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Homotopy Perturbation Method for Solving Nonlinear Partial Differential Equations

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Abstract: *Homotopy Perturbation Method (HPM) is an elegant and powerful method to solve linear and nonlinear partial differential equations. As we know to get an exact solution of nonlinear partial differential equation is very difficult, so any kind of perturbative approach is acceptable depending on its criteria. Homotopy Perturbation Method provides an analytical solution by using the initial conditions. It is interesting to note that only a few terms are required to obtain a most accurate approximate solution. In this paper we have applied this technique and got most accurate result considering only four terms. A graphical representation of the result has been shown which provides us the most accurate physical situation and accuracy of the solution. The HPM allows us to find the solution of the nonlinear partial differential equations which will be calculated in the form of a series with easily computable components. From the calculation and its graphical representation it is clear that how the solution of the original equation and its behavior depends on the initial conditions.*

Keywords: *Homotopy perturbation method, Approximate solutions, Exact solution, Fisher's equation, Initial condition.*

I. Introduction

The homotopy perturbation method was introduced by the Chinese researcher Dr. Ji Huan HE in 1998 [1,2,3,4,5,6,7]. Recently this method became popular and acceptable as an elegant tool in the hands of researchers because of its simplicity and give rise highly effective solutions of complicated problems in many diverse areas of science and technology. Many physical problems can be described by mathematical models that involve partial differential equations. In other words, a mathematical model is a simplified description of physical reality. The behavior of each model is governed by the input data for the particular problem: the boundary or initial conditions, the coefficient functions of the partial differential equation and the forcing function. This input data cause the solution of the model problem to possess highly localized properties in space in time or in both. Thus, the investigation of the exact or approximate solution helps us to understand the means of these mathematical models and the real physical significance can be understood from the graphical representation of the solution. The main goal of this paper is to apply the Homotopy Perturbation Method (HPM) to obtain an approximate solution of some nonlinear partial differential equations with initial conditions.

The perturbation technique is one of the analytical methods to solve non-linear differential equations. This technique is widely used by engineers to solve some practical problems. Most often we obtain many interesting and important results by using this technique. However, the perturbation methods have their own limitations. Firstly, all perturbation techniques are based on small or large parameters so that at least one unknown must be expressed in a series of small parameters. But unfortunately, not every non-linear differential equation has such a small parameter. Secondly, even if there exists such a parameter, the results given by perturbation methods are valid, in most cases, only for the small values of the parameter. Mostly, the simplified linear equations have different properties from the original non-linear differential equation and sometimes some initial or boundary conditions are superfluous for the simplified linear equations. As a result, the corresponding initial approximations are perhaps far from exact. Clearly, these limitations of perturbations techniques come from the small parameter assumption. So it seems necessary to develop a kind of new non-linear analytical method which does not require small parameters at all. Ji Huan He has described a non-linear analytical technique which does not require small parameters and thus can be applied to solve non-linear problems without small or large parameters. This technique is based on homotopy, which is an important part of topology. Using one interesting property of homotopy, one can transform any non-linear problem into an infinite number of linear problems, no matter whether or not there exists a small or large parameter.

To illustrate the general procedure let us consider a general nonlinear partial differential equation of the form

$$u_t = \mathrm{F}(x, t, u, u_x, u_{xx}) \qquad (x, t) \in (a, b) \times (0, \mathrm{T}) \qquad (1.1)$$

With the initial condition

$$u(x,0)=u_0 = f(x) \qquad x \in (a,b)$$

Where f is a function of variables and F is a function of differential operators and variables. This type of operator equations can be solved using approximate analytical schemes such as Adomian Decomposition Method (ADM), Variational Iteration Method (VIM), Homotopy Perturbation Method (HPM) and Homotopy Analysis Method (HAM), tanh-expansion method. These schemes generate an infinite series of solutions and do not have the problem of rounding error. The solution obtained by using these methods shows the applicability, accuracy and efficiency in solving a large class of nonlinear equations in physics, engineering and various branches of mathematics.

Our paper is organized as follows .In section II we have Generalized HE'S Homotopy Perturbation Method (HPM). In section III we have applied HPM for obtaining analytical approximate or exact solution of three different types of partial differential equations with initial conditions. In section IV the graphical representation of each solution has also been compared the approximate solution that we found for these problems with the exact solution. Finally, in section V the conclusion is provided.

II. Generalized He's Homotopy Perturbation Method (Hpm)

In this section we have illustrated the basic idea of HPM to apply in non-linear equations. Let us consider the following nonlinear differential equation of the form

$$A(u) - f(r) = 0,\ r \in \Omega \tag{2.1}$$

Subject to the boundary conditions:

$$B(u, \partial u/\partial n) = 0,\ r \in \Gamma, \tag{2.2}$$

where A is a general differential operator, B a boundary operator, $f(r)$ a known analytical function and Γ is the boundary of the domain Ω . In general one can divide the operator A into two parts: linear and non- linear. That means

$A = L + N$

where L is linear and N is non-linear.

