respectively. An edge associated with a vertex pair $\{v_i, v_i\}$ is called a self-loop. The number of edges associated with the vertex is the degree of the vertex, and δ(v) denotes the degree of the vertex v. If there is more than one edge associated with a given pair of vertices, then these edges are called parallel edges or multiple edges. A graph that does not have self-loop or parallel edges is called a simple graph. We consider simple graphs only. For an associative ring R, prime graph ofR(denoted by $PG(R)$) was introduced in Satyanarayana, Syam Prasad and Nagaraju [22].For a commutative ring R, the notion of 'zero divisor graph' is given in Beck [1988]. In this paper, we consider the associative rings (need not be commutative) and provided some examples on the zero divisor graphs of Z_n where n is a positive integer.

1.2 Definitions:

(i) A graph G (V, E) is said to be a **star graph** if there exists a fixed vertex v (called the center of the star graph)such that E = {vu / u ∈ V and u ≠ v}. A star graph is said to be an **n-star graph** if the number of vertices of the graph is n.

(ii) In a graph G, a subset S of V(G) is said to be a ***dominating set*** if every vertex not in S has a neighbour in S. The ***domination number***, denoted by γ(G) is defined as min {|S| / S is a dominating set in G}.

(iii) In a connected graph, a closed walk running through every vertex of G exactly once (except the starting vertex at which the walk terminates) is called as **Hamiltonian circuit.** A graph containing a Hamiltonian circuit is called as **Hamiltonian graph.**

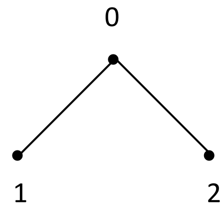
1.3 Theorem: (Th. 13.8, page 361, [18]) A given connected graph G is an Eulerian graph if and only if all the vertices of G are of even degree.
For other preliminary results and notations we use [18], [20] or [21]

II. Prime Graph of a Ring

2.1 Definition: (Satyanarayana, Syam Prasad and Nagaraju [22]) Let R be an associative ring. A graph G(V, E) is said to be a ***prime graph*** of R (denoted by PG(R)) if V = R and E = { $\overline{xy}$ / xRy = 0 or yRx = 0, and x $\neq$ y}.

For convenience of the reader we included the following example.

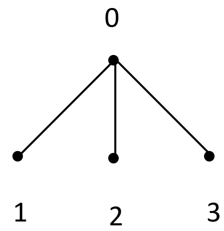
2.2Example (Example 9.4.2 of Satyanarayana and Syam Prasad [20]):Consider$\mathbb{Z}_n$, the ring of integersmodulo n. (i) Let us construct the graph PG(R), where R = $\mathbb{Z}_3$. We know that R = $\mathbb{Z}_3$ = {0, 1, 2}. So V(PG(R)) = {0, 1, 2}. Since 0R1 = 0, 0R2 = 0 there exists an edge between 0 and 1, and also an edge between 0 and 2. There are no other edges, as there are no two non-zero elements x, y $\in$ R with xRy = 0. So E(PG(R)) = { 01, 02 }. Now PG(R) is given in Figure 2.2 (i).



PG(R) = PG($\mathbb{Z}_3$)

Fig. 2.2 (i)

(ii) Let us construct the graph PG(R), where R = $\mathbb{Z}_4$. We know that R = $\mathbb{Z}_4$ = {0, 1, 2, 3}. So V(PG(R)) = {0, 1, 2, 3}. Since 0R1 = 0, 0R2 = 0, 0R3 = 0, we have that 01, 02, 03 $\in$E(PG(R)). There are no other edges, as there are no two distinct non-zero elements x, y $\in$ R such that xRy = 0. So E(PG(R)) = { 01, 02, 03 }. Now PG(R) is given in Figure 2.2 (ii).



PG(R) = PG($\mathbb{Z}_4$)

Fig. 2.2 (ii)

III. Zero Divisor Graph of an Associative Ring

In this section, we wish to studyzero divisor graph of an associative ring

3.1 Definition: (Vasantha kandasamy and Florentin Smarandache [23])A graph $G = (V, E)$ is said to be the **zero divisor graph** of a commutative ring R if $V = R$ and E = $\left\{ \overline{xy} / x \neq y,\ x, y \in R,\ x \neq 0 \neq y,\ xy = 0 \right\} \cup \left\{ \overline{x0} \ / \ 0 \neq x \in R \right\}$ where $\overline{xy}$ denotes an edge between $x, y \in V$.

This definition 'zero divisor graph' is same as that of Beck [1988].

3.2 Notation: (i) We denote zero divisor graph of ring R by ZDG(R)

(ii) In the graph ZDG(R), we have that $V(ZDG(R)) = R$ and

$$E(ZDG(R)) = \left\{ \overline{xy} / x \neq y,\ x, y \in R, x \neq 0 \neq y,\ xy = 0 \right\} \cup \left\{ \overline{x0} \ / 0 \neq x \in R \right\}$$

3.3 Example:(Vasantha kandasamy and Florentin Smarandache [23])

Consider Z_n , the ring of integers modulo n.