Hence, equation (2.1) can now be rewritten as

$$L(u) + N(u) - f(r) = 0,\ r \in \Omega \tag{2.3}$$

By the homotopy technique, one can construct a homotopy in the following way $v(r,p) : \Omega \times [0,1] \to R$ which satisfies

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, p \in [0,1], r \in \Omega, \tag{2.4}$$

or $$H(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0 \tag{2.5}$$

where $p \in [0,1]$ is an embedding parameter, u_0 is an initial approximation of equation (1.1) which satisfies the boundary conditions. Obviously, from equations (2.4) and (2.5) we will have:

$$H(v,0) = L(v) - L(u_0) = 0 \tag{2.6}$$

$$H(v,1) = A(v) - f(r) = 0 \tag{2.7}$$

The changing process of p from zero to unity is just that of $v(r,p)$ from $u_0(r)$ to $u(r)$. In topology, this is called deformation and $L(v) - L(u_0)$ and $A(v) - f(r)$ are called homotopy. According to the HPM, we can first use the embedding parameter p as a "small parameter" and assume that the solution of equations (2.6) and (2.7) can be written as a power series in p

$$v = v_0 + pv_1 + p^2 v_2 + ... \tag{2.8}$$

Setting $p = 1$ results in the approximate solution of equation (2.1):

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + ... \tag{2.9}$$

The combination of the perturbation method and the homotopy method is called the homotopy perturbation method (HPM), which has eliminated the limitations of the traditional perturbation methods. On the other hand, this technique can have full advantage of the traditional perturbation techniques. The series (2.9) is convergent for most cases.

However, the convergent rate depends on the nonlinear operator $A(v)$:

1. The second derivative of $N(v)$ with respect to v must be small because the parameter may be relatively large, i.e. $p \to 1$.

2. The norm of $L^{-1}\partial N/\partial v$ must be smaller than one so that the series converges.

According to above Homotopy perturbation method, we can write equation (1.1) as

$$H(v,p) \equiv (1-p)\left(\frac{\partial v}{\partial t}-\frac{\partial u_0}{\partial t}\right)+p\left[\frac{\partial v}{\partial t}-F\left(x,t,v,\frac{\partial v}{\partial x},\frac{\partial^2 v}{\partial x^2}\right)\right]=0, \quad p\in[0,1] \tag{2.10}$$

Or

$$H(v,p) \equiv \left(\frac{\partial v}{\partial t}-\frac{\partial u_0}{\partial t}\right)+p\frac{\partial u_0}{\partial t}-pF\left(x,t,v,\frac{\partial v}{\partial x},\frac{\partial^2 v}{\partial x^2}\right)=0, \quad p\in[0,1] \tag{2.11}$$

Where $p\in[0,1]$ is an embedding parameter and u_0 is the initial approximation of Equation (1.1) which satisfies the boundary conditions. It is obvious that when $p \to 1$ then $v \to u$ and equation (2.10) becomes equation (1.1)

Now for $p=0$ and $p=1$ equation (2.10) and (2.11) we will have the following form

$$H(v,0) \equiv \frac{\partial v}{\partial t}-\frac{\partial u_0}{\partial t}=0 \tag{2.12}$$

$$H(v,1) \equiv \frac{\partial v}{\partial t}-F\left(x,t,v,v_x,v_{xx}\right)=0 \tag{2.13}$$

According to the (HPM), we can first use the embedding parameter *p* as a small parameter and assume that the solutions of Equations (2.10) and (2.11) can be written as a power series in *p*:

$$v=\sum_{i=0}^{\infty} v_i\, p^i \tag{2.14}$$

by substituting (2.14) into (2.11) we have

$$\sum_{i=0}^{\infty} p^i \frac{\partial v_i}{\partial t}-\frac{\partial u_0}{\partial t}=p\left(-\frac{\partial u_0}{\partial t}+F\left(x,t,\sum_{i=0}^{\infty} v_i p^i,\sum_{i=0}^{\infty} p^i\frac{\partial v_i}{\partial x},\sum_{i=0}^{\infty} p^i\frac{\partial^2 v_i}{\partial x^2}\right)\right)=0 \tag{2.15}$$

setting $p=1$ we get the approximate solution of Equation (1.1)

$$u=\lim_{p\to 1} v=v_0+v_1+v_2+v_3+\dots\dots\dots \tag{2.16}$$

III. Solution Of Some Nonlinear Partial Differential Equations

In this section we have applied the HPM for obtaining the analytical approximate solution of these Nonlinear different partial differential equations with the different initial conditions.

Example 1

Let us consider the wave-like equation [10] in the following form

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+u\frac{\partial u}{\partial x}+u-u^2 \tag{3.1}$$

Now let us try to find the solution using HPM with the following initial condition:

$$u(x,0)=1+e^{x} \tag{3.2}$$

Equation (3.1) is a nonlinear partial differential equation. So, in principle to get an exact solution is very difficult. However, here we can see that

$$u(x,t)=1+e^{(x+t)} \tag{3.3}$$

is an exact solution of (3.1)

Now let us try to solve this equation using HPM

In order to solve equation (3.1) using HPM, equation (2.11) can be constructed as follows

$$H(v,p)=\frac{\partial \mathrm{v}}{\partial t}-\frac{\partial u_0}{\partial t}+\mathrm{p}\frac{\partial \mathrm{u}_0}{\partial \mathrm{t}}-p[\frac{\partial^2 \mathrm{v}}{\partial \mathrm{x}^2}+v\frac{\partial v}{\partial x}+v-v^2]=0 \tag{3.4}$$

Suppose the solution of (4.1) has the form

$$v=\sum_{i=0}^{\infty} v_i\, p^i \tag{3.5}$$

Substituting (3.5) into equation (3.4) yields

$$\frac{\partial(\mathrm{v}_0+v_1p+v_2p^2+.......)}{\partial t}-\frac{\partial u_0}{\partial t}=\mathrm{p}(-\frac{\partial \mathrm{u}_0}{\partial \mathrm{t}})+\mathrm{p}(\frac{\partial^2(\mathrm{v}_0+v_1p+v_2p^2+......)}{\partial \mathrm{x}^2})$$

$$+\mathrm{p}(\mathrm{v}_0+v_1p+v_2p^2+.......)\frac{\partial}{\partial x}(\mathrm{v}_0+v_1p+v_2p^2+.......)$$

$$+\mathrm{p}(\mathrm{v}_0+v_1p+v_2p^2+.......)-\mathrm{p}(\mathrm{v}_0+v_1p+v_2p^2+.......)^2$$

Comparing coefficient of terms with identical powers of p leads to :

$$\mathrm{p}^0\ :\ \frac{\partial \mathrm{v}_0}{\partial \mathrm{t}}-\frac{\partial u_0}{\partial t}=0 \tag{3.6}$$

$$\mathrm{p}^1\ :\ \frac{\partial \mathrm{v}_1}{\partial t}=-\frac{\partial \mathrm{u}_0}{\partial \mathrm{t}}+\frac{\partial^2 \mathrm{v}_0}{\partial \mathrm{x}^2}+v_0\frac{\partial v_0}{\partial x}+v_0-v_0^2 \tag{3.7}$$

$$\mathrm{p}^2:\ \frac{\partial v_2}{\partial t}=\frac{\partial^2 v_1}{\partial x^2}+v_1\frac{\partial v_0}{\partial x}+v_0\frac{\partial v_1}{\partial x}+v_1-2v_0v_1 \tag{3.8}$$

$$p^3:\ \frac{\partial v_3}{\partial t}=\frac{\partial^2 v_2}{\partial x^2}+v_2\frac{\partial v_0}{\partial x}+v_0\frac{\partial v_2}{\partial x}+v_1\frac{\partial v_1}{\partial x}+v_2-2v_0v_2-v_1^2 \tag{3.9}$$

and so on.

Solving equation (3.6) and using the initial condition we have

$$\frac{\partial v_0}{\partial t}=0$$

The solution of which is

$$v_0=A \tag{3.10}$$

Again using the initial conditions at $\mathrm{t}=0$ we get

$$\mathrm{v}_0=1+e^x \tag{3.11}$$

Using this solution we can find the solution of (3.7). That means (3.7) now becomes

$$\frac{\partial \mathrm{v}_1}{\partial t}=\frac{\partial^2 \mathrm{v}_0}{\partial \mathrm{x}^2}+v_0\frac{\partial v_0}{\partial x}+v_0-v_0^2$$

The solution of which is

$\mathrm{v}_1=te^x$ where initial condition has been used (3.12)

Now, let us solve equation (3.8) using the solutions of v_0 and v_1

we get

$$\frac{\partial \mathrm{v}_2}{\partial t}=te^x$$

Integrating and using the initial conditions we get the solution of v_2 as

$$\mathrm{v}_2 = \frac{1}{2} t^2 e^x \tag{3.13}$$

Our final step is to find the solution for v_3 from (3.9), using the solution v_0, v_1 and v_2 in (3.9) we get,

$$\frac{\partial v_3}{\partial t} = \frac{1}{2} t^2 e^x$$

Integrating and using the initial conditions we get the solution

$$v_3 = \frac{1}{3!} t^2 e^x \tag{3.14}$$

Similarly following the above procedure we can find the other solutions.
We have already mentioned in (2.16) that the approximate or exact solution of (3.1) is

$$u = v = v_0 + v_1 + v_2 + v_3 + \dots\dots\dots\dots$$

Substituting the results that we obtained in v_0, v_1, v_2 and v_3

$$u = 1 + e^x \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots\dots\dots\dots\right) \tag{3.15}$$

which leads to the exact solution

$$u(x,t) = 1 + e^{x+t} \tag{4.16}$$

We see that this approximate solution is exactly same as exact solution. Hence, we can conclude that this HPM is a powerful method to get the most accurate result of nonlinear differential equation. However, these are cases where the approximate solution is not exactly match with the exact solution then how accuracy is maintained can be seen from the graphical representation of the solution and also the difference between exact and approximate solution. To understand this situation we are displaying a few graphs of the solution.