Consider ZDG(R) with $R = Z_{10}$. We know that $R = Z_{10} = \{0,1,2,3,\ldots,9\}$,

DOI: 10.9790/5728-1205067578

So $V(ZDG(R)) = \{0,1,2,3,4,5,6,7,8,9\}$. Since $5.8 = 5.4 = 5.6 = 0$ (mod 10), there exist edges between the vertices 5 and 8; 5 and 4; also between 5 and 6. Since '0' is adjacent to all the elements in R, we get $\overline{01}, \overline{02}, \overline{03}, \overline{04}, \overline{05}, \overline{06}, \overline{07}, \overline{08}, \overline{09} \in E(ZDG(R))$.

Therefore, $\mathrm{E}(\mathrm{ZDG(R)}) = \{\overline{01}, \overline{02}, \overline{03}, \overline{04}, \overline{05}, \overline{06}, \overline{07}, \overline{08}, \overline{09}, \overline{25}, \overline{58}, \overline{54}, \overline{56}\}$.

Now ZDG(R)given by the figure 3.3.

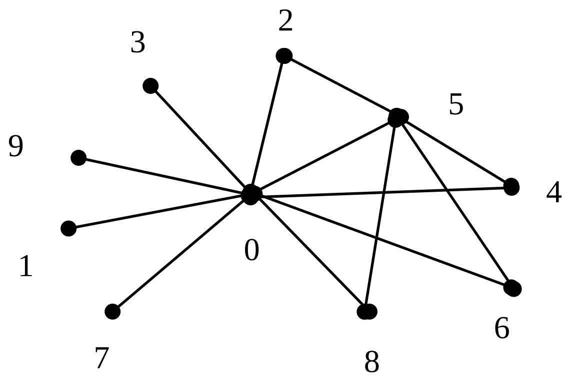


Figure-3.3

3.4 Observations: (i) ZDG(Z_{10}) contains 10-star graph as its subgraph;(ii) The domination number is 1; (iii) Since $\overline{02}$, $\overline{25}$, $\overline{50}$ forms a triangle, we conclude that the graph cannot be a bipartite graph; (iv) ZDG(Z_{10}) is not an Eulerian graph (by using the Th. 13.8, p 361 of [18]); and (v) Since ZDG(Z_{10}) contains pendent vertices, it contains no Hamiltonian circuit.

The following definition is an extension of the concept " zero divisor graph" to Associative rings.

3.5 Definition: Consider an associative ring R(need not be commutative) with identity 1. The zero divisor graph (in notation, ZDG(R)) is defined as $V(ZDG(R))$= R and $E(ZDG(R)) = \{\overline{ab}\ /\ a,\ b \in R, either\ ab = 0\ or\ ba=0,\ a\neq b$

3.6 Note: In case of commutative rings, the above concept coincides with the zero divisor graph defined in commutative rings by Beck [1]

3.7 Theorem: For an associative ring R we have that $PG(R)\ is\ a\ subgraph\ of\ ZDG(R)$.

Proof: We know that $V(PG(R))$= R = $V(ZDG(R))$.

Let $\overline{uv} \in$ E($PG(R)$). Then $uRv = 0\ or\ vRu = 0$. Since 1∈ R we have that either $uv = 0\ or\ vu = 0$.

By definition 3.5 we have that $\overline{uv} \in$ E($ZDG(R)$. This shows that PG(R) is a subgraph of ZDG(R).

3.8 Corollary: If R is a commutative ring then PG(R) = ZDG(R).

Proof: Let $\overline{vu} \in$ E($ZDG(R)$with $u, v \in V(ZDG(R))$ = R. Then $vu = 0$ or $uv = 0$. Suppose that $vu = 0$.This implies $vux = 0$ for all $x \in R$. Since R is commutative $vxu = 0$ for $x \in R$ and so $vRu = 0$. This shows that $\overline{vu} \in$ E($PG(R)$). Hence ZDG(R) is a subgraph of PG(R).

By theorem 3.7 we have that PG(R) = ZDG(R).

3.9 Remark: In case of associative ring which is not commutative, the converse of the theorem 3.7 need not be true. This was made clear by the example presented in the next section.

IV. An Example

In this section, we present an example of an associative ring for which PG(R) ≠ ZDG(R).

4.1 Example: Let F =Z_2be the field of integers modulo 2. Write R = Set of 3×3 matrices over the field F. We know that R is an associative ring with respect to usual matrix addition and multiplication. Consider the two elements $x, y \in R$mentioned below

$$x = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ y = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since $xy = 0$ and $x \neq y$ we conclude that $\overline{xy} \in E(ZDG(R))$.

Consider $z = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \in R$.

Now $xzy = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq 0$Since$0 \neq xzy \in xRy$, we have that $xRy \neq 0$.

Also $yzx = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \neq 0$Since $0 \neq yzx \in yRx$, we have that $yRx \neq 0$

Since $xRy \neq 0$ and $yRx \neq 0$, we have that $\overline{xy} \notin E(PG(R))$

Hence $E(ZDG(R)) \not\subseteq E(PG(R))$.

Thus we verified that for the ring of 3×3 matrices over Z_2, the two graphs: prime graph and zero divisor graph are not equal.

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