Example 2

To apply HPM in more complicated problem let us consider Fisher's equation [11]

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 6u(1-u) \tag{3.17}$$

and try to find the solution with the following initial condition:

$$\mathrm{u(x,0)} = -\frac{1}{4}\left[\mathrm{sec}h^2\left(-\sqrt{\frac{1}{4c}}x\right) - 2\tanh\left(-\sqrt{\frac{1}{4c}}x\right) - 2\right] \tag{3.18}$$

The properties of Fisher's equation have been designed theoretically by many authors. The analysis of traveling wave solution of Fisher's equation has been studied by many computational approaches. Traveling wave fronts have important applications in various fields of science and engineering.
The exact solution of the problem (3.17) is given as

$$\mathrm{u(x,\,t)} = -\frac{1}{4}\left[\mathrm{sec}h^2\left(-\sqrt{\frac{1}{4c}}x + \frac{5}{2}t\right) - 2\tanh\left(-\sqrt{\frac{1}{4c}}x + \frac{5}{2}t\right) - 2\right] \tag{3.19}$$

This equation states that the change of labeled particles at a given time depends on the infection rate $6u(1-u)$ and the diffusion in the neighboring area.

The term $6u$ measures the infection rate which is proportional to the product of the density of the infected and uninfected particles. The term $-6u^2$ shows how fast the infected particles are diffusing. The amplitude of the wave is proportional to 1.

In order to solve equation (3.17) using HPM a homotopy- perturbation method can be constructed as follows

$$H(v,p)=\frac{\partial \mathrm{v}}{\partial t}-\frac{\partial u_0}{\partial t}+\mathrm{p}\frac{\partial \mathrm{u}_0}{\partial \mathrm{t}}-p[\frac{\partial^2 \mathrm{v}}{\partial \mathrm{x}^2}+6v-6v^2]=0 \tag{3.20}$$

Suppose the solution of (3.17) has the form

$$v=\sum_{i=0}^{\infty} v_i\, p^{i} \tag{3.21}$$

Substituting (3.21) into equation (3.20)

$$\frac{\partial(\mathrm{v}_0+v_1 p+v_2 p^2+.......)}{\partial t}-\frac{\partial u_0}{\partial t}=\mathrm{p}\{-\frac{\partial \mathrm{u}_0}{\partial \mathrm{t}}+(\frac{\partial^2(\mathrm{v}_0+v_1 p+v_2 p^2+......)}{\partial \mathrm{x}^2})$$

$$+6((\mathrm{v}_0+v_1 p+v_2 p^2+.......)-6(\mathrm{v}_0+v_1 p+v_2 p^2+.......)^2\}$$

and comparing coefficient of terms with identical powers of p, leads to :

$$\mathrm{p}^0 \quad : \quad \frac{\partial \mathrm{v}_0}{\partial \mathrm{t}}-\frac{\partial u_0}{\partial t}=0 \tag{3.22}$$

$$\mathrm{p}^1 \ : \quad \frac{\partial \mathrm{v}_1}{\partial t}=-\frac{\partial \mathrm{u}_0}{\partial \mathrm{t}}+\frac{\partial^2 \mathrm{v}_0}{\partial \mathrm{x}^2}+6v_0-6v_0^2 \tag{3.23}$$

$$\mathrm{p}^2 : \quad \frac{\partial \mathrm{v}_2}{\partial t}=\frac{\partial^2 v_1}{\partial x^2}+6v_1-6v_0 v_1 \tag{3.24}$$

$$p^3 : \quad \frac{\partial v_3}{\partial t}=\frac{\partial^2 v_2}{\partial x^2}+6v_2-6v_1^2 \tag{3.25}$$

Solving the above equations and using the initial condition we get

$$\mathrm{v}_0=-\frac{1}{4}[\sec h^2(-\sqrt{\frac{1}{4c}}x)-2\tanh(-\sqrt{\frac{1}{4c}}x)-2] \tag{3.26}$$

$$\mathrm{v}_1=\mathrm{t}[\{\frac{3}{2}(2-\sec h^2(\frac{1}{2}\sqrt{\frac{1}{c}}x)-2\tanh(\frac{1}{2}\sqrt{\frac{1}{c}}x)\}-\frac{3}{8}\{(2-\sec h^2(\frac{1}{2}\sqrt{\frac{1}{c}}x)$$

$$-2\tanh(\frac{1}{2}\sqrt{\frac{1}{c}}x)\}^2+\frac{1}{4}\{\frac{\sec h^4(\frac{1}{2}\sqrt{\frac{1}{c}}x)}{2c}+\frac{\sec h^2(\frac{1}{2}\sqrt{\frac{1}{c}}x)\tanh(\frac{1}{2}\sqrt{\frac{1}{c}}x)}{c} \tag{3.27}$$

$$-\frac{\sec h^2(\frac{1}{2}\sqrt{\frac{1}{c}}x)\ \tanh^2(\frac{1}{2}\sqrt{\frac{1}{c}}x)}{c}\}]$$

$$v_2 = \frac{1}{64c^2} t^2 \operatorname{sech}^6(\frac{1}{2}\sqrt{\frac{1}{c}}x)(\cosh(\frac{1}{2}\sqrt{\frac{1}{c}}x) - \sinh(\frac{1}{2}\sqrt{\frac{1}{c}}x))\ \{3(-5-8c$$
$$+48c^2)\cosh(\frac{1}{2}\sqrt{\frac{1}{c}}x) + (7+30c+18c^2)\cosh(\frac{3}{2}\sqrt{\frac{1}{c}}x) \tag{3.28}$$
$$+6\{-7+19c+3c^2+(3+6c+6c^2)\cosh(\sqrt{\frac{1}{c}}x)\}\sinh(\frac{1}{2}\sqrt{\frac{1}{c}}x)\}$$

$$v_3 = \frac{1}{768c^3} t^3 \operatorname{sech}^8(\frac{1}{2}\sqrt{\frac{1}{c}}x)(\cosh(\frac{1}{2}\sqrt{\frac{1}{c}}x) - \sinh(\frac{1}{2}\sqrt{\frac{1}{c}}x))\ \{2(245+63c$$
$$-1098c^2+810c^3)\ \cosh(\frac{1}{2}\sqrt{\frac{1}{c}}x) + (-385-570c+1044c^2$$
$$+216c^3)\cosh(\frac{3}{2}\sqrt{\frac{1}{c}}x) + 31\cosh(\frac{5}{2}\sqrt{\frac{1}{c}}x) + +144c\cosh(\frac{5}{2}\sqrt{\frac{1}{c}}x)$$
$$+252c^2\ \cosh(\frac{5}{2}\sqrt{\frac{1}{c}}x) + 108c^3\cosh(\frac{5}{2}\sqrt{\frac{1}{c}}x) + 1926\sinh(\frac{1}{2}\sqrt{\frac{1}{c}}x)$$
$$-3678c\ \sinh(\frac{1}{2}\sqrt{\frac{1}{c}}x) + 1476c^2\sinh(\frac{1}{2}\sqrt{\frac{1}{c}}x) + 756c^3\sinh(\frac{1}{2}\sqrt{\frac{1}{c}}x)$$
$$-561\sinh(\frac{3}{2}\sqrt{\frac{1}{c}}x) - 42c\sinh(\frac{3}{2}\sqrt{\frac{1}{c}}x) + 468c^2\sinh(\frac{3}{2}\sqrt{\frac{1}{c}}x)$$
$$+1080c^3\sinh(\frac{3}{2}\sqrt{\frac{1}{c}}x) + 33\sinh(\frac{5}{2}\sqrt{\frac{1}{c}}x) + 144c\ \sinh(\frac{5}{2}\sqrt{\frac{1}{c}}x)$$
$$+180c^2\sinh(\frac{5}{2}\sqrt{\frac{1}{c}}x) + 108c^3\sinh(\frac{5}{2}\sqrt{\frac{1}{c}}x)\} \tag{3.29}$$

According to HPM we can write the solution of (4.17):

$$u(x,t) = \lim_{p\to 1} v = \lim_{p\to 1}[v_0 + pv_1 + p^2v_2 + p^3v_3 + \dots\dots\dots]$$

Setting $p = 1$ the above equation becomes,

$$u(x,t) = v_0 + v_1 + v_2 + v_3 + \dots\dots\dots\dots\dots$$

Then the approximate solution in a series form can be written as

$$u(x,t) = -\frac{1}{4}[\operatorname{sech}^2(-\sqrt{\frac{1}{4c}}x) - 2\tanh(-\sqrt{\frac{1}{4c}}x) - 2] + t[\{\frac{3}{2}(2-\operatorname{sech}^2(\frac{1}{2}\sqrt{\frac{1}{c}}x) - 2\tanh(\frac{1}{2}\sqrt{\frac{1}{c}}x)\}$$

$$-\frac{3}{8}\{(2-\operatorname{sech}^2(\frac{1}{2}\sqrt{\frac{1}{c}}x) - 2\tanh(\frac{1}{2}\sqrt{\frac{1}{c}}x))\}^2 + \frac{1}{4}\{\frac{\operatorname{sech}^4(\frac{1}{2}\sqrt{\frac{1}{c}}x)}{2c} + \frac{\operatorname{sech}^2(\frac{1}{2}\sqrt{\frac{1}{c}}x)\tanh(\frac{1}{2}\sqrt{\frac{1}{c}}x)}{c}$$

$$-\frac{\operatorname{sech}^2(\frac{1}{2}\sqrt{\frac{1}{c}}x)\ \tanh^2(\frac{1}{2}\sqrt{\frac{1}{c}}x)}{c}\}] + \frac{1}{64c^2}t^2\operatorname{sech}^6(\frac{1}{2}\sqrt{\frac{1}{c}}x)\{\cosh(\frac{1}{2}\sqrt{\frac{1}{c}}x) - \sinh(\frac{1}{2}\sqrt{\frac{1}{c}}x)\}$$

$$\{3(-5-8c+48c^2)\cosh(\frac{1}{2}\sqrt{\frac{1}{c}}x) + (7+30c+18c^2)\cosh(\frac{3}{2}\sqrt{\frac{1}{c}}x) + 6\{-7+19c+3c^2$$

$$+(3+6c+6c^2)\cosh(\sqrt{\frac{1}{c}}x)\}\sinh(\frac{1}{2}\sqrt{\frac{1}{c}}x)\} + \frac{1}{768c^3}t^3\operatorname{sech}^8(\frac{1}{2}\sqrt{\frac{1}{c}}x)(\cosh(\frac{1}{2}\sqrt{\frac{1}{c}}x) - \sinh(\frac{1}{2}\sqrt{\frac{1}{c}}x))$$

$$\{2(245+63c-1098c^2+810c^3)\ \cosh(\frac{1}{2}\sqrt{\frac{1}{c}}x) + (-385-570c+1044c^2+216c^3)\cosh(\frac{3}{2}\sqrt{\frac{1}{c}}x)$$

$$+31\cosh(\frac{5}{2}\sqrt{\frac{1}{c}}x) + +144c\cosh(\frac{5}{2}\sqrt{\frac{1}{c}}x) + 252c^2\ \cosh(\frac{5}{2}\sqrt{\frac{1}{c}}x) + 108c^3\ \cosh(\frac{5}{2}\sqrt{\frac{1}{c}}x)$$

$$+1926\sinh(\frac{1}{2}\sqrt{\frac{1}{c}}x) - 3678c\ \sinh(\frac{1}{2}\sqrt{\frac{1}{c}}x) + 1476c^2\sinh(\frac{1}{2}\sqrt{\frac{1}{c}}x) + 756c^3\sinh(\frac{1}{2}\sqrt{\frac{1}{c}}x)$$

$$-561\sinh(\frac{3}{2}\sqrt{\frac{1}{c}}x) - 42c\sinh(\frac{3}{2}\sqrt{\frac{1}{c}}x) + 468c^2\sinh(\frac{3}{2}\sqrt{\frac{1}{c}}x) + 1080c^3\sinh(\frac{3}{2}\sqrt{\frac{1}{c}}x) + 33\sinh(\frac{5}{2}\sqrt{\frac{1}{c}}x)$$

$$+144c\ \sinh(\frac{5}{2}\sqrt{\frac{1}{c}}x) + 180c^2\sinh(\frac{5}{2}\sqrt{\frac{1}{c}}x) + 108c^3\sinh(\frac{5}{2}\sqrt{\frac{1}{c}}x)\}$$

$$+\ldots\ldots\ldots\ldots\ldots\ldots\ldots \tag{3.30}$$

This is the approximate solution of the Fisher's non-linear partial differential equation (3.17).

IV. Figures And Table

Graphical Representation of The Solution (3.16) of Example 1

To get a clear idea of the solution if we draw a few graphs taking some range of x values and t values we get three dimensional graphs. Similarly for a particular values of t and some range of x values we get 2-dimensional graphs. Some of the graphs obtained from solution (3.16) has been depicted below.

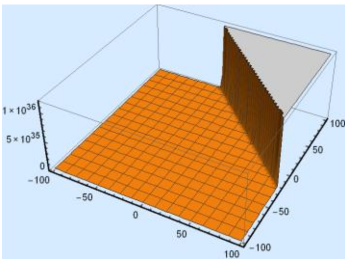
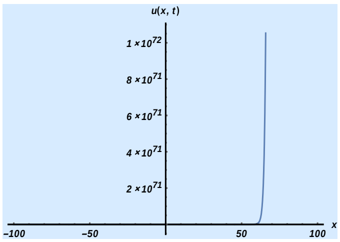
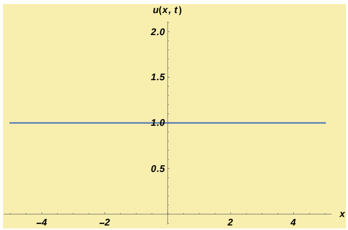


Figure 1: The surface of $u(x,t)$ for $t \in (-100,100)$, $x \in (-100,100)$



1(a) Corresponding 2D figure for t=100



1(b) Corresponding 2D figure for t=-100

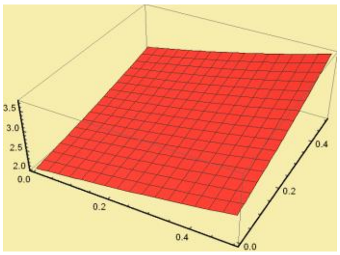
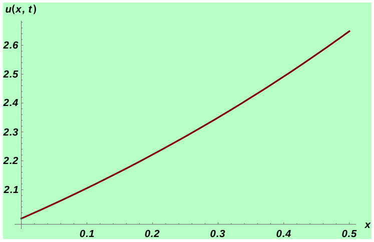
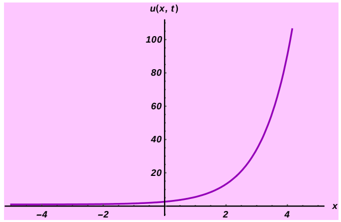


Figure 2: The surface of $u(x,t)$ for $t \in (0,.5)$, $x \in (0,.5)$



2 (a) Corresponding 2D figure for t=0



2(b) Corresponding 2D figure for t=.5

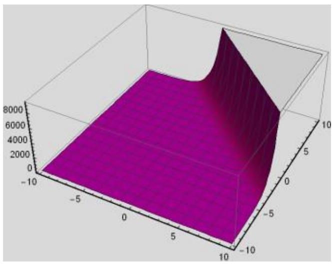
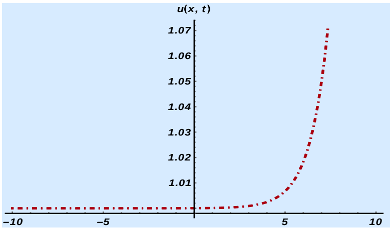
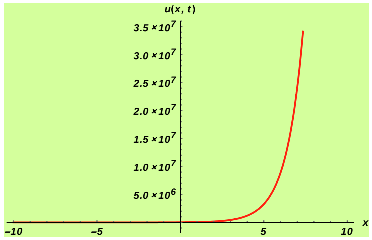


Figure 3: The surface of $u(x,t)$ for $t \in (-10,10)$, $x \in (-10,10)$



3 (a) Corresponding 2D figure for t=-10



3 (b) Corresponding 2D figure for t=10

Result and Discussion for Example 1: The graphs in Fig. 1, 2, 3 represent the solution of the given differential equation for different ranges of x and t values and corresponding two dimensional Fig. have been shown in 1(a), 1(b), 2(a), 2(b),3(a) and 3(b). We see that this approximate solution is exactly same as exact solution. Hence, we can conclude that this HPM is a powerful method to get the most accurate result of nonlinear differential equation. However, these are cases where the approximate solution is not exactly match with the exact solution then how accuracy is maintained can be seen from the graphical representation of the solution and also the difference between exact and approximate solution. To understand this situation we are displaying a few graphs of the solution.

Graphical Representation of the Solution (3.17) of The Example 2

The computed results are presented graphically by three dimensional and corresponding two dimensional graphs for $c = 1$. The surfaces are drawn for different ranges of variables but the two dimensional figures are drawn for a fixed value of t.

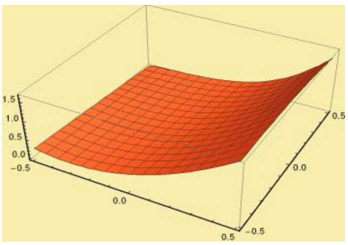
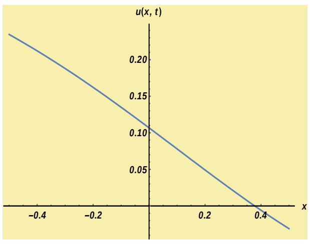
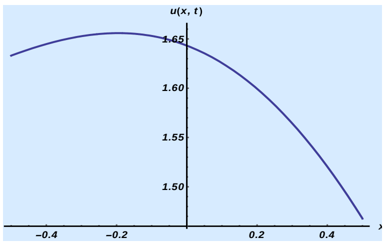


Figure 4 : The surface of $u(x,t)$ for $t \in (-.5,.5)$, $x \in (-.5,.5)$



4(a)Corresponding 2D figure for t= - 0.5



4(b) Corresponding 2D figure for t= 0.5

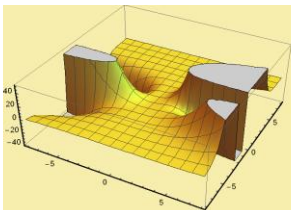
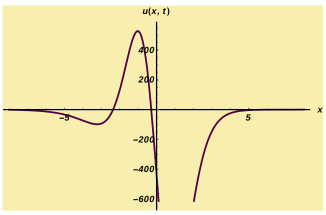
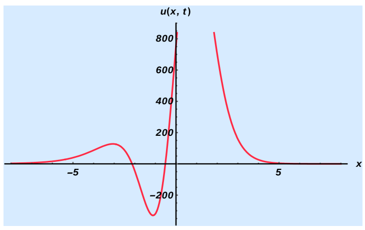


Figure 5: The surface of $u(x,t)$ for $t \in (-8,8)$, $x \in (-8,8)$



5(a) Corresponding 2D figure for t= - 8



5(b) Corresponding 2D figure for t= 8

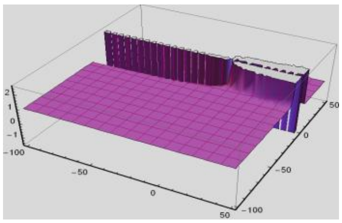
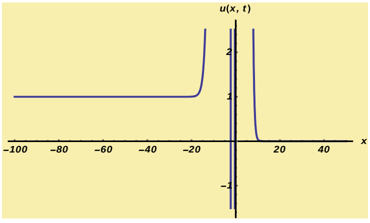
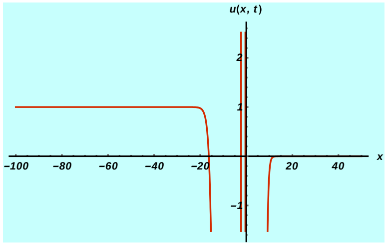


Figure 6: The surface of $u(x,t)$ for $t \in (-100,50), x \in (-100,50)$



6(a) Corresponding 2D figure for t=50



6(b) Corresponding 2D figure for t=-100

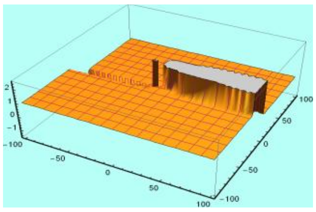
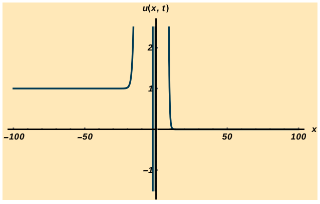
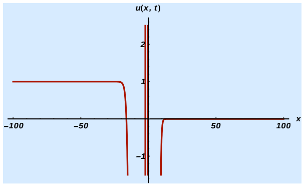


Figure 7 : The surface of $u(x,t)$ for $t \in (-100,100), x \in (-100,100)$



7(a) Corresponding 2D figure for t=100



7(b) Corresponding 2D figure for t= - 100

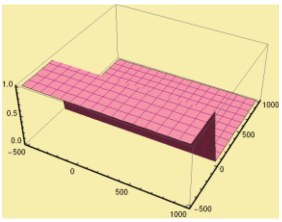
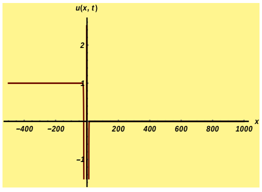
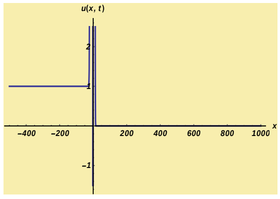


Figure 8 : The surface of $u(x,t)$ for $t \in (-500,1000),\ x \in (-500,1000)$



8(a) Corresponding 2D figure for t= - 500



8(b) Corresponding 2D figure for t= 1000

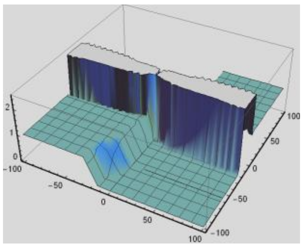


Figure 9 : The absolute error $t \in (-100,100)$ and $x \in (-100,100)$

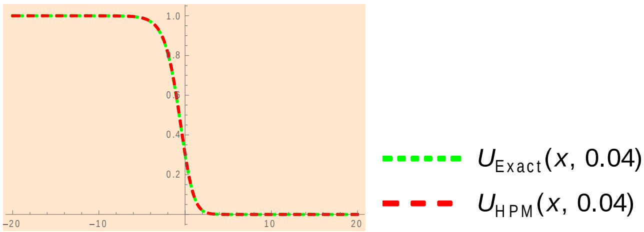


Figure 10: The curves of Exact solution $U_{Exact}(x,t)$ and the approximate solution $U_{HPM}(x,t)$ for $t = 0.04$ and $x \in (-20,20)$ respectively.

Table 1: comparison of approximate and exact solution at $t = .04$ for example 2.

x	Approximate	Exact	Error
0	3.04073×10^{-1}	3.02317×10^{-1}	1.75591×10^{-3}
1	9.63639×10^{-2}	9.61158×10^{-2}	2.48068×10^{-4}
2	2.01296×10^{-2}	2.01217×10^{-2}	7.90735×10^{-6}

3	3.28473×10^{-3}	3.28606×10^{-3}	1.32711×10^{-6}
4	4.78478×10^{-4}	4.7879×10^{-4}	3.12095×10^{-7}
5	6.65788×10^{-5}	6.66276×10^{-5}	4.88045×10^{-8}
6	9.10391×10^{-6}	9.11084×10^{-6}	6.92829×10^{-9}
7	1.23678×10^{-6}	1.23774×10^{-6}	9.53647×10^{-10}
8	1.67615×10^{-7}	1.67745×10^{-7}	1.29857×10^{-10}
9	2.2696×10^{-8}	2.27136×10^{-8}	1.7613×10^{-11}
10	3.07215×10^{-9}	3.07454×10^{-9}	2.38571×10^{-12}

Result and Discussion for the Example 2

The approximate results are presented graphically for various ranges of variables x and t in Fig. 4,5,6,7 and 8 and also have been shown corresponding two dimensional in Fig. 4(a), 4(b),5(a), 5(b), 6(a),6(b), 7(a), 7(b) and 8(a),8(b) for fixed values of t. Fig. 9 shows the absolute error for $x \in (-100,100)$ and $t \in (-100,100)$ by three dimensional graph. Table1 shows the comparison between homotopy perturbation method and the exact solution for $t = .04$ and $x \in [0,10]$.The errors are very very small in this table. It is clear from figure10 that the approximate solution and the exact solution are very close for the chosen values of t. The results provide very strong evidence that is the homotopy perturbation technique is easy to get approximate solution of nonlinear equation. It is to be noted that four terms only were used in evaluating the approximate solutions.

V. Conclusion

In this paper we have successfully developed HPM for solving nonlinear different types of partial differential equations with different types of initial conditions. It is clearly seen that HPM is a very powerful and efficient technique for finding solutions for wide classes of nonlinear partial differential equations in the form of analytical expressions. One of the importance advantages of the HPM is that it solves the nonlinear equations without any need for discretization, perturbation, transformation or linearization. The results of the numerical examples are presented to observe highly accuracy of the solution by HPM. It was demonstrated that the HPM is highly accurate and is a effective tool for solution of nonlinear partial differential equations.

